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Erratum: "Hamiltonian magnetohydrodynamics: Lagrangian, Eulerian, and dynamically accessible stability—Theory" [Phys. Plasmas 20, 092104 (2013)]

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An algebraic mistake in the rendering of the Energy Casimir stability condition for a symmetric magnetohydrodynamics plasma configuration with flows made in the article Andreussi *et al.* "Hamiltonian magnetohydrodynamics: Lagrangian, Eulerian, and dynamically accessible stability—Theory," Phys. Plasmas **20**, 092104 (2013) is corrected. © *2015 AIP Publishing LLC*.
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In Ref. 1, the stability conditions of magnetized plasma flows were obtained by exploiting the Hamiltonian structure of the magnetohydrodynamics (MHD) equations. Three kinds of energy principles were considered: an energy principle in Lagrangian variables, an energy-Casimir principle for symmetric equilibria in an Eulerian variable noncanonical formulation of MHD, and, finally, an energy principle based on dynamically accessible variations, i.e., on variations that explicitly preserve invariants of the system. General criteria for stability were obtained, along with comparisons between the three different approaches.

In particular, it was shown that the stability of helically-symmetric equilibria can be assessed by considering the second variation of the energy-Casimir functional

$$\mathfrak{F}[Z] = \int_V \left[\frac{\rho v_\perp^2}{2} + \frac{\rho v_h^2}{2} + \rho U + \frac{k^2|\nabla\psi|^2}{8\pi} + \frac{B_h^2}{8\pi} - \rho\mathcal{J} - kB_h\mathcal{H} - \left(k^4[l]\sin 2\alpha\right)\mathcal{H}^- - \frac{1}{k}\rho v_h\mathcal{G} - \mathbf{v}\cdot\mathbf{B}\,\mathcal{F}\right] d^3r, \quad (1)$$

where Z denotes the set of Eulerian variables $Z = (\rho, \mathbf{v}_\perp, v_h, \psi, B_h)$ and $\mathcal{J}, \mathcal{H}, \mathcal{H}^-, \mathcal{G}$ and $\mathcal{F}$ are arbitrary functions of the flux variable ψ (Casimir functionals). For the explicit definitions of the quantities in Eq. (1), see Sec. III of Ref. 1 and in particular, Eqs. (61–63).

The second variation of $\mathfrak{F}$ yields

$$\begin{aligned}\delta^2\mathfrak{F}[Z] = \int_V \Bigg[&\rho(\delta\mathbf{v})^2 + 2\left(\mathbf{v} - \frac{1}{k}\mathcal{G}\mathbf{h}\right)(\delta\mathbf{v})(\delta\rho) + (\rho U_{\rho\rho} + 2U_\rho)(\delta\rho)^2 \\ &- 2\mathcal{F}(\delta\mathbf{B})\cdot(\delta\mathbf{v}) + \frac{1}{4\pi}(\delta\mathbf{B})^2 - 2\left(\mathcal{F}'\mathbf{B} + \frac{1}{k}\rho\mathcal{G}'\mathbf{h}\right)\cdot(\delta\mathbf{v})(\delta\psi) \\ &+ 2\left(\rho U_{\rho s}\mathcal{S}' + U_s\mathcal{S}' - \mathcal{J}' - \frac{1}{k}v_h\mathcal{G}'\right)(\delta\rho)(\delta\psi) - 2(\mathbf{v}\mathcal{F}' + k\mathcal{H}'\mathbf{h})\cdot(\delta\mathbf{B})(\delta\psi) \\ &+ \left(\rho T\mathcal{S}'' + \rho U_{ss}\mathcal{S}'^2 - \rho\mathcal{J}'' - kB_h\mathcal{H}'' - \left(k^4[l]\sin 2\alpha\right)\mathcal{H}' - \frac{1}{k}\rho v_h\mathcal{G}'' - \mathbf{v}\cdot\mathbf{B}\,\mathcal{F}''\right)(\delta\psi)^2\Bigg] d^3r. \quad (2)\end{aligned}$$

Although the above expression is given correctly in Ref. 1, when reformulating it in terms of the physical equilibrium quantities, an algebraic error was made that affects Eq. (74) of Ref. 1 and modifies the form of the following expressions in Sec. III. In the present erratum, we provide the correct form of these equations. In addition, a few notational changes are introduced to make the present text more selfcontained.

By grouping as in Ref. 1 the variation of the plasma equilibrium quantities $\delta\rho, \delta\mathbf{v}, \delta\mathbf{B}, \delta\psi$ into convenient linear combinations $\delta\mathbf{S}, \delta Q, \delta\mathbf{R}_\perp, \delta R_h$, and $\delta\psi$, but keeping the components of $\delta\mathbf{R}$ along the symmetry direction $\mathbf{h}$ and perpendicular to it separate, after some algebra, we obtain the following from Eq. (1):

$$\delta^2\mathfrak{F}[Z] = \int_V [a_1|\delta\mathbf{S}|^2 + a_2(\delta Q)^2 + a_3(\delta R_h)^2 + a_4|\delta\mathbf{R}_\perp|^2 + a_5(\delta\psi)^2] d^3r, \quad (3)$$

where

$$a_1 = \frac{1}{\rho}, \quad a_2 = \frac{\rho}{c_s^2 - M^2 c_a^2},$$
$$a_3 = \frac{4\pi\left(c_s^2 - M^2 c_a^2\right)}{c_s^2 - M^2\left(c_s^2 + c_a^2\right) + \dfrac{M^4}{4\pi\rho}B_\perp^2}, \tag{4}$$

$$a_4 = \frac{4\pi}{(1 - M^2)} \frac{c_s^2 - M^2\left(c_s^2 + c_a^2\right) + \dfrac{M^4}{4\pi\rho}B_\perp^2}{c_s^2 - M^2\left(c_s^2 + c_a^2\right)}, \tag{5}$$

$$a_5 = -\Upsilon - a_1\left|\frac{\delta \mathbf{S}}{\delta\psi}\right|^2_{\tilde{Z}_S} - a_2\left|\frac{\delta Q}{\delta\psi}\right|^2_{\tilde{Z}_S} - a_3\left|\frac{\delta R_h}{\delta\psi}\right|^2_{\tilde{Z}_S} - a_4\left|\frac{\delta \mathbf{R}_\perp}{\delta\psi}\right|^2_{\tilde{Z}_S}, \tag{6}$$

c_s and c_a are the sound and the Alfvén speed, M is the poloidal Alfvén Mach number defined by Eq. (71) of Ref. 1

$$\Upsilon = -\Big(\rho T \mathcal{S}'' + \rho U_{ss}\mathcal{S}'^2 - \rho\mathcal{J}'' - kB_h\mathcal{H}'' - \left(k^4[l]\sin 2\alpha\right)\mathcal{H}' - \frac{1}{k}\rho v_h \mathcal{G}'' - \mathbf{v}\cdot\mathbf{B}\,\mathcal{F}''\Big), \tag{7}$$

and the derivatives with respect to ψ at constant $\tilde{Z}_S$ are defined in the second paragraph below Eq. (75). In Eq. (3), the combinations $\delta\mathbf{S}$, δQ, and δR_h correspond to the variations of

$$\mathbf{S} = \rho\mathbf{v} - \mathcal{F}\mathbf{B} - \frac{1}{k}\rho\mathcal{G}\mathbf{h}, \tag{8}$$

$$Q = \frac{B^2}{2\rho^2}\mathcal{F}^2 + U + \frac{p}{\rho} - \mathcal{J} - \frac{1}{2k^2}\mathcal{G}^2, \tag{9}$$

$$R_h = \frac{1 - M^2}{4\pi}B_h - k\mathcal{H} - \frac{1}{k}\mathcal{F}\mathcal{G}, \tag{10}$$

where, in the last equation, $M^2 = 4\pi\mathcal{F}^2/\rho(\psi, \mathbf{B})$ is considered as a function of ψ and $\mathbf{B}$. The variation of Eqs. (8)–(10) yields

$$\delta\mathbf{S} = \rho\delta\mathbf{v} + \left(\mathbf{v} - \frac{1}{k}\mathcal{G}\mathbf{h}\right)\delta\rho - \mathcal{F}\delta\mathbf{B} - \left(\mathcal{F}'\mathbf{B} + \frac{1}{k}\rho\mathcal{G}'\mathbf{h}\right)\delta\psi, \tag{11}$$

$$\delta Q = \frac{1}{\rho}\left(c_s^2 - M^2 c_a^2\right)\delta\rho + \frac{M^2}{4\pi\rho}\mathbf{B}\cdot\delta\mathbf{B} - \mathcal{M}\delta\psi, \tag{12}$$

$$\delta R_h = \frac{c_s^2 - M^2\left(c_a^2 + c_s^2\right) + \dfrac{M^4}{4\pi\rho}B_\perp^2}{4\pi\left(c_s^2 - M^2 c_a^2\right)}\delta B_h - \frac{M^4}{4\pi\left(c_s^2 - M^2 c_a^2\right)}\frac{B_h}{4\pi\rho}\mathbf{B}_\perp\cdot\delta\mathbf{B}_\perp - \left[2\frac{\mathcal{F}\mathcal{F}'}{\rho}B_h + \mathcal{N} - \frac{M^2 B_h}{4\pi\left(c_s^2 - M^2 c_a^2\right)}\mathcal{M}\right]\delta\psi, \tag{13}$$

where

$$\mathcal{M} := \frac{1}{k^2}\mathcal{G}\mathcal{G}' + \mathcal{J}' - (\rho U_{\rho s} + U_s)\mathcal{S}' - \frac{\mathcal{F}\mathcal{F}'}{\rho^2}B^2, \tag{14}$$

$$\mathcal{N} := k\mathcal{H}' + \frac{1}{k}\mathcal{F}'\mathcal{G} + \frac{1}{k}\mathcal{F}\mathcal{G}', \tag{15}$$

while for $\delta\mathbf{R}_\perp$, we obtain

$$\delta\mathbf{R}_\perp = \frac{1}{a_4}\delta\mathbf{B}_\perp + \frac{M^2}{a_4\sqrt{4\pi\rho\left[c_s^2 - M^2\left(c_s^2 + c_a^2\right)\right]}}(\mathbf{B}_\perp\times\delta\mathbf{B}_\perp) - \left[2\frac{\mathcal{F}\mathcal{F}'}{\rho} - \frac{M^2}{4\pi\left(c_s^2 - M^2 c_a^2\right)}\mathcal{M} - \frac{\dfrac{M^4}{4\pi\rho}B_h}{c_s^2 - M^2\left(c_s^2 + c_a^2\right) + \dfrac{M^4}{4\pi\rho}B_\perp^2}\left|\frac{\delta R_h}{\delta\psi}\right|^2_{\tilde{Z}_S}\right]\mathbf{B}_\perp\delta\psi. \tag{16}$$

Note that Eq. (3) differs from Eq. (74) of Ref. 1 insofar as the term proportional to $\delta\mathbf{R}$ in Eq. (74) is now split into two terms with different coefficients (a_3 and a_4) that take the place of, and correct, the coefficient a_3 of Eq. (74) in Ref. 1.

Setting the variations $\delta\mathbf{S}$, δQ, and δR_h equal to zero (as follows from the minimization of $\delta^2\mathfrak{F}[Z]$, provided a_1, a_2, and a_3 are positive) yields

$$\delta^2\mathfrak{F}[Z] = \int_V \Bigg[\frac{1}{a_4}(\delta\mathbf{B}_\perp)^2 + \frac{M^4}{4\pi\rho a_4\left[c_s^2 - M^2\left(c_s^2 + c_a^2\right)\right]} \times(\mathbf{B}_\perp\times\delta\mathbf{B}_\perp)^2 + 2\frac{\delta\mathbf{R}_\perp}{\delta\psi}\bigg|_{\tilde{Z}_S}\cdot(\delta\mathbf{B}_\perp)(\delta\psi) + \left(a_5 + a_4\left|\frac{\delta\mathbf{R}_\perp}{\delta\psi}\right|^2_{\tilde{Z}_S}\right)(\delta\psi)^2\Bigg]d^3r\,. \tag{17}$$

Since the variation of the poloidal magnetic field is related to the variation of ψ by the equation

$$\delta\mathbf{B}_\perp = \nabla\delta\psi\times k\mathbf{h}, \tag{18}$$

where k is a helical metric factor that reduces to the inverse of the cylindrical radius in the limit of azimuthal symmetry, we can rewrite Eq. (3) as

$$\delta^2\mathfrak{F}[Z] = \int_V [b_1|\nabla\delta\psi|^2 + b_2(\delta\psi)^2 + b_3|\nabla\delta\psi\times\mathbf{e}_\psi|^2]d^3r\,, \tag{19}$$

where $\mathbf{e}_\psi = \nabla\psi/|\nabla\psi|$. The third term on the r.h.s. of Eq. (19) is absent in Eq. (80) of Ref. 1. In order to obtain Eq. (19), we used the relationship

$$\int_V 2\frac{\delta\mathbf{R}_\perp}{\delta\psi}\bigg|_{\tilde{Z}_S}\cdot(\delta\mathbf{B}_\perp)(\delta\psi)d^3r = \int_V \nabla\cdot\left(\frac{\delta\mathbf{R}_\perp}{\delta\psi}\bigg|_{\tilde{Z}_S}\times k\mathbf{h}\right)(\delta\psi)^2 d^3r, \tag{20}$$

and we have defined the following functions of the equilibrium configuration:

$$b_1 = k^2 \frac{1-\mathrm{M}^2}{4\pi} \frac{c_s^2 - \mathrm{M}^2\left(c_a^2 + c_s^2\right)}{c_s^2 - \mathrm{M}^2\left(c_a^2 + c_s^2\right) + \dfrac{\mathrm{M}^4}{4\pi\rho} B_\perp^2}, \tag{21}$$

$$b_2 = -\Upsilon - a_1 \left|\frac{\delta \mathbf{S}}{\delta\psi}\right|^2_{\tilde{Z}_S} - a_2 \left|\frac{\delta Q}{\delta\psi}\right|^2_{\tilde{Z}_S} - a_3 \left|\frac{\delta R_h}{\delta\psi}\right|^2_{\tilde{Z}_S} + \nabla\cdot\left(\left.\frac{\delta \mathbf{R}_\perp}{\delta\psi}\right|_{\tilde{Z}_S} \times k\mathbf{h}\right), \tag{22}$$

$$b_3 = k^2 \frac{1-\mathrm{M}^2}{4\pi} \frac{\dfrac{\mathrm{M}^4}{4\pi\rho} B_\perp^2}{c_s^2 - \mathrm{M}^2\left(c_a^2 + c_s^2\right) + \dfrac{\mathrm{M}^4}{4\pi\rho} B_\perp^2}. \tag{23}$$

We note that

$$k^2 \frac{1-\mathrm{M}^2}{4\pi} = b_1 + b_3\,, \tag{24}$$

which can be convenient if we rewrite Eq. (19) as

$$\delta^2\mathfrak{F}[Z] = \int_V \left[b_1 |\nabla_\perp \delta\psi|^2 + b_2 (\delta\psi)^2 + (b_1 + b_3)|\nabla\delta\psi \times \mathbf{e}_\psi|^2\right] d^3 r\,, \tag{25}$$

where $\nabla_\perp \delta\psi$ denotes the component of the gradient of the variation $\delta\psi$ perpendicular to the equilibrium flux function ψ.

The Euler-Lagrange equation associated with the extrema of Eq. (19) is

$$\nabla\cdot\left[b_1\, I + b_3\,(I - \mathbf{e}_\psi \mathbf{e}_\psi)\right]\cdot\nabla\delta\psi - b_2\,\delta\psi = 0\,, \tag{26}$$

where I is the identity tensor and $(I - \mathbf{e}_\psi\mathbf{e}_\psi)$ is the projector on the tangent plane to the ψ-surfaces. Equation (26) should take the place of Eq. (81) in Ref. 1—it represents the generalized form of the Newcomb equation[2] for MHD symmetric equilibria with flow.

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