# A hierarchy of noncanonical Hamiltonian systems

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**Abstract**

The dynamics of an ideal fluid or plasma is constrained by topological invariants such as the circulation of (canonical) momentum. In the Hamiltonian formalism, topological invariants restrict the orbits to submanifolds of the phase space. While the coadjoint orbits have a natural symplectic structure, the global geometry of the degenerate (constrained) Poisson manifold can be very complex. Some invariants are represented by the center of the Poisson algebra (i.e., the Casimir elements such as the helicities), and then, the global structure of phase space is delineated by Casimir leaves. However, a general constraint is not necessarily *integrable*, which precludes the existence of an appropriate Casimir element; the circulation is an example of such an invariant. In this work, we formulate a systematic method to embed a Hamiltonian system in an extended phase space; we introduce *mock fields* and extend the Poisson algebra. A mock field defines a new Casimir element, a *cross helicity*, which represents topological constraints which are not integrable in the original phase space. Changing the perspective, a singularity of the extended system may be viewed as a subsystem on which the mock fields (though they are *actual fields*, when viewed from the extended system) vanishes, i.e., the original system. This hierarchical relation of degenerate Poisson manifolds enables us to see the "interior" of a singularity as a sub Poisson manifold. The theory can be applied to describe the bifurcation and instabilities in a wide class of general Hamiltonian systems [Yoshida & Morrison, Fluid Dyn. Res. **46** (2014), 031412],

# 1 Introduction

The aim of this work is to delineate the topological constraints on an ideal fluid (or a plasma) by constructing *Casimir elements* and foliating the phase space; the state vector is constrained to move on a submanifold that is an intersection of the level-sets of the Casimir elements (which we call Casimir leaves).[1] Casimir elements are the invariants (first integrals) of the Hamiltonian mechanics, but are not pertinent to symmetries of a specific Hamiltonian; instead, Casimir elements are the attributes of the underlying Poisson algebra.

The "integrability" of the determining equation of Casimir elements is the mathematical issue to be explored. A general constraint is not necessarily integrable, which precludes the existence of an appropriate Casimir element; the circulation is an example of such an invariant. In this work, we formulate a systematic method to embed a Hamiltonian system in an extended phase space; we introduce *mock fields* and extend the Poisson algebra. A mock field defines a new Casimir element, a *cross helicity*, which represents topological constraints which are not integrable in the original phase space (see Morrison[2] for an early precursor to this idea.).

The singularities of the determining equation of Casimir elements are also of particular interest, which give rise to *singular Casimir elements*.[3, 4] Viewing from the extended phase space, the original Poisson manifold is the singularity of the extended system. One can probe into the singularity (which contains an infinite number of degrees of freedom) by the help of the singular Casimir elements.

# 2 Noncanonical Hamiltonian system (Poisson algebra)

A general Hamiltonian system may be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{z} = \mathcal{J}(\boldsymbol{z})\partial_{\boldsymbol{z}} H(\boldsymbol{z}), \tag{1}$$

where $\boldsymbol{z}$ is the state vector, a member of the phase space $X$ (here a Hilbert space endowed with an inner product $\langle\ ,\ \rangle$; in the later discussion, $X$ will be a function space, and then we will denote the state vector by $u$), $H(\boldsymbol{z})$ is the Hamiltonian (here a real-valued functional on $X$), $\partial_{\boldsymbol{z}}$ is the gradient in $X$, and $\mathcal{J}$ is the Poisson operator. We allow $\mathcal{J}$ to be a function of $\boldsymbol{z}$ on $X$, and write it as $\mathcal{J}(\boldsymbol{z})$. We assume that the bilinear product

$$\{F, G\} = \langle \partial_{\boldsymbol{z}} F(\boldsymbol{z}), \mathcal{J}\partial_{\boldsymbol{z}} G(\boldsymbol{z})\rangle$$

is antisymmetric and satisfies the Jacobi identity; then $\{\ ,\ \}$ is a Poisson bracket. By a "general Hamiltonian system" we mean a Poisson algebra $C^\infty_{\{\ ,\ \}}(X)$.

A *canonical* Hamiltonian system is endowed with a *symplectic* Poisson operator where

$$\mathcal{J}_c = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

However, our interest is in *noncanonical* systems endowed with Poisson operators $\mathcal{J}$ that are inhomogeneous and degenerate (i.e., $\mathrm{Ker}(\mathcal{J}(\boldsymbol{z}))$ contains nonzero elements, and its dimension may change depending on the position in $X$). Since $\mathcal{J}$ is antisymmetric, $\mathrm{Ker}(\mathcal{J}(\boldsymbol{z})) = \mathrm{Coker}(\mathcal{J}(\boldsymbol{z}))$, and hence, every orbit is topologically constrained on the orthogonal complement of $\mathrm{Ker}(\mathcal{J}(\boldsymbol{z}))$.

A functional $C(\boldsymbol{z})$ such that $\{C, G\} = 0$ for all $G$ is called a Casimir element (or an element of the *center* of the Poisson algebra). If $\mathrm{Ker}(\mathcal{J}) = \{0\}$, the case for a canonical Hamiltonian system, then there is only a trivial element $C = \text{constant}$ in the center. Evidently, a Casimir element $C(\boldsymbol{z})$ is a solution to the differential equation

$$\mathcal{J}(\boldsymbol{z})\partial_{\boldsymbol{z}} C(\boldsymbol{z}) = 0. \tag{2}$$

When the phase space $X$ has a finite dimension $(n)$, (2) is a first-order partial differential equation. If $\mathrm{Ker}(\mathcal{J}(\boldsymbol{z}))$ has a constant dimension $\nu$ in an open set $X_\nu \subseteq X$, and $n - \nu$ is an even number, we can integrate (2) in $X_\nu$ to obtain $\nu$ independent solutions, i.e., $\mathrm{Ker}(\mathcal{J}(\boldsymbol{z}))$ is locally spanned by the gradients of $\nu$ Casimir elements (Lie-Darboux theorem). The intersection of all Casimir leaves (the level-sets of Casimir elements) is the effective phase space, on which $\mathcal{J}(\boldsymbol{z})$ reduces to a symplectic Poisson operator.

However, the general (global) integrability of (2) is a mathematical challenge. Let us see a simple example that epitomizes the problems.

**Example 1** *Let us consider a two-dimensional system* $\boldsymbol{z} = {}^t(x, y) \in X = \mathbb{R}^2$ *with*

$$\mathcal{J} = x\mathcal{J}_c.$$

*The plane* $x = 0$ *is a singularity at which* $\mathrm{Rank}(\mathcal{J})$ *drops by two. The kernel of* $J$ *consists of two eigenvectors:*

$$\mathrm{Ker}(\mathcal{J}) = \{\alpha_x \boldsymbol{\nu}_x + \alpha_y \boldsymbol{\nu}_y;\ \alpha_x, \alpha_y \in \mathbb{R}\},$$
$$\begin{cases} \boldsymbol{\nu}_x = \delta(x)\boldsymbol{e}_x, \\ \boldsymbol{\nu}_y = \delta(x)\boldsymbol{e}_y. \end{cases} \tag{3}$$

*The first component* $\boldsymbol{\nu}_x$ *of the kernel is* integrable *to produce a singular Casimir element:*

$$\boldsymbol{\nu}_x = \partial_{\boldsymbol{z}} C_{\mathrm{ex}}(\boldsymbol{z}), \quad C_{\mathrm{ex}}(\boldsymbol{z}) = Y(x),$$

*where $Y(x)$ is the Heaviside step function (with the gap filled). We call $C_{\rm ex}$ an* exterior Casimir element*. However, the second component $\boldsymbol{\nu}_y$ is not a closed 1-form, thus it is not representable as a gradient of some scalar function, i.e., we cannot integrate $\boldsymbol{\nu}_y$ to produce a Casimir element. Characterization of this odd element of* $\mathrm{Ker}(\mathcal{J})$ *is one of the main issues of this work.*

The problems and our strategy of study are summarized as follows:

1. To see the essence of the problem, let us assume that $X$ is a cotangent bundle $T^*M$ of a finite-dimensional manifold $M$. Suppose that $\boldsymbol{w}$ is a non-zero element of $\mathrm{Ker}(\mathcal{J})$. For $\boldsymbol{w} \in T^*M$ to be written as $\boldsymbol{w} = \partial_{\boldsymbol{z}} C$ $(= \mathrm{d}C)$ with a Casimir element (0-form) $C$, $\boldsymbol{w}$ must be an exact 1-from (or, for local integrability, it must be a closed 1-form). This is a rather strong condition; one can easily construct a counter example that violates the integrability; see Example 1. Our strategy of improving the integrability of (2) is to embed the Poisson manifold in higher-dimensional spaces; by adding extra components to $\boldsymbol{w}$, we may make it exact in a higher-dimension space.
2. The point where the rank of $\mathcal{J}(\boldsymbol{z})$ changes is a singularity of (2), from which singular (hyperfunction) solutions are generated.[4] However, the hyperfunction Casimir elements fall short of spanning $\mathrm{Ker}(\mathcal{J})$; see Example 1. This problem of "Casimir deficit" suggests that the interior of the singularity cannot be well described by hyperfunctions (which are, in fact, the cohomology class of the sheaf of holomorphic functions). We may yet describe the interior of the singularity as a subsystem (or, a Poisson submanifold), on which a non-integrable element of $\mathrm{Ker}(\mathcal{J})$ may be integrable to define a Casimir element of a "reduced" Poisson operator.
3. Because models of fluids and plasmas are formulated on an infinite-dimensional phase space, we have to develop an infinite-dimensional theory. For these systems (2) is a functional differential equation, and a singularity may cause an infinite-dimensional rank change. The reader is referred to Yoshida, Morrison & Dobarro[3] for an example of a singular Casimir element generated by singularities in a function space.

# 3 A hierarchy of vortex dynamics systems

In Table 1 we compare the Hamiltonian formalisms of well-known examples of two-dimensional vortex dynamics systems.[5, 6]. We denote by $\omega = -\Delta\varphi$ the vorticity with $\Delta$ being the Laplacian and

Table 1: Hierarchy of two-dimensional vortex systems. Here $[a, b] = \partial_y a \partial_x b - \partial_x a \partial_y b$.

| system | state vector | Poisson operator |
|---|---|---|
| (I) | $\omega$ | $\mathcal{J}_{\mathrm{I}} = [\omega, \circ]$ |
| (II) | $\begin{pmatrix} \omega \\ \psi \end{pmatrix}$ | $\mathcal{J}_{\mathrm{II}} = \begin{pmatrix} [\omega, \circ] & [\psi, \circ] \\ [\psi, \circ] & 0 \end{pmatrix}$ |
| (III) | $\begin{pmatrix} \omega \\ \psi \\ \check{\psi} \end{pmatrix}$ | $\mathcal{J}_{\mathrm{III}} = \begin{pmatrix} [\omega, \circ] & [\psi, \circ] & [\check{\psi}, \circ] \\ [\psi, \circ] & 0 & 0 \\ [\check{\psi}, \circ] & 0 & 0 \end{pmatrix}$ |

$\varphi \in H_0^1(\Omega) \cap H^2(\Omega)$ for the two-dimensional Eulerian velocity field $\boldsymbol{V} = {}^t(\partial_y \varphi, -\partial_x \varphi)$. Given a Hamiltonian

$$H_{\mathrm{E}}(\omega) = -\frac{1}{2} \int \mathrm{d}^2 x \, \omega \, (\Delta^{-1} \omega),$$

the system (I) is the vorticity equation for Eulerian flow,

$$\partial_t \omega + \boldsymbol{V} \cdot \nabla \omega = 0.$$

The Casimir elements of the system (I) are

$$C_0 = \int \mathrm{d}^2 x \, f(\omega),$$

where $f$ is an arbitrary $C^2$ function.

If $\psi$ is the Gauss potential of a magnetic field, i.e., $\boldsymbol{B} = {}^t(\partial_y \psi, -\partial_x \psi)$, and the Hamiltonian is

$$H_{\mathrm{RMHD}}(\omega, \psi) = -\frac{1}{2} \int \mathrm{d}^2 x \, \left[ \omega \, (\Delta^{-1} \omega) + \psi \, (\Delta \psi) \right],$$

the system (II) is the reduced MHD system,

$$\begin{aligned} \partial_t \omega + \boldsymbol{V} \cdot \nabla \omega &= \boldsymbol{J} \times \boldsymbol{B}, \\ \partial_t \psi + \boldsymbol{V} \cdot \nabla \psi &= 0. \end{aligned}$$

In the system (II), $C_0$ is no longer a constant of motion, being replaced by

$$\begin{aligned} C_1 &= \int \mathrm{d}^2 x \, \omega g(\psi), \\ C_2 &= \int \mathrm{d}^2 x \, f(\psi). \end{aligned}$$

However, if the Hamiltonian $H$ is independent of $\psi$, the dynamics of $\omega$ is unaffected by $\psi$, while both $\omega$ and $\psi$ obey the same evolution equation. Then, we call $\psi$ a *mock field* which can be chosen arbitrarily without changing the dynamics of the *actual field* $\omega$. At the special choice of $\psi = \omega$, both $C_1$ and $C_2$ evaluate as $C_0$, i.e., $C_0$ is subsumed by $C_1$ (so-called cross helicity)[7] and $C_2$ as their special value (indeed, $C_1$ and $C_2$ carry more information about system (I); in Sec. 4, we will see how the mock field probes the topological constraints). The constancy of $C_0$ is, then, due to the symmetry $\partial_\psi H = 0$. A modification of the Hamiltonian to involve $\psi$ destroys the constancy of $C_0$; the electromagnetic interaction is a physical example of such a modification.

We can extend the phase space further to obtain a system (III) by adding another field $\check{\psi}$ that obeys the same evolution equation as $\psi$. In the reduced MHD system, $\check{\psi}$ is a mock field, i.e., it does not have a direct physical meaning; however, in the original RMHD context such a field physically correspond to the pressure in the high-beta MHD model.[5] The Casimir elements of this further extended system are

$$
\begin{aligned}
C_2 &= \int \mathrm{d}^2x\, f(\psi), \\
C_3 &= \int \mathrm{d}^2x\, h(\psi\check{\psi}), \\
C_4 &= \int \mathrm{d}^2x\, \check{f}(\check{\psi}).
\end{aligned}
$$

Replacing $C_1$, we obtain new Casimir elements $C_3$ and $C_4$.

## 4 Integrability of topological constraints

An interesting consequence of extending the system from (I) to (II) is found in the *integrability* of the $\mathrm{Ker}(\mathcal{J}_\mathrm{I})$. In (I),

$$
\mathrm{Ker}(\mathcal{J}_\mathrm{I}(\omega)) = \{\psi;\, [\omega, \psi] = 0\},
$$

which implies that $\psi$ and $\omega$ are related, invoking a certain scalar $\zeta(x, y)$, by

$$
\psi = \eta(\zeta), \quad \omega = \xi(\zeta). \tag{4}
$$

As far as $\xi$ is a monotonic function, we may write $\psi = \eta(\xi^{-1}(\omega))$, which we can integrate to obtain the Casimir element $C_0(\omega)$ with $f(\omega)$ such that $f'(\omega) = \eta(\xi^{-1}(\omega))$. Other elements of $\mathrm{Ker}(\mathcal{J}_\mathrm{I}(\omega))$ that are given by nonmonotonic $\xi$ are not integrable to define Casimir elements. Yet, we can integrate such elements as $C_1(\omega, \psi)$ in the extended space of (II). In fact, every member of $\mathrm{Ker}(\mathcal{J}_\mathrm{I}(\omega))$ can be represented as $\partial_\omega C_1 = g(\psi)$ by choosing $\psi$ in $\mathrm{Ker}(\mathcal{J}_\mathrm{I}(\omega))$.

Similarly, in the system (II), we encounter the deficit of the Casimir element $C_2 = \int \mathrm{d}^2x f(\psi)$ in covering all elements ${}^t(0, \chi) \in \mathrm{Ker}(\mathcal{J}_\mathrm{II}(\omega, \psi))$ such that $[\psi, \chi] = 0$. By the help of a mock filed $\check{\psi}$, we can integrate every element of $\mathrm{Ker}(\mathcal{J}_\mathrm{II}(\omega, \psi))$ as $C_3$.

## 5 Submanifold of singularity

In the preceding section, we extended the phase space to improve the integrability of the topological constraints (kernel elements). Here we reverse the perspective, and see the "singularity" as a submanifold of a larger system. We will introduce a new notion of *exterior Casimir elements* and *interior Casimir elements*; the former is a hyperfunction that identifies the singularity as a cohomology, while the latter turns out to be a Casimir element of the subsystem, which is invisible in the larger (extended) system.

Let us start with the system (II). Apparently, the rank of $\mathcal{J}_{\rm II}$ drops (by infinite dimension) at the submanifold $\psi = 0$. This singularity is a leaf of the singular Casimir element (which we call an exterior Casimir element)

$$C_{\rm ex} = Y(\|\psi\|^2),$$

where $\|\psi\|^2 = \int {\rm d}x^2 |\psi|^2$. Notice that $C_{\rm ex}$ is a special singular form of the Casimir element $C_2(\psi)$. This hyperfunction Casimir element has only one leaf $\psi = 0$ (i.e., the equation $C_{\rm ex} = c$ has a solution, iff $c \in (0, 1)$, and then the level-set is the singularity $\psi = 0$), in marked contrast to other regular Casimir elements that densely foliate the phase space.

At the singularity $\psi = 0$, $C_{\rm in} = \int {\rm d}x^2 f(\omega)$ satisfies (denoting the state vector by $u = {}^t(\omega, \psi)$)

$$\mathcal{J}_{\rm II} \partial_u C_{\rm in}|_{\psi=0} = 0.$$

Notice that $C_{\rm in}$ is nothing but the Casimir element $C_0$ of the system (I), which, however, is not a Casimir element of the system (II).

From these observations, we draw the following conclusion: a singularity of a Hamiltonian system (where ${\rm Rank}(\mathcal{J})$ drops) defines a submanifold, on which one can introduce a sub Hamiltonian system. The Casimir elements of the subsystem convert into the *interior Casimir elements* of the larger system. Simultaneously, the submanifold is identified as a singular leaf of the *exterior Casimir element* which is the hyperfunction of the cross helicity describing the coupling of the submanifold and the mock fields. The pair of the interior and exterior Casimir elements constitute *singular Casimir elements.*

## 6 Conclusion

By embedding a Poisson manifold of a noncanonical Hamiltonian system into a higher-dimensional phase space, we can delineate topological structures within a simpler picture. Here we invoked the two-dimensional vortex systems to explain the systematic method of constructing a hierarchy of Poisson manifolds; we introduce *mock fields* and extend the Poisson algebra so that the mock fields are Lie-dragged by the flow vector. A mock field defines a new Casimir element, a *cross helicity*, which represents topological constraints including the circulation.

Unearthing a Casimir element brings about immense advantage in the study of dynamics and equilibria — the so-called energy-Casimir method becomes ready available. The theory can be applied to a wider class of fluid and plasma dynamics; see Yoshida & Morrison (2014).[8]

## References


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