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Hamiltonian closures for fluid models with four moments by dimensional analysis

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Abstract

Fluid reductions of the Vlasov–Ampère equations that preserve the Hamiltonian structure of the parent kinetic model are investigated. Hamiltonian closures using the first four moments of the Vlasov distribution are obtained, and all closures provided by a dimensional analysis procedure for satisfying the Jacobi identity are found. Two Hamiltonian models emerge, for which the explicit closures are given, along with their Poisson brackets and Casimir invariants.

Keywords: Vlasov equation, Hamiltonian reduction, Jacobi identity

(Some figures may appear in colour only in the online journal)

1. Introduction

The Vlasov–Ampère set of equations is a suitable framework for describing the dynamics of systems interacting through electrostatic forces. In this work, we focus on the study of electrostatic plasmas even though the results may be applied to more general systems described in part by the Vlasov equation. We consider a one-dimensional plasma made of electrons of unit mass and negative unit electric charge, evolving in a neutralizing background of static ions. The evolution of the distribution function of the electrons f, defined on phase space with coordinates (x, v), and electric field E is given by the Vlasov–Ampère equations

$$\partial_t f = -v\partial_x f + \widetilde{E}\partial_v f, \tag{1}$$

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$$\partial_t E = -4\pi\tilde{j}, \tag{2}$$

where $\tilde{E}$ and $\tilde{j}$ are the fluctuating parts of the electric field E and the current density $j = -\int vf \, \mathrm{d}v$ respectively. We assume vanishing boundary conditions at infinity in the velocity v so that integrals such as the charge and current densities are well-defined. In this work, we limit ourselves to the study of systems of unit length in the spatial domain x with periodic boundary conditions. The fluctuating part of the electric field is defined by $\tilde{E} = E - \int_0^1 E \, \mathrm{d}x$. The system is fully nonlinear, but has a form that builds in the preservation of the spatial average of E and maintains momentum conservation.

The use of fluid reductions to describe the dynamics of a plasma is ubiquitous in plasma physics. Indeed, this usually allows one to decrease the complexity of the problem at hand and to gain physical insight into the phenomenon under investigation since the dimension of phase space is reduced. Fluid reductions of the Vlasov–Ampère equations are done by introducing fluid quantities such as the fluid moments

$$P_n = \int v^n f(x, v, t) \, \mathrm{d}v. \tag{3}$$

The associated dynamical equations are then obtained by multiplying equation (1) by v^n and integrating with respect to the velocity. This leads to

$$\partial_t P_n = -\partial_x P_{n+1} - nP_{n-1}\tilde{E}, \tag{4}$$

$$\partial_t E = 4\pi\tilde{P}_1, \tag{5}$$

for all $n \in \mathbb{N}$. In order for this system to be reduced, one has to truncate the infinite sequence of equation (4). Truncating this system at order N, that is considering $(P_0, P_1, ..., P_N, E)$ as dynamical field variables, one can see from equation (4) that the time evolution of P_N depends on P_{N+1}. As a consequence, it is necessary to express P_{N+1} in terms of $(P_0, P_1, ..., P_N, E)$ in order to close equations (4) and (5) and thereby obtain a fluid reduction.

Many models have been proposed based on as many closures with various requirements (see, e.g., [1–4]). A usual procedure consists in assuming a particular form for the distribution function f (e.g., Dirac, Maxwellian,...) depending on a finite number of parameters, and expressing the closure with respect to these parameters [5]. Alternatively, closures have been constructed in order to recover certain kinetic effects [6–12]. In any event, a reduction by closure should be such that, if the parent model possesses a Hamiltonian structure [13–16], then the resulting fluid model should also have one, after discarding all the terms that are supposed to provide dissipation. A closure procedure ignoring this aspect could potentially lead to the introduction of some nonphysical dissipation [17, 18]. Consequently, here we use a procedure that preserves the Hamiltonian structure of the parent kinetic (Vlasov–Ampère) system, which is one of its most important structural features. Specifically, in this work we present a model for the first four fluid moments of the distribution function, namely the density ρ, the fluid velocity u, the pressure P and the heat flux q. This allows us to account for the time evolution of the heat flux, which is of great importance for the study of transport phenomena inside the plasma. For such a model with four moments, one has to find a closure for the fifth order moment of the distribution function, namely P_4. Here, we determine all the closures, obtained from a procedure based on dimensional analysis, that preserve the Hamiltonian structure of the parent model [14, 19] given by equations (1) and (2). We show that there are only two such Hamiltonian closures. The equations of motion of one of these two models are identical to the ones obtained with a bi-delta reduction [20–22], i.e., assuming that the Vlasov distribution has the form

$$f(x, v, t) = \omega_1 \delta(v - \mu_1) + \omega_2 \delta(v - \mu_2),$$

where $\omega_{1,2}$ and $\mu_{1,2}$ depend on space and time. It should be noted here that we obtain these equations without any assumption on the special form of the distribution function. We provide the explicit expressions of the Hamiltonian and the Poisson bracket for the two Hamiltonian models. In addition, we derive the global Casimir invariants, which are specific invariants resulting from the knowledge of the Poisson bracket. These conserved quantities can be used, e.g., to ensure the validity of a numerical simulation of the equations of motion.

The paper is organized as follows. In section 2 we describe the methodology used for the derivation of the two Hamiltonian reduced models. We start from the definitions of the appropriate variables, namely, the reduced fluid moments. Subsequently, we introduce our method, based on dimensional analysis, which leads to models that obey the Jacobi identity. We show that there are only two such models. In section 3, we analyze the two resulting Hamiltonian closures, providing explicit expressions for their Hamiltonians, Poisson brackets, and Casimir invariants.

2. Method

2.1. Reduced moments

Our purpose is to build a Hamiltonian fluid model for the first four moments of the distribution function, namely the density ρ, the fluid velocity u, the pressure P, the heat flux q and the electric field E. These models will be referred to as 4 + 1 field models, where the 4 refers to the four first moments of the Vlasov distribution (or equivalently to ρ, u, P and q) and the 1 refers to the electric field E. We begin by considering the Poisson structure of the parent model with (f, E) as dynamical field variables. It was shown in [23] that the system of equations (1)–(2) possesses a Hamiltonian structure with Poisson bracket

$$\{F, G\} = \int f \left[\partial_x F_f \partial_v G_f - \partial_x G_f \partial_v F_f + 4\pi \left(\widetilde{F_E} \partial_v G_f - \widetilde{G_E} \partial_v F_f \right) \right] \mathrm{d}x\mathrm{d}v, \tag{6}$$

where F_f (respectively F_E) denotes the functional derivative of F with respect to f (respectively E). In addition, bracket (6) is bilinear and satisfies the Leibniz rule and the Jacobi identity. The Hamiltonian of the system is given by

$$\mathcal{H} = \int f \frac{v^2}{2} \,\mathrm{d}x\mathrm{d}v + \int \frac{E^2}{8\pi} \,\mathrm{d}x, \tag{7}$$

where the first term accounts for the kinetic energy of the particles and the second one corresponds to the energy of the electric field. Together with bracket (6), this Hamiltonian leads to equations (1) and (2) by using $\partial_t f = \{f, \mathcal{H}\}$ and $\partial_t E = \{E, \mathcal{H}\}$. We recall that such a bracket has Casimir invariants, i.e., functionals C that Poisson-commute with any other functionals of the Poisson algebra, $\{C, F\} = 0$ for all F. Bracket (6) has the following global (i.e., independent of the coordinates x and v) Casimir invariants

$$C_1 = \int \varphi(f) \,\mathrm{d}x\mathrm{d}v,$$

$$C_2 = \int E \, \mathrm{d}x,$$

for any scalar function φ, and a local Casimir invariant

$$C_{\mathrm{L}} = \partial_x E + 4\pi \int f \, \mathrm{d}v,$$

which is equivalent to Gauss's law.

The change from the kinetic to the fluid description is done by performing the change of variables defined by equation (3) in bracket (6) and Hamiltonian (7). The latter becomes

$$\mathcal{H} = \frac{1}{2} \int \left(P_2 + \frac{E^2}{4\pi} \right) \mathrm{d}x.$$

Making use of the chain rule to transform the functional derivatives, bracket (6) becomes [24–26]

$$\{F, G\} = \int j \left[P_{i+j-1} \left(G_j \partial_x F_i - F_j \partial_x G_i \right) + 4\pi P_{j-1} \left(G_j \widetilde{F_E} - F_j \widetilde{G_E} \right) \right] \mathrm{d}x, \tag{8}$$

where F_n denotes the functional derivative of F with respect to P_n, and summation is implicit over the repeated indices i and j. Because we want to construct a Hamiltonian model for the first four moments of the distribution function, we consider functionals of the kind $F[P_0, P_1, P_2, P_3, E]$. However, the Poisson bracket (8) of two functionals of this kind depends explicitly on two additional moments, namely P_4 and P_5. In order to close the system, these two additional moments need to be expressed in terms of $P_{n\leqslant 3}$ and E. As a result, the Jacobi identity is no longer satisfied in general, and the resulting truncated and closed bracket is not of Poisson type. Consequently, the resulting system is not Hamiltonian, or in other terms, the reduction procedure potentially includes dissipation. We notice that the closure has to be performed on two moments, P_4 and P_5, which slightly differs from what has been stated in the introduction, concerning the closure performed on the equations of motion directly, where only one additional moment, P_4, needs to be closed. However we shall see in section 2.2 that the expression of P_5 is entirely determined by P_4.

We introduce the reduced fluid moments, which we find to be more suitable variables for our purpose

$$\rho = \int f \, \mathrm{d}v, \qquad u = \frac{1}{\rho} \int v f \, \mathrm{d}v, \qquad S_n = \frac{1}{\rho^{n+1}} \int (v - u)^n f \, \mathrm{d}v, \tag{9}$$

for all $n \geqslant 2$. The first and second ones correspond respectively to the usual density and fluid velocity. The higher-order moments are the central fluid moments with a specific scaling with respect to the density. The change from the usual fluid moments P_n to the reduced fluid moments (ρ, u, S_n), used hereafter, is invertible so that the results, even though they are expressed in a different set of coordinates, are equivalent. This change is given by

$$\rho = P_0, \qquad u = \frac{P_1}{P_0}, \qquad S_n = \frac{1}{P_0^{n+1}} \sum_{m=0}^{n} \binom{n}{m} \left(\frac{-P_1}{P_0} \right)^{n-m} P_m,$$

for all $n \geqslant 2$. The inverse of this transformation is given by

$$P_0 = \rho, \qquad P_1 = \rho u, \qquad P_n = \rho\left[u^n + \sum_{m=2}^{n}\binom{n}{m}\rho^m u^{n-m} S_m\right].$$

Explicitly for the first four moments of the distribution function, this change of variables is given by

$$\rho = P_0, \qquad u = \frac{P_1}{P_0},$$
$$S_2 = \frac{1}{P_0^3}\left(P_2 - \frac{P_1^2}{P_0}\right), \quad S_3 = \frac{1}{P_0^4}\left(P_3 - 3\frac{P_1 P_2}{P_0} + 2\frac{P_1^3}{P_0^2}\right),$$

with the inverse

$$P_0 = \rho, \quad P_1 = \rho u, \quad P_2 = \rho\left(u^2 + \rho^2 S_2\right), \quad P_3 = \rho\left(u^3 + 3\rho^2 u S_2 + \rho^3 S_3\right).$$

In terms of the moments, Hamiltonian (7) is

$$\mathcal{H} = \frac{1}{2}\int\left(\rho u^2 + \rho^3 S_2 + \frac{E^2}{4\pi}\right)\mathrm{d}x. \tag{10}$$

The first part of Hamiltonian (10) accounts for the kinetic energy of the system while its second part corresponds to the internal energy. The last term, which accounts for the electric energy, remains unchanged compared to equation (7). By considering functionals of the kind $F[\rho, u, S_2, S_3, E]$ and using the chain rule for the functional derivatives (see appendix C for more details), bracket (6) takes the form

$$\begin{aligned}\{F, G\} = \int\Big[&G_u\partial_x F_\rho - F_u\partial_x G_\rho + 4\pi\left(G_u\widetilde{F_E} - F_u\widetilde{G_E}\right)\\ &- \frac{1}{\rho}\left(G_u F_i - F_u G_i\right)\partial_x S_i + \alpha_{ij}\frac{F_i}{\rho}\frac{G_j}{\rho} + \partial_x\left(\frac{F_i}{\rho}\right)\beta_{ij}\frac{G_j}{\rho}\Big]\,\mathrm{d}x,\end{aligned} \tag{11}$$

where F_i denotes the functional derivative of F with respect to S_i. From now on and unless otherwise stated, summation from 2 to 3 over repeated indices is implicit. The matrices α and β have indices ranging from 2 to 3 such that

$$\alpha = \partial_x\begin{pmatrix} 2S_3 & 2S_4 - 3S_2^2 \\ 3S_4 - 6S_2^2 & 3S_5 - 12S_2S_3\end{pmatrix}, \quad \beta = \begin{pmatrix} 4S_3 & 5S_4 - 9S_2^2 \\ 5S_4 - 9S_2^2 & 6S_5 - 24S_2S_3\end{pmatrix}. \tag{12}$$

We notice that $\partial_x\beta = \alpha + \alpha^t$, a property that ensures that bracket (11) is antisymmetric. From definitions (12) we see that the closure requires reexpression of S_4 and S_5, i.e., one has to express these two reduced moments with respect to the dynamical variables (ρ, u, S_2, S_3, E) such that bracket (11) satisfies the Jacobi identity.

We remark that bracket (11) has several subalgebras. Trivial ones include $F[\rho]$ (i.e., the algebra of functionals of the type $F[\rho]$), $F[u]$, $F[E]$, $F[\rho, E]$, and non-trivial ones include $F[\rho, u]$, $F[\rho, S_2, S_3]$, $F[u, E]$, $F[\rho, u, S_2, S_3]$, $F[\rho, u, E]$ and $F[\rho, S_2, S_3, E]$. The most interesting one is the subalgebra of functionals $F[\rho, S_2, S_3]$ for which ρ becomes a Casimir invariant. The existence of this subalgebra is the reason for considering the reduced fluid moments S_n.

2.2. The Hamiltonian constraints

In order to be a Poisson bracket, bracket (11) must satisfy the Jacobi identity

$$\{F, \{G, H\}\} + \{H, \{F, G\}\} + \{G, \{H, F\}\} = 0.$$

Here we determine the conditions on S_4 and S_5 resulting from the Jacobi identity. We begin by assuming that S_4 and S_5 depend on ρ, u, S_2, S_3, E and their derivatives $\partial_x^n\rho$, $\partial_x^n u$, $\partial_x^n S_2$, $\partial_x^n S_3$, $\partial_x^n E$ for n lower than some order ν. Using the result obtained in appendix A, we conclude that S_4 and S_5 do not depend on ρ, u, E and their derivatives $\partial_x^n\rho$, $\partial_x^n u$, $\partial_x^n E$. In addition, we show in appendix B that in order for the Jacobi identity to be satisfied, we need to impose

$$\gamma_{ljm}\gamma_{kij} = \gamma_{kjm}\gamma_{lij},$$

for all i, k, l and m ranging from 2 to 3, where the summation is implicit on j, and

$$\gamma_{ljm}\left(S_k, \partial_x S_k, \ldots, \partial_x^{\nu-1} S_k\right) = \frac{\partial \alpha_{lj}}{\partial \partial_x^{\nu} S_m}.$$

For instance, for $l = 2$, $m = 3$, $i = 2$ and $k = 3$, we end up with $\gamma_{233}\gamma_{323} = 0$ since $\gamma_{222} = 0$ and $\gamma_{223} = 0$ for $\nu \geqslant 2$. From equation (12), we have $3\gamma_{233} = 2\gamma_{323}$, therefore $\gamma_{233} = 0$, or equivalently

$$\frac{\partial S_4}{\partial \partial_x^{\nu-1} S_3} = 0.$$

Using equation (B.5) leads to

$$S_3 \frac{\partial \alpha_{23}}{\partial \partial_x^{\nu} S_2} = 0.$$

Since this has to be true for any value of S_3, we thus conclude that $\gamma_{232} = 0$, i.e.,

$$\frac{\partial S_4}{\partial \partial_x^{\nu-1} S_2} = 0.$$

Concerning S_5, equation (B.5) for $l = i = 3$ leads to

$$\beta_{kj} \frac{\partial S_5}{\partial \partial_x^{\nu-1} S_j} = 0. \tag{13}$$

There are two solutions to equation (13). The first solution is given by

$$\frac{\partial S_5}{\partial \partial_x^{\nu-1} S_2} = 0, \qquad \frac{\partial S_5}{\partial \partial_x^{\nu-1} S_3} = 0.$$

The second solution requires $\det \beta = 0$, which, using equation (12), can be written as

$$S_5 = 4S_2S_3 + \frac{\left(5S_4 - 9S_2^2\right)^2}{4S_3}.$$

Since S_4 does not depend on $\partial_x^{\nu-1} S_2$ and $\partial_x^{\nu-1} S_3$, we again have

$$\frac{\partial S_5}{\partial \partial_x^{\nu-1} S_2} = 0, \qquad \frac{\partial S_5}{\partial \partial_x^{\nu-1} S_3} = 0.$$

In what follows we will see that the second solution does not lead to a Hamiltonian closure. By induction on ν down to $\nu = 2$ we show that S_4 and S_5 have to be functions of S_2 and S_3

only. These conditions are necessary but not sufficient, i.e., for any functions S_4 and S_5 of S_2 and S_3, bracket (11) does not satisfy the Jacobi identity in general.

We compute in appendix C the necessary and sufficient conditions on the closures for a fluid bracket of the type (11) to satisfy the Jacobi identity. For four fluid moments, equations (C.9) and (C.10) are

$$\frac{\partial S_5}{\partial S_2} = 4S_3 + \frac{\partial S_4}{\partial S_3}\left(\frac{\partial S_4}{\partial S_2} - 3S_2\right), \tag{14}$$

$$\frac{\partial S_5}{\partial S_3} = \left(\frac{\partial S_4}{\partial S_3}\right)^2 + \frac{\partial S_4}{\partial S_2}, \tag{15}$$

$$6S_5 = 4S_3\left(3S_2 + \frac{\partial S_4}{\partial S_2}\right) - \frac{\partial S_4}{\partial S_3}\left(9S_2^2 - 5S_4\right). \tag{16}$$

We see from equation (16) that the expression of S_5 is fully determined by S_4. By introducing the expression for S_5 given by equation (16) into equations (14) and (15), we end up with the following two nonlinear second order partial differential equations:

$$4S_3\frac{\partial^2 S_4}{\partial S_2^2} - \frac{\partial^2 S_4}{\partial S_2 \partial S_3}\left(9S_2^2 - 5S_4\right) - \frac{\partial S_4}{\partial S_2}\frac{\partial S_4}{\partial S_3} = 12S_3, \tag{17}$$

$$4S_3\frac{\partial^2 S_4}{\partial S_3 \partial S_2} - \frac{\partial^2 S_4}{\partial S_3^2}\left(9S_2^2 - 5S_4\right) + 12S_2 = \left(\frac{\partial S_4}{\partial S_3}\right)^2 + 2\frac{\partial S_4}{\partial S_2}. \tag{18}$$

Provided that these two equations are satisfied, bracket (11) is a Poisson bracket and the resulting system is Hamiltonian. Solving these equations in general is challenging; consequently, in what follows we restrict ourselves to the set of solutions provided by dimensional analysis [27].

2.3. Closures based on dimensional analysis

We consider all the closures for the fifth-order moment $S_4 = g(S_2, S_3)$ that satisfy the constraints given by equations (17) and (18) based on a dimensional analysis argument. In order to proceed, we assume that the closure $S_4 = g(S_2, S_3)$ does not depend on any further dimensional parameters. This would not be the case for, e.g., diffusion-like closures (Fourier's law, Fick's law, etc...) that introduce phenomenological parameters resulting from various hypotheses based on characteristic scales of the dynamics of the system. Indeed, in diffusion processes, diffusion coefficients replace information on the particle interactions, thus removing small scale dynamics. Instead, we would like our reduction procedure to be very general and not to depend on the geometry of the system. Consequently, we seek Hamiltonian closures where $S_4 = g(S_2, S_3)$ do not depend on any further dimensional parameters.

It can be shown from equation (9) that the dimensions of the S_n's, denoted $[S_n]$, are not independent. Indeed, for all $n \geqslant 2$ we have $[S_n] = \mathrm{A}^n$, where $\mathrm{A} = \mathrm{L}^2\mathrm{T}^{-1}$ with L and T denoting the units of length and time, respectively. As a consequence, the closure $S_4 = g(S_2, S_3)$ involves three quantities and a unique physical dimension A. Making use of the Buckingham π theorem [27], there exists two dimensionless quantities, denoted ζ and ξ, such that $S_4 = g(S_2, S_3)$ reduces to $\zeta = R(\xi)$. Therefore, this procedure eliminates one of the variables in the closure. Defining $\zeta = S_4/S_2^2$ and $\xi = S_3/S_2^{3/2}$ and inserting these expressions into equations (17)–(18), we get the following two constraints:

$$3\xi R''\left(6\xi^2 + 9 - 5R\right) + R'\left(3\xi R' - 18\xi^2 + R - 9\right) + 16\xi R = 24\xi, \tag{19}$$

$$R''\left(6\xi^2 + 9 - 5R\right) + R'(R' - 5\xi) + 4R = 12. \tag{20}$$

To solve equations (19)–(20), we compute their values for $\xi = 0$. Defining $R_0 = R(0)$, $R_0' = R'(0)$, and $R_0'' = R''(0)$, equations (19) and (20) become

$$R_0'\left(R_0 - 9\right) = 0, \tag{21}$$

$$R_0''\left(9 - 5R_0\right) + R_0'^2 + 4R_0 = 12. \tag{22}$$

Equation (21) has two solutions: $R_0' = 0$ and $R_0 = 9$. Equation (22) then reads $R_0'' = 4(3 - R_0)/(9 - 5R_0)$ for $R_0' = 0$ and $R_0'^2 = 12(3R_0'' - 2)$ for $R_0 = 9$. We now differentiate equations (19) and (20) with respect to ξ and evaluate them at $\xi = 0$. This gives us

$$R_0''\left(9 - 7R_0\right) + 2R_0'^2 + 8R_0 = 12, \tag{23}$$

$$R_0'''\left(9 - 5R_0\right) - R_0'\left(3R_0'' + 1\right) = 0. \tag{24}$$

Using $R_0 = 9$, equation (23) together with equation (22) leads to $R_0'' = -2/3$ and $R_0'^2 = -48$. As a consequence, this solution is not real and of no interest for our purpose. The other solutions satisfy $(R_0, R_0', R_0'', R_0''') = (0, 0, 4/3, 0)$ and $(R_0, R_0', R_0'', R_0''') = (1, 0, 2, 0)$, where $R_0''' = R'''(0)$. Since the solution is unique for a given set of initial conditions, there exist only two solutions to equations (19) and (20). Moreover, one can see that $R(-\xi)$ is also a solution of these equations. Consequently, the two solutions R are even with respect to ξ. These two solutions are described in the next section.

3. Hamiltonian fluid models with 4 + 1 fields

3.1. Model with normal variables

The first solution to equations (19)–(20) corresponds to the branch $(R_0, R_0', R_0'', R_0''') = (1, 0, 2, 0)$ found in section 2.3, and is given by

$$R(\xi) = 1 + \xi^2.$$

This leads to

$$S_4 = S_2^2 + \frac{S_3^2}{S_2}, \tag{25}$$

$$S_5 = 2S_2S_3 + \frac{S_3^3}{S_2^2}. \tag{26}$$

These functions are plotted in figures 1 and 2. By defining the skewness $S = S_3/S_2^{3/2}$ and the kurtosis $K = S_4/S_2^2$ of the distribution function f, equation (25) becomes $K = 1 + S^2$. This relation is a particular case of more general parabolic relations that appear in various natural systems including plasma edge turbulence [30–33]. Here we show that this relation results from the Hamiltonian structure of the system. The Hamiltonian and the Poisson bracket

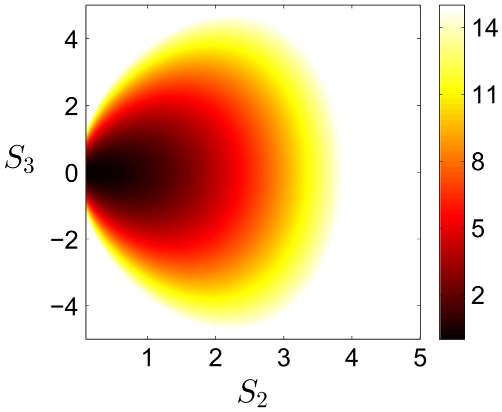

Figure 1. Color map of S_4 (in A^4 units as defined in section 2.3) given by equation (25) as a function of S_2 (in A^2 units) and S_3 (in A^3 units).

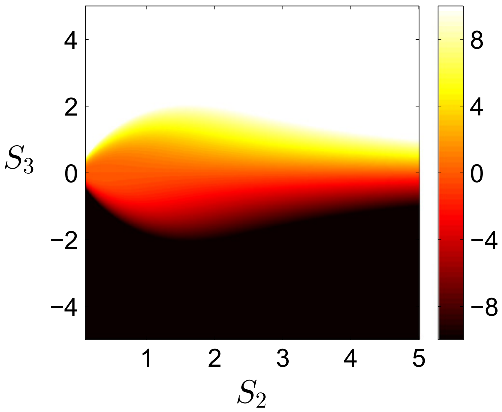

Figure 2. Color map of S_5 (in A^5 units as defined in section 2.3) given by equation (26) as a function of S_2 (in A^2 units) and S_3 (in A^3 units).

resulting from this closure are given respectively by equations (10) and (11) with α and β given by equation (12) and by replacing the closures S_4 and S_5 by equations (25) and (26).

In order to further characterize this Poisson bracket, we investigate its Casimir invariants. These are functionals $C[\rho, u, S_2, S_3, E]$ that commute with any other functionals F, i.e., $\{F, C\} = 0$ for all F. In particular, C commutes with the field variables. As a consequence, we must impose

$$\{\rho, C\} = -\partial_x C_u = 0,$$

which leads to

$$C\left[\rho, u, S_2, S_3, E\right] = K_1 \int u \, \mathrm{d}x + D\left[\rho, S_2, S_3, E\right],$$

where K_1 is constant. Imposing that C commutes with u leads to

$$\{u, C\} = -\partial_x D_\rho - 4\pi\widetilde{D_E} + \frac{1}{\rho} D_i \partial_x S_i = 0,$$

whose solution is given by

$$D[\rho, S_2, S_3, E] = K_2 \int \rho\phi(S_2, S_3)\,\mathrm{d}x + K_3 \int E\,\mathrm{d}x,$$

where K_2 and K_3 are constant. Imposing that C commutes with S_i leads to

$$\{S_i, C\} = -\frac{K_1}{\rho}\partial_x S_i + K_2 \frac{\alpha_{ij}}{\rho}\phi_j - \frac{K_2}{\rho}\partial_x\left(\beta_{ij}\phi_j\right) = 0, \tag{27}$$

where $\phi_i = \partial\phi/\partial S_i$. We then solve the associated homogeneous equation ($K_1 = 0$). Again making use of the Buckingham π theorem, we assume that there exist a real number a and a function ψ such that $\phi = S_2^a \psi(S_3/S_2^{3/2}) = S_2^a \psi(\xi)$. The resulting equations are

$$8(a-1)a\xi\psi + \left[(9-14a)\xi^2 - 8a\right]\psi' + 3\left(4+\xi^2\right)\xi\psi'' = 0, \tag{28}$$

$$2a\psi + (4a-3)\xi\psi' - \left(4+\xi^2\right)\psi'' = 0, \tag{29}$$

$$4a[2 - 4a + (5a-7)\xi^2]\psi + 9\xi^2[(3-4a)\xi\psi' + (4+\xi^2)\psi''] = 0, \tag{30}$$

$$8a\xi\psi + \left[(10a-9)\xi^2 - 8a\right]\psi' = 3\xi\left(4+\xi^2\right)\psi''. \tag{31}$$

Combining equations (28)–(29) leads to

$$a\left[\xi\psi - \left(\xi^2+4\right)\psi'\right] = 0.$$

A first solution is given by $a = 0$. Inserting this constraint into equations (28)–(31) provides the solution $\psi(\xi) = \xi/\sqrt{4+\xi^2}$. The second solution reads $a = 1/2$ and $\psi(\xi) = \sqrt{4+\xi^2}$. An additional invariant can be computed by solving equation (27) in the non-homogeneous case ($K_1 \neq 0$). Eventually, we show that this Poisson bracket possesses five global (independent of the space coordinate x) Casimir invariants, i.e., as many Casimir invariants as field variables, given by

$$C_1 = \int \rho\,\mathrm{d}x, \qquad C_2 = \int E\,\mathrm{d}x,$$

$$C_3 = \int \rho\frac{\sqrt{4S_2^3 + S_3^2}}{S_2}\,\mathrm{d}x, \qquad C_4 = \int \rho\frac{S_3}{\sqrt{4S_2^3 + S_3^2}}\,\mathrm{d}x,$$

$$C_5 = \int\left(u + \frac{\rho}{2}\frac{S_3}{S_2}\right)\mathrm{d}x,$$

where C_1 and C_2 are Casimir invariants inherited from the Vlasov–Ampère equations. From these expressions for the global Casimir invariants, we introduce the normal variables ρ, $M = u + \rho S_3/(2S_2)$, $Q_2 = \rho\sqrt{4S_2^3 + S_3^2}/S_2$, $Q_3 = \rho S_3\big/\sqrt{4S_2^3 + S_3^2}$ and E. Consequently, bracket (11) takes the particularly simple (normal) form

$$\{F, G\} = \int\left[G_M\partial_x F_\rho - F_M\partial_x G_\rho + 4\pi\left(G_M\widetilde{F_E} - F_M\widetilde{G_E}\right) - 2G_3\partial_x F_2 - 2G_2\partial_x F_3\right]\mathrm{d}x.$$

The resulting model is referred to as a Hamiltonian four moments model with normal variables because of the existence of (normal) variables such that the coefficients in the

Poisson bracket are constant. Hamiltonian (10) becomes

$$\mathcal{H} = \frac{1}{2}\int\left(\rho M^2 - M\frac{Q_3}{Q_2} + \frac{\rho}{4}Q_2^2 + \frac{E^2}{4\pi}\right)\mathrm{d}x,$$

and the Casimir invariants C_3, C_4 and C_5 become

$$C_3 = \int Q_2\ \mathrm{d}x, \qquad C_4 = \int Q_3\ \mathrm{d}x, \qquad C_5 = \int M\ \mathrm{d}x.$$

As mentioned in section 1, one may be interested in using the pressure P and the heat flux q as dynamical variables, instead of S_2 and S_3. Indeed, even though the reduced moments appear to be very convenient, their physical meaning may not be as clear as the usual pressure and heat flux quantities. The latter quantities can be expressed in terms of the reduced moments in the following way:

$$P = \rho^3 S_2 = P_2 - \frac{P_1^2}{P_0}, \qquad q = \frac{\rho^4}{2}S_3 = \frac{P_3}{2} - \frac{3}{2}\frac{P_1 P_2}{P_0} + \frac{P_1^3}{P_0^2},$$

in terms of which the closures take the form

$$S_4 = \frac{1}{\rho^5}\left(\frac{P^2}{\rho} + \frac{4q^2}{P}\right), \qquad S_5 = \frac{4q}{\rho^6}\left(\frac{P}{\rho} + \frac{2q^2}{P^2}\right).$$

Expressed in terms of these variables, bracket (11) becomes

$$\begin{aligned}\{F, G\} = \int\Bigg[& G_u\partial_x F_\rho - F_u\partial_x G_\rho + \frac{3P}{\rho}\left(G_u\partial_x F_P - F_u\partial_x G_P\right)\\ & + 4\pi\left(G_u\widetilde{F_E} - F_u\widetilde{G_E}\right) + \frac{2}{\rho}\left(G_u F_P - F_u G_P\right)\partial_x P + \frac{4q}{\rho}\left(G_u\partial_x F_q - F_u\partial_x G_q\right)\\ & + \frac{3}{\rho}\left(G_u F_q - F_u G_q\right)\partial_x q + \rho^4\bar{\alpha}_{22}F_P G_P + \rho^5\bar{\alpha}_{23}F_P G_q + \rho^5\bar{\alpha}_{32}F_q G_P\\ & + \rho^6\bar{\alpha}_{33}F_q G_q + \rho^2\bar{\beta}_{22}G_P\partial_x\left(\rho^2 F_P\right) + \rho^3\bar{\beta}_{23}G_q\partial_x\left(\rho^2 F_P\right) + \rho^2\bar{\beta}_{32}G_P\partial_x\left(\rho^3 F_q\right)\\ & + \rho^3\bar{\beta}_{33}G_q\partial_x\left(\rho^3 F_q\right)\Bigg]\mathrm{d}x,\end{aligned}$$

where

$$\bar{\alpha} = \partial_x\begin{pmatrix} 4q/\rho^4 & 4q^2/\left(\rho^5 P\right) - P^2/\left(2\rho^6\right) \\ 6q^2/\left(\rho^5 P\right) - 3P^2/\left(2\rho^6\right) & 6q^3/\left(\rho^6 P^2\right) - 3Pq/\rho^7 \end{pmatrix},$$

and

$$\bar{\beta} = \begin{pmatrix} 8q/\rho^4 & 10q^2/\left(\rho^5 P\right) - 2P^2/\rho^6 \\ 10q^2/\left(\rho^5 P\right) - 2P^2/\rho^6 & 12q^3/\left(\rho^6 P^2\right) - 6Pq/\rho^7 \end{pmatrix}.$$

Hamiltonian (10) takes the simple form

$$\mathcal{H} = \frac{1}{2}\int\left(\rho u^2 + P + \frac{E^2}{4\pi}\right)\mathrm{d}x,$$

and the equations of motion, obtained from $\partial_t F = \{F, \mathcal{H}\}$, are

$$\begin{aligned}
\partial_t \rho &= -\partial_x(\rho u),\\
\partial_t u &= -u\partial_x u - \frac{1}{\rho}\partial_x P - \widetilde{E},\\
\partial_t P &= -u\partial_x P - 3P\partial_x u - 2\partial_x q,\\
\partial_t q &= -u\partial_x q - 4q\partial_x u - 2\partial_x\left(\frac{q^2}{P}\right) + \frac{1}{4\rho^3}\partial_x\left(\rho^2 P^2\right),\\
\partial_t E &= 4\pi\widetilde{\rho u}.
\end{aligned}$$

We notice that these equations are identical (at least the ones concerning ρ, u, P and q) to the equations obtained with a bi-delta reduction [22, 34–36]. Therefore, as a by-product of our reduction procedure, we have proved here that the bi-delta reduction is Hamiltonian. This can also be shown by effecting a chain rule calculation relating the Vlasov–Poisson bracket [19] to that of fluid streams [37]. As a consequence, one can verify that all the even fluid moments, namely P_{2n} for all $n \in \mathbb{N}$, are positive. Indeed for $f = \rho_1\delta(v - u_1) + \rho_2\delta(v - u_2)$ we have $P_{2n} = \int v^{2n} f \,\mathrm{d}v = \rho_1 u_1^{2n} + \rho_2 u_2^{2n} > 0$.

A benefit of knowing the Hamiltonian structure of the reduced model is the ability to use the Poisson bracket to obtain the additional invariants, e.g., Casimir invariants, that can be tricky to derive directly from the equations of motion. For example, the global Casimir invariants C_3, C_4 and C_5 for the present system are seen to be

$$C_3 = \int \sqrt{\frac{P}{\rho} + \frac{q^2}{P^2}}\,\mathrm{d}x, \quad C_4 = \int \rho\, q\sqrt{\frac{\rho}{P^3 + \rho q^2}}\,\mathrm{d}x, \quad C_5 = \int \left(u + \frac{q}{P}\right)\mathrm{d}x.$$

We note that these invariants can be used to check the validity of numerical algorithms used for the integration of the equations of motion.

3.2. Model without normal variables

We consider the second solution to equations (19)–(20), corresponding to $(R_0, R_0', R_0'', R_0''') = (0, 0, 4/3, 0)$. As mentioned in section 2.3, the solution R is even. Thus we introduce $R(\xi) = \bar{R}(\eta)$, where $\eta = \xi^2$. Then, equations (19)–(20) become

$$\sqrt{\eta}\left[3\eta\bar{R}''\left(6\eta + 9 - 5\bar{R}\right) + \bar{R}'\left(9 + 3\eta\bar{R}' - 7\bar{R}\right) + 4\bar{R} - 6\right] = 0, \tag{32}$$

$$2\eta\bar{R}''\left(6\eta + 9 - 5\bar{R}\right) + \bar{R}'\left(9 + 2\eta\bar{R}' - 5\bar{R} + \eta\right) + 2\bar{R} - 6 = 0. \tag{33}$$

By linearly combining these equations to eliminate terms in $\bar{R}''$, and introducing the new variable $\mu = -(\bar{R} - 3\eta - 9)/5$, we end up with an Abel equation of the second kind (see, for instance, [28]):

$$\mu\mu' - \mu = -\frac{6\eta + 24}{25},$$

which has the parametric solution

$$\eta(\tau) = K\frac{(2 - 5\tau)^2}{(3 - 5\tau)^3} - 4, \qquad \mu(\tau) = K\tau\frac{(2 - 5\tau)^2}{(3 - 5\tau)^3}, \tag{34}$$

where K is some constant to be determined. Inserting equation (34) into equations (32)–(33) implies these equations are satisfied if and only if $K = 27$. This leads to an explicit expression for the closure $\zeta = S_4/S_2^2 = R(\xi)$ given by

$$R(\xi) = 3\frac{\left[4 + 4t(\xi)^2 - t(\xi)\left(\xi^2 - 8\right) - 8\xi^2\right]\left[2 + 2t(\xi)^2 - t(\xi)\left(\xi^2 - 2\right) - 4\xi^2\right]}{\left[1 - 2\xi^2 + 3t(\xi) + t(\xi)^2\right]^2}, \tag{35}$$

where

$$t(\xi) = \left(\frac{\sqrt{\xi^2\left(4 + \xi^2\right)^3} - 2 - 10\xi^2 + \xi^4}{2}\right)^{1/3}.$$

Furthermore, by using equation (16), S_5 is given by $S_5 = S_2 S_3 T(\xi)$ with

$$T(\xi) = 2\frac{3\xi^2 - R(\xi)^2 - 7R(\xi)}{R(\xi) - 3\xi^2 - 9}. \tag{36}$$

In summary, the Hamiltonian and the Poisson bracket resulting from this closure are given respectively by equation (10) and (11) with α and β given by equation (12) with

$$S_4 = S_2^2 R\left(\frac{S_3}{S_2^{3/2}}\right), \tag{37}$$

$$S_5 = S_2 S_3 T\left(\frac{S_3}{S_2^{3/2}}\right), \tag{38}$$

where R and T are given by equations (35)–(36). The dependence of the functions R and T in their arguments is not trivial. In order to help the reader visualize the closure relations corresponding to equations (37)–(38), we provide, in figure 3 (respectively figure 4), color maps showing the dependence of S_4 (respectively S_5) on S_2 and S_3. As a side note, we remark that as S_3 tends toward 0, S_4 also goes to 0, as shown in figure 3. Thus, with this closure, symmetric distribution functions cannot exist. Consequently, the physical relevance of this solution in the context of plasma physics is questionable. This is a peculiarity that is not present in the model of section 3.1 where S_4 does not go to 0 as S_3 goes to 0 as seen in equation (25). As a consequence, the model with normal variables allows symmetric distribution functions as it could be expected. Furthermore, we notice the difference in the amplitude of the closures between the two models (up to two orders of magnitude) by comparing figures 1 and 3 and figures 2 and 4.

By using a calculation analogous to the one performed in the section 3.1, we show that this model does not have Casimir invariants of the entropy-type [29], i.e., of the form $\int \rho\phi(S_2, S_3)\,\mathrm{d}x$. Making use of the Buckingham π theorem, we assume that there exist a real number a and a function ψ such that $\phi = S_2^a \psi(S_3/S_2^{3/2}) = S_2^a \psi(\xi)$. The equations that have to be solved are

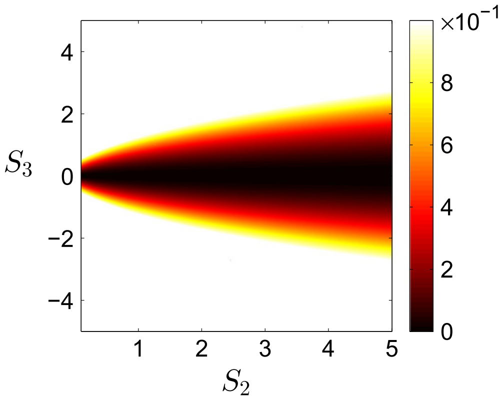

Figure 3. Color map of S_4 (in A^4 units as defined in section 2.3) given by equation (37) as a function of S_2 (in A^2 units) and S_3 (in A^3 units).

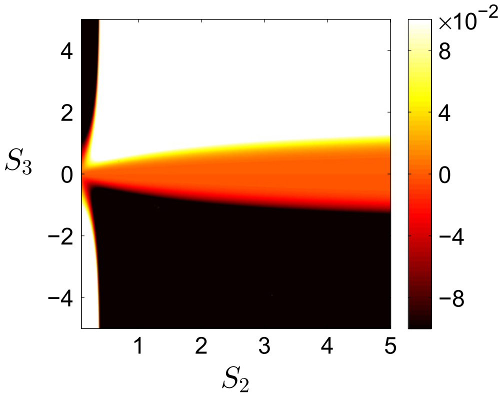

Figure 4. Color map of S_5 (in A^5 units as defined in section 2.3) given by equation (38) as a function of S_2 (in A^2 units) and S_3 (in A^3 units).

$$
\begin{aligned}
&8(a-1)a\xi\psi + \psi'\left(3 - 18\,a + 30\xi^2 - 24\,a\xi^2 - 3R + 10\,aR - 9\xi R'\right)\\
&\quad + 3\xi\left(9 + 6\xi^2 - 5R\right)\psi'' = 0,\\
&2a\psi + \psi'(-9\xi + 4a\xi + 3R') + \left(-9 - 6\xi^2 + 5R\right)\psi'' = 0,\\
&4a\psi\,[3 - 9\,a + (-1 + 5a)R - 3\xi R'] = 3\xi\left[\psi'(1 - 4a + (-17 + 20a)R\right.\\
&\quad \left. - 8(a-1)T - 6\xi R' + 6\xi T') - 3\xi(7 + 5\,R - 4T)\psi''\right],\\
&4a\psi R' + \psi'[3 - 18a + 5(-3 + 2a)R + 6\,T - 6\xi R' + 6\xi T']\\
&\quad = 3\xi(7 + 5\,R - 4T)\psi''.
\end{aligned}
$$

Combining the first two equations leads to

$$
2a(4a-1)\xi\psi + \left[3\left(1 - 6a + \xi^2 - 4a\xi^2\right) + (-3 + 10a)R\right]\psi' = 0,
$$

whose solution is given by

$$\psi(\xi) = \psi_0 \exp\left(\int^{\xi} \frac{2ay(4a-1)}{3(6a-1) + 3y^2(4a-1) + R(y)(3-10a)}\, \mathrm{d}y\right),$$

where ψ_0 is a constant. Inserting this expression into the previous equations provides the necessary constraints $a = 0$ and $\psi(\xi) = \psi_0$, which shows that this model does not have Casimir invariants of the entropy-type. The Poisson bracket resulting from this closure has only two global Casimir invariants, given by

$$C_1 = \int \rho\ \mathrm{d}x \qquad \text{and} \qquad C_2 = \int E\ \mathrm{d}x,$$

which are also Casimir invariants of the Vlasov–Ampère equations. Consequently, unlike the previous model, we cannot define normal variables for this closure.

3.3. Comparison with Hamiltonian fluid models with 3 + 1 fields

The same analysis done for 4 + 1 fields can be carried out for fluid models with 3 + 1 fields, namely with the field variables (P_0, P_1, P_2, E) or equivalently (ρ, u, S_2, E). This was partly done in [29] (in the absence of electric field), where it was shown that Hamiltonian fluid models are given by closures S_3 that only depend on S_2. This is also evident from appendix C, where the conditions given by equation (B.7) are automatically satisfied (since there is only one value for the indices). The fact that the closure for 3 + 1 Hamiltonian fluid models only depends on S_2 is similar to the fact that the closures for 4 + 1 fluid models are given by S_4 and S_5 as functions of only S_2 and S_3.

A specific closure $S_3(S_2)$, which corresponds to the dimensional analysis performed in the present work, is given by

$$S_3 = \lambda S_2^{3/2},$$

where λ is a dimensionless constant. The Poisson bracket for this closure is

$$\begin{aligned}\{F, G\}_3 = \int \Bigg\{ & G_u \partial_x F_\rho - F_u \partial_x G_\rho + 4\pi\left(G_u \widetilde{F_E} - F_u \widetilde{G_E}\right) \\ & - \frac{1}{\rho}\left(G_u F_2 - F_u G_2\right)\partial_x S_2 + 2\lambda S_2^{3/2}\left[\frac{G_2}{\rho}\partial_x\left(\frac{F_2}{\rho}\right) - \frac{F_2}{\rho}\partial_x\left(\frac{G_2}{\rho}\right)\right]\Bigg\}\ \mathrm{d}x.\end{aligned}$$

It should be noted that the dimensional analysis provides a family of models (labeled by λ). However there are only three fundamentally different models: one for $\lambda = 0$ and the others for $\lambda = \pm 1$, since all of the other models can be rescaled to $\lambda = \pm 1$ by an appropriate rescaling of S_2, e.g., $\bar{S_2} = S_2/\lambda^2$. Moreover, the two models $\lambda = 1$ and $\lambda = -1$ are linked by symmetry [29]. The model for $\lambda = 0$ has the two following global Casimir invariants:

$$C_1 = \int \rho\ \mathrm{d}x \qquad \text{and} \qquad C_2 = \int E\ \mathrm{d}x,$$

in addition to the family of Casimir invariants

$$C = \int \rho \kappa\left(S_2\right)\ \mathrm{d}x,$$

for any scalar function κ. The two Casimir invariants C_1 and C_2 are identical to the ones for 4 + 1 fields. Concerning the model with $\lambda = 1$, the Poisson bracket with 3 + 1 fields has C_1 and C_2 as Casimir invariants, and also has two additional Casimir invariants

$$C_3 = \int \rho S_2^{1/4} \mathrm{d}x \qquad \text{and} \qquad C_4 = \int \left(u - 2\rho S_2^{1/2} \right) \mathrm{d}x.$$

Therefore, in total it has four Casimir invariants, i.e., as many as the number of field variables. The common point between this 3 + 1 model with $\lambda = 1$ and the 4 + 1 field model with normal field variables is that both have a generalized velocity as Casimir invariant. It should also be noticed that the 3 + 1 fluid model has one Casimir invariant of the entropy type, i.e., of the form $\int \rho \phi(S_2)\ \mathrm{d}x$, whereas the 4 + 1 fluid model has two of this type.

4. Summary

In summary, starting from the one-dimensional Vlasov–Ampère equations, we build two Hamiltonian models with the first four moments of the Vlasov distribution function and the electric field as dynamical variables. Our reduction method relies on the preservation of the Hamiltonian structure of the Vlasov–Ampère model. The closures we obtain are derived from a dimensional analysis argument. We show that there are only two Hamiltonian closures obtained by this method. A fundamental difference between these two models is characterized by their Casimir invariants: one model has only two global Casimir invariants (preserved from the Vlasov–Ampère system), whereas the second model has three additional ones, two of the entropy-type and one generalized velocity.

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Appendix A. Independence of α and β of bracket (11) on ρ, u, and E

In this appendix, we consider the following bracket defined on functionals of the form $F[\rho, u, S_2, ..., S_N, E]$ for $N \geqslant 2$:

$$\{F, G\} = \int \left[G_u \partial_x F_\rho - F_u \partial_x G_\rho + 4\pi \left(G_u \widetilde{F_E} - F_u \widetilde{G_E} \right) - \frac{1}{\rho} \left(G_u F_i - F_u G_i \right) \partial_x S_i \right.$$
$$\left. + \ \alpha_{ij} \frac{F_i}{\rho} \frac{G_j}{\rho} + \partial_x \left(\frac{F_i}{\rho} \right) \beta_{ij} \frac{G_j}{\rho} \right] \mathrm{d}x, \tag{A.1}$$

where α and β are matrices satisfying $\beta^t = \beta$ and $\partial_x \beta = \alpha + \alpha^t$, assuring antisymmetry of the bracket. Here we assume *a priori* that α and β depend on both the dynamical variables ρ, u, S_k (for $k \geqslant 2$) and E, and their derivatives $\partial_x^n \rho$, $\partial_x^n u$, $\partial_x^n S_k$ and $\partial_x^n E$ for $n \geqslant 1$. Repeated indices are implicitly summed from 2 to N, unless specified. We seek necessary conditions on α and β for bracket (A.1) to satisfy the Jacobi identity

$$\{F, \{G, H\}\} + \{H, \{F, G\}\} + \{G, \{H, F\}\} = 0.$$

In this appendix, we prove that α and β do not depend on the variables ρ, u and E and their derivatives $\partial_x^n \rho$, $\partial_x^n u$ and $\partial_x^n E$ for $n \geqslant 1$.

First we split bracket (A.1) into two parts

$$\{F,\, G\} = \{F,\, G\}^J + \{F,\, G\}^*,$$

where the first part,

$$\{F,\, G\}^J = \int \left[G_u \partial_x F_\rho - F_u \partial_x G_\rho + 4\pi \left(G_u \widetilde{F_E} - F_u \widetilde{G_E} \right) - \frac{1}{\rho} \left(G_u F_i - F_u G_i \right) \partial_x S_i \right] \mathrm{d}x,$$

satisfies the Jacobi identity [14, 15]. The Jacobi identity is then equivalent to

$$\left\{ F,\, \{G,\, H\}^J \right\}^* + \left\{ F,\, \{G,\, H\}^* \right\}^J + \left\{ F,\, \{G,\, H\}^* \right\}^* + \circlearrowleft_{(F,G,H)} = 0, \tag{A.2}$$

where $\circlearrowleft_{(F,G,H)}$ denotes the summation over circular permutation of any three functionals F, G and H. Using the lemma stating that only the functional derivatives with respect to the explicit dependence on the variables need be taken into account for the Jacobi identity [14], the first term becomes

$$\begin{aligned} \left\{ F,\, \{G,\, H\}^J \right\}^* = \int \Bigg[& \frac{\alpha_{ij}}{\rho} \frac{F_i}{\rho} \left[H_u \partial_x \left(\frac{G_j}{\rho} \right) - G_u \partial_x \left(\frac{H_j}{\rho} \right) + \frac{G_j}{\rho} \partial_x H_u - \frac{H_j}{\rho} \partial_x G_u \right] \\ & + \frac{\beta_{ij}}{\rho} \partial_x \left(\frac{F_i}{\rho} \right) \left(\frac{G_j}{\rho} \partial_x H_u - \frac{H_j}{\rho} \partial_x G_u \right) \Bigg] \mathrm{d}x, \end{aligned} \tag{A.3}$$

where we have used the fact that β is symmetric. The second term in equation (A.2) is

$$\begin{aligned} \left\{ F,\, \{G,\, H\}^* \right\}^J = \int \Bigg[& \{G,\, H\}^*_u \partial_x F_\rho + \{G,\, H\}^*_\rho \partial_x F_u + 4\pi \left(\{G,\, H\}^*_u \widetilde{F_E} - \{G,\, H\}^*_E \widetilde{F_u} \right) \\ & - \frac{1}{\rho} \left(\{G,\, H\}^*_u F_i - F_u \{G,\, H\}^*_i \right) \partial_x S_i \Bigg] \mathrm{d}x, \end{aligned} \tag{A.4}$$

where

$$\begin{aligned} \{G,\, H\}^*_\rho &= \left(\frac{\alpha_{ji}}{\rho} - \frac{\alpha_{ij}}{\rho} \right) \frac{G_i}{\rho} \frac{H_j}{\rho} + \frac{\beta_{ij}}{\rho} \left[\partial_x \left(\frac{H_j}{\rho} \right) \frac{G_i}{\rho} - \partial_x \left(\frac{G_i}{\rho} \right) \frac{H_j}{\rho} \right] \\ &\quad + (-1)^n \partial_x^n \left(\left[\frac{\partial \alpha_{ij}}{\partial \partial_x^n \rho} \frac{G_i}{\rho} + \frac{\partial \beta_{ij}}{\partial \partial_x^n \rho} \partial_x \left(\frac{G_i}{\rho} \right) \right] \frac{H_j}{\rho} \right) \\ \{G,\, H\}^*_u &= (-1)^n \partial_x^n \left(\left[\frac{\partial \alpha_{ij}}{\partial \partial_x^n u} \frac{G_i}{\rho} + \frac{\partial \beta_{ij}}{\partial \partial_x^n u} \partial_x \left(\frac{G_i}{\rho} \right) \right] \frac{H_j}{\rho} \right), \\ \{G,\, H\}^*_k &= (-1)^n \partial_x^n \left(\left[\frac{\partial \alpha_{ij}}{\partial \partial_x^n S_k} \frac{G_i}{\rho} + \frac{\partial \beta_{ij}}{\partial \partial_x^n S_k} \partial_x \left(\frac{G_i}{\rho} \right) \right] \frac{H_j}{\rho} \right), \\ \{G,\, H\}^*_E &= (-1)^n \partial_x^n \left(\left[\frac{\partial \alpha_{ij}}{\partial \partial_x^n E} \frac{G_i}{\rho} + \frac{\partial \beta_{ij}}{\partial \partial_x^n E} \partial_x \left(\frac{G_i}{\rho} \right) \right] \frac{H_j}{\rho} \right). \end{aligned}$$

We consider the terms of the type $(F_u,\, G_i,\, H_j)$ in the Jacobi identity (A.2). These terms only come from equations (A.3) and (A.4). By using successive integrations by parts and assuming that the boundary conditions are such that the associated boundary integrals vanish, the Jacobi identity for these terms becomes

$$\int\left\{\left[\frac{\partial\alpha_{ij}}{\partial\partial_x^n\rho}\frac{G_i}{\rho}+\frac{\partial\beta_{ij}}{\partial\partial_x^n\rho}\partial_x\left(\frac{G_i}{\rho}\right)\right]\frac{H_j}{\rho}\partial_x^{n+1}F_u-\frac{H_j}{\rho}\partial_x\left(\alpha_{ij}\frac{G_i}{\rho}\frac{F_u}{\rho}\right)\right.$$
$$-\alpha_{ji}\partial_x\left(\frac{G_i}{\rho}\right)\frac{H_j}{\rho}\frac{F_u}{\rho}-4\pi\left[\frac{\partial\alpha_{ij}}{\partial\partial_x^n E}\frac{G_i}{\rho}+\frac{\partial\beta_{ij}}{\partial\partial_x^n E}\partial_x\left(\frac{G_i}{\rho}\right)\right]\frac{H_j}{\rho}\partial_x^n\widetilde{F_u}$$
$$\left.+\left[\frac{\partial\alpha_{ij}}{\partial\partial_x^n S_k}\frac{G_i}{\rho}+\frac{\partial\beta_{ij}}{\partial\partial_x^n S_k}\partial_x\left(\frac{G_i}{\rho}\right)\right]\frac{H_j}{\rho}\partial_x^n\left(\frac{F_u}{\rho}\partial_x S_k\right)\right\}\mathrm{d}x+\circlearrowleft_{(F,G,H)}=0. \tag{A.5}$$

Choosing $F=\int u\ \mathrm{d}x$, $G=\int\rho S_l\ \mathrm{d}x$ and $H=\rho S_m$, equation (A.5) leads to the necessary condition

$$\frac{1}{\rho}\partial_x\alpha_{lm}=\frac{\alpha_{lm}}{\rho^2}\partial_x\rho+\frac{\partial\alpha_{lm}}{\partial\partial_x^n S_k}\partial_x^n\left(\frac{1}{\rho}\partial_x S_k\right). \tag{A.6}$$

However, we have by definition

$$\partial_x\alpha_{lm}=\frac{\partial\alpha_{lm}}{\partial\partial_x^n\rho}\partial_x^{n+1}\rho+\frac{\partial\alpha_{lm}}{\partial\partial_x^n u}\partial_x^{n+1}u+\frac{\partial\alpha_{lm}}{\partial\partial_x^n S_k}\partial_x^{n+1}S_k+\frac{\partial\alpha_{lm}}{\partial\partial_x^n E}\partial_x^{n+1}E+\frac{\partial\alpha_{lm}}{\partial x},$$

where the summation over n is implicit and $\partial\alpha_{lm}/\partial x$ denotes the derivative of α_{lm} with respect to its explicit dependence on x. Eventually, equation (A.6) writes

$$\frac{1}{\rho}\left(\frac{\partial\alpha_{lm}}{\partial\partial_x^n\rho}\partial_x^{n+1}\rho+\frac{\partial\alpha_{lm}}{\partial\partial_x^n u}\partial_x^{n+1}u+\frac{\partial\alpha_{lm}}{\partial\partial_x^n S_k}\partial_x^{n+1}S_k+\frac{\partial\alpha_{lm}}{\partial\partial_x^n E}\partial_x^{n+1}E+\frac{\partial\alpha_{lm}}{\partial x}\right)$$
$$=\frac{\alpha_{lm}}{\rho^2}\partial_x\rho+\frac{\partial\alpha_{lm}}{\partial\partial_x^n S_k}\partial_x^n\left(\frac{1}{\rho}\partial_x S_k\right). \tag{A.7}$$

By canceling the only term that depend on $\partial_x^{\nu+1}\rho$ in equation (A.7), we can show that

$$\frac{\partial\alpha_{lm}}{\partial\partial_x^\nu\rho}=0.$$

By performing an induction on ν down to $\nu=0$, we can show that α does not depend on ρ and its derivatives. Because the dynamical variables are independent, using the same reasoning we prove that α cannot depend on u, E and their derivatives, nor can it depend explicitly on x. The same result can be obtained for β by choosing $G=\int\rho S_l x\ \mathrm{d}x$. Therefore a necessary (however not sufficient) condition for bracket (A.1) to satisfy the Jacobi identity is that α and β do not depend explicitly on x, ρ, u and E, as well as the derivatives $\partial_x^n\rho$, $\partial_x^n u$ and $\partial_x^n E$ for all $n\in\mathbb{N}$.

Appendix B. Dependence of α and β of bracket (11) on S_k

In this appendix, we derive some necessary conditions on the dependence of α and β (and their derivatives) of bracket (11) on S_k. Following appendix A, we consider two sets of functionals

$$(F,G,H)=\left(\int ux\ \mathrm{d}x,\int\rho S_l\ \mathrm{d}x,\rho S_m\right),$$

and

$$(F,\, G,\, H) = \left(\int ux\ \mathrm{d}x,\ \int \rho S_l x\ \mathrm{d}x,\ \rho S_m\right),$$

which we insert into equation (A.5). Thus we find the necessary conditions

$$\alpha_{lm} = n\frac{\partial \alpha_{lm}}{\partial \partial_x^n S_k}\partial_x^n S_k, \qquad n\frac{\partial \beta_{lm}}{\partial \partial_x^n S_k}\partial_x^n S_k = 0, \tag{B.1}$$

for all l and m, where we recall the implicit summation over repeated indices. We assume that α and β depend on the derivatives of S_k up to order ν, where

$$\nu = \max\left\{n \in \mathbb{N} \text{ s.t. } \partial\alpha/\partial\partial_x^n S \neq 0 \ \text{ or } \ \partial\beta/\partial\partial_x^n S \neq 0\right\}.$$

From the first of equations (B.1) we have

$$\partial_x \beta_{lm} = \sum_{n=0}^{\nu} \frac{\partial \beta_{lm}}{\partial \partial_x^n S_k}\partial_x^{n+1} S_k = \alpha_{lm} + \alpha_{ml} = \sum_{n=0}^{\nu} n\left[\frac{\partial \alpha_{lm}}{\partial \partial_x^n S_k} + \frac{\partial \alpha_{ml}}{\partial \partial_x^n S_k}\right]\partial_x^n S_k. \tag{B.2}$$

Differentiating equation (B.2) with respect to $\partial^{\nu+1} S_j$ leads to

$$\frac{\partial \beta_{lm}}{\partial \partial_x^{\nu} S_j} = 0.$$

As a consequence, the highest derivatives of S appear in α; thus ν becomes

$$\nu = \max\left\{n \in \mathbb{N} \text{ s.t. } \partial\alpha/\partial\partial_x^n S \neq 0\right\}.$$

The Jacobi identity (A.2) reduces to:

$$\{F,\, \{G,\, H\}\} + \circlearrowleft_{(F,G,H)} = \int\left[\alpha_{ij}\frac{F_i}{\rho}\frac{\{G,\, H\}_j^*}{\rho} + \beta_{ij}\frac{\{G,\, H\}_j^*}{\rho}\partial_x\left(\frac{F_i}{\rho}\right)\right]\mathrm{d}x$$
$$+ \circlearrowleft_{(F,G,H)} = 0. \tag{B.3}$$

This identity corresponds to the Jacobi identity for the subalgebra of observables $F[\rho,\, S_2\, ,...,\, S_N]$. Expanding equation (B.3) gives

$$\{F,\, \{G,\, H\}\} + \circlearrowleft_{(F,G,H)} = \int\left\{\frac{\alpha_{ij}}{\rho}\frac{F_i}{\rho}(-1)^n\partial_x^n\left(\left[\frac{\partial \alpha_{kl}}{\partial \partial_x^n S_j}\frac{G_k}{\rho} + \frac{\partial \beta_{kl}}{\partial \partial_x^n S_j}\partial_x\left(\frac{G_k}{\rho}\right)\right]\frac{H_l}{\rho}\right)\right.$$
$$+ (-1)^{n+1}\frac{F_i}{\rho}\partial_x\left[\frac{\beta_{ij}}{\rho}\partial_x^n\left(\left[\frac{\partial \alpha_{kl}}{\partial \partial_x^n S_j}\frac{G_k}{\rho} + \frac{\partial \beta_{kl}}{\partial \partial_x^n S_j}\partial_x\left(\frac{G_k}{\rho}\right)\right]\frac{H_l}{\rho}\right)\right]$$
$$+ \partial_x^n\left[\frac{\alpha_{lj}}{\rho}\frac{H_l}{\rho} + \frac{\beta_{lj}}{\rho}\partial_x\left(\frac{H_l}{\rho}\right)\right]\frac{\partial \alpha_{ik}}{\partial \partial_x^n S_j}\frac{F_i}{\rho}\frac{G_k}{\rho}$$

$$- \frac{F_i}{\rho}\partial_x\left(\partial_x^n\left[\frac{\alpha_{lj}}{\rho}\frac{H_l}{\rho} + \frac{\beta_{lj}}{\rho}\partial_x\left(\frac{H_l}{\rho}\right)\right]\frac{\partial\beta_{ik}}{\partial\partial_x^n S_j}\frac{G_k}{\rho}\right)$$
$$+ \partial_x^n\left[\frac{\alpha_{kj}}{\rho}\frac{G_k}{\rho} + \frac{\beta_{kj}}{\rho}\partial_x\left(\frac{G_k}{\rho}\right)\right]\left[\frac{\partial\alpha_{li}}{\partial\partial_x^n S_j}\frac{H_l}{\rho} + \frac{\partial\beta_{li}}{\partial\partial_x^n S_j}\partial_x\left(\frac{H_l}{\rho}\right)\right]\frac{F_i}{\rho}\Bigg\}\,\mathrm{d}x.$$

Choosing consecutively

$$(F,\, G,\, H) = \left(\rho S_i,\, \int\rho S_k\ \mathrm{d}x,\, \int\rho S_l\ \mathrm{d}x\right),$$

$$(F,\, G,\, H) = \left(\rho S_i,\, \int\rho S_k x\ \mathrm{d}x,\, \int\rho S_l\ \mathrm{d}x\right),$$

and

$$(F,\, G,\, H) = \left(\rho S_i,\, \int\rho S_k\ \mathrm{d}x,\, \int\rho S_l x\ \mathrm{d}x\right),$$

we get the following three conditions:

$$\left[\frac{\partial\alpha_{ik}}{\partial\partial_x^\nu S_j} - \frac{\partial\beta_{ik}}{\partial\partial_x^{\nu-1} S_j}\right]\alpha_{lj} + \alpha_{kj}\frac{\partial\alpha_{li}}{\partial\partial_x^\nu S_j} = 0, \tag{B.4}$$

$$\beta_{kj}\frac{\partial\alpha_{li}}{\partial\partial_x^\nu S_j} = 0, \tag{B.5}$$

$$\beta_{lj}\frac{\partial\beta_{ik}}{\partial\partial_x^{\nu-1} S_j} = 0. \tag{B.6}$$

Due to the fact that $\partial_x\beta = \alpha + \alpha^t$, equations (B.5) and (B.6) are redundant. We assume that α depends only linearly on $\partial_x^\nu S_k$. We show in appendix C that this is the case for fluid brackets. As a consequence, we write

$$\alpha_{lj}\left(S,\, \partial_x S,\ldots,\partial_x^\nu S\right) = A_{lj}\left(S,\, \partial_x S,\ldots,\partial_x^{\nu-1} S\right) + \gamma_{ljm}\left(S,\, \partial_x S,\ldots,\partial_x^{\nu-1} S\right)\partial_x^\nu S_m.$$

By inserting this expression into equation (B.4), for the Jacobi identity we need to impose

$$\gamma_{ljm}\left[\gamma_{ikj} - \frac{\partial\beta_{ik}}{\partial\partial_x^{\nu-1} S_j}\right] + \gamma_{kjm}\gamma_{lij} = 0,$$

for all $(i,\, k,\, l,\, m)$ to make the term proportional to $\partial_x^\nu S_m$ vanish. However, thanks to equation (B.2) we have

$$\frac{\partial\beta_{ik}}{\partial\partial_x^{\nu-1} S_j} = \gamma_{ikj} + \gamma_{kij}.$$

This eventually leads to the following conditions:

$$\gamma_{ljm}\gamma_{kij} = \gamma_{kjm}\gamma_{lij}. \tag{B.7}$$

These commutation relations remind us of the conditions for Lie–Poisson brackets based on Lie algebra extensions to satisfy the Jacobi identity of [38]. These conditions on the tensor γ are necessary but not sufficient.

Appendix C. Jacobi identity for fluid models

In this appendix we find necessary and sufficient conditions for the Jacobi identity for bracket (11). We start from the one-dimensional Vlasov–Ampère bracket given by equation (6) and perform a change of variables, from f to $(\rho, u, S_{n\geqslant 2})$ defined by

$$\rho = \int f \, \mathrm{d}v, \qquad \rho u = \int v f \, \mathrm{d}v, \qquad \rho^{n+1} S_n = \int (v-u)^n f \, \mathrm{d}v.$$

Using the following chain rule expression for the functional derivative with respect to f,

$$F_f = \bar{F}_\rho + \frac{v-u}{\rho}\bar{F}_u + \bar{F}_n\left[\frac{(v-u)^n}{\rho} - \frac{n+1}{\rho}S_n - n\frac{S_{n-1}}{\rho}\frac{(v-u)}{\rho}\right],$$

and after some algebra, we show that the Poisson bracket (6) reduces to equation (A.1) with α and β given by

$$\alpha_{nm} = n\partial_x S_{n+m-1} - nS_{n-1}\partial_x S_m - n(m+1)S_m\partial_x S_{n-1} - nmS_{m-1}\partial_x S_n, \tag{C.1}$$

$$\beta_{nm} = (m+n)S_{n+m-1} - m(n+1)S_n S_{m-1} - n(m+1)S_m S_{n-1}, \tag{C.2}$$

where $n, m \geqslant 2$ and $S_1 = 0$. The resulting bracket is of Poisson type. Next, we truncate the matrices α and β such that $\alpha_{mn} = 0$ and $\beta_{mn} = 0$ for $m > N$ and $n > N$. The matrices α and β depend on S_n for $n = 2, \ldots, 2N-1$. We restrict ourselves to the case where α and β are functions of $(S_2, \ldots, S_N)$, i.e., we introduce $N-1$ closures $S_k = S_k(S_2, \ldots, S_N)$ for $k = N+1, \ldots, 2N-1$. In this truncation/reduction, the bracket is no longer of Poisson type in general. In this appendix, we establish the necessary and sufficient conditions for the Jacobi identity to be satisfied. From appendices A and B, this Jacobi identity is seen to be

$$\begin{aligned}
\{F, \{G, H\}\} + \circlearrowleft_{(F,G,H)} = \int \Bigg\{ & \partial_x\left(\frac{F_i}{\rho}\right)\frac{1}{\rho}\partial_x\left(\frac{G_k}{\rho}\right)\frac{H_l}{\rho}\left[\beta_{ij}\frac{\partial \beta_{kl}}{\partial S_j} - \beta_{ij}\frac{\partial \alpha_{kl}}{\partial \partial_x S_j}\right.\\
&\left. - \beta_{kj}\frac{\partial \alpha_{li}}{\partial \partial_x S_j}\right] + \frac{F_i}{\rho}\frac{1}{\rho}\frac{G_k}{\rho}\frac{H_l}{\rho}\left[\alpha_{ij}\frac{\partial \alpha_{kl}}{\partial S_j} + \alpha_{lj}\frac{\partial \alpha_{ik}}{\partial S_j} + \alpha_{kj}\frac{\partial \alpha_{li}}{\partial S_j}\right.\\
&\left. - \alpha_{ij}\partial_x\left(\frac{\partial \alpha_{kl}}{\partial \partial_x S_j}\right) - \alpha_{lj}\partial_x\left(\frac{\partial \alpha_{ik}}{\partial \partial_x S_j}\right) - \alpha_{kj}\partial_x\left(\frac{\partial \alpha_{li}}{\partial \partial_x S_j}\right)\right]\\
& + \frac{F_i}{\rho}\frac{1}{\rho}\partial_x\left(\frac{G_k}{\rho}\right)\frac{H_l}{\rho}\left[\alpha_{ij}\frac{\partial \beta_{kl}}{\partial S_j} + \beta_{kj}\frac{\partial \alpha_{li}}{\partial S_j} - \alpha_{ij}\frac{\partial \alpha_{kl}}{\partial \partial_x S_j} - \alpha_{lj}\frac{\partial \alpha_{ik}}{\partial \partial_x S_j}\right.\\
&\left. - \beta_{kj}\partial_x\left(\frac{\partial \alpha_{li}}{\partial \partial_x S_j}\right)\right] + \partial_x\left(\frac{F_i}{\rho}\right)\frac{1}{\rho}\frac{G_k}{\rho}\frac{H_l}{\rho}\left[\beta_{ij}\frac{\partial \alpha_{kl}}{\partial S_j} + \alpha_{lj}\frac{\partial \beta_{ik}}{\partial S_j}\right.\\
&\left. - \beta_{ij}\partial_x\left(\frac{\partial \alpha_{kl}}{\partial \partial_x S_j}\right) - \alpha_{lj}\frac{\partial \alpha_{ik}}{\partial \partial_x S_j} - \alpha_{kj}\frac{\partial \alpha_{li}}{\partial \partial_x S_j}\right]\\
& + \frac{1}{\rho}\partial_x\left(\frac{H_l}{\rho}\right)\frac{F_i}{\rho}\frac{G_k}{\rho}\left[\beta_{lj}\frac{\partial \alpha_{ik}}{\partial S_j} - \alpha_{ij}\frac{\partial \alpha_{kl}}{\partial \partial_x S_j} + \alpha_{kj}\frac{\partial \beta_{li}}{\partial S_j}\right.
\end{aligned}$$

$$- \beta_{lj}\partial_x\left(\frac{\partial\alpha_{ik}}{\partial\partial_x S_j}\right) - \alpha_{kj}\frac{\partial\alpha_{li}}{\partial\partial_x S_j}\Bigg] + \partial_x\left(\frac{F_i}{\rho}\right)\frac{1}{\rho}\partial_x\left(\frac{H_l}{\rho}\right)$$

$$\frac{G_k}{\rho}\left[\beta_{lj}\frac{\partial\beta_{ik}}{\partial S_j} - \beta_{ij}\frac{\partial\alpha_{kl}}{\partial\partial_x S_j} - \beta_{lj}\frac{\partial\alpha_{ik}}{\partial\partial_x S_j}\right]$$

$$+ \frac{1}{\rho}\partial_x\left(\frac{G_k}{\rho}\right)\partial_x\left(\frac{H_l}{\rho}\right)\frac{F_i}{\rho}\left[\beta_{kj}\frac{\partial\beta_{li}}{\partial S_j} - \beta_{lj}\frac{\partial\alpha_{ik}}{\partial\partial_x S_j} - \beta_{kj}\frac{\partial\alpha_{li}}{\partial\partial_x S_j}\right]\Bigg\}\,\mathrm{d}x. \qquad \text{(C.3)}$$

Choosing $F = \rho S_i$, $G = \int\rho S_k\,\mathrm{d}x$ and $H = \int\rho S_l\,\mathrm{d}x$, equation (C.3) reduces to

$$\frac{\alpha_{ij}}{\rho}\left(\frac{\partial\alpha_{lk}}{\partial S_j} - \partial_x\frac{\partial\alpha_{lk}}{\partial\partial_x S_j}\right) + \partial_x\left(\frac{\beta_{ij}}{\rho}\left[\partial_x\left(\frac{\partial\alpha_{lk}}{\partial\partial_x S_j}\right) - \frac{\partial\alpha_{lk}}{\partial S_j}\right] - \frac{\alpha_{kj}}{\rho}\frac{\partial\beta_{il}}{\partial S_j}\right)$$

$$+ \frac{\alpha_{kj}}{\rho}\frac{\partial\alpha_{il}}{\partial S_j} + \frac{1}{\rho}\frac{\partial\alpha_{il}}{\partial\partial_x S_j}\partial_x\alpha_{kj} + \frac{\alpha_{lj}}{\rho}\frac{\partial\alpha_{ki}}{\partial S_j} + \frac{\partial\alpha_{ki}}{\partial\partial_x S_j}\partial_x\left(\frac{\alpha_{lj}}{\rho}\right) = 0. \qquad \text{(C.4)}$$

Equation (C.4) can be split into two terms with only one depending on ρ. To make the term that depends on ρ vanish, we have to impose

$$\beta_{ij}\left[\partial_x\left(\frac{\partial\alpha_{lk}}{\partial\partial_x S_j}\right) - \frac{\partial\alpha_{lk}}{\partial S_j}\right] - \alpha_{kj}\frac{\partial\beta_{il}}{\partial S_j} + \frac{\partial\alpha_{il}}{\partial\partial_x S_j}\alpha_{kj} + \frac{\partial\alpha_{ki}}{\partial\partial_x S_j}\alpha_{lj} = 0, \qquad \text{(C.5)}$$

for all (i, l, k). In addition, canceling the term in equation (C.4) that does not depend on ρ leads to

$$\alpha_{ij}\left(\frac{\partial\alpha_{lk}}{\partial S_j} - \partial_x\frac{\partial\alpha_{lk}}{\partial\partial_x S_j}\right) + \alpha_{kj}\left(\frac{\partial\alpha_{il}}{\partial S_j} - \partial_x\frac{\partial\alpha_{il}}{\partial\partial_x S_j}\right) + \alpha_{lj}\left(\frac{\partial\alpha_{ki}}{\partial S_j} - \partial_x\frac{\partial\alpha_{ki}}{\partial\partial_x S_j}\right) = 0, \qquad \text{(C.6)}$$

for all (i, l, k). With these constraints, equation (C.3) becomes

$$\{F, \{G, H\}\} + \circlearrowleft_{(F,G,H)} = \int\Bigg\{\partial_x\left(\frac{F_i}{\rho}\right)\frac{1}{\rho}\partial_x\left(\frac{G_k}{\rho}\right)\frac{H_l}{\rho}\Bigg[\beta_{ij}\frac{\partial\beta_{kl}}{\partial S_j}$$

$$- \beta_{ij}\frac{\partial\alpha_{kl}}{\partial\partial_x S_j} - \beta_{kj}\frac{\partial\alpha_{li}}{\partial\partial_x S_j}\Bigg]$$

$$+ \partial_x\left(\frac{F_i}{\rho}\right)\frac{1}{\rho}\partial_x\left(\frac{H_l}{\rho}\right)\frac{G_k}{\rho}\left[\beta_{lj}\frac{\partial\beta_{ik}}{\partial S_j} - \beta_{ij}\frac{\partial\alpha_{kl}}{\partial\partial_x S_j} - \beta_{lj}\frac{\partial\alpha_{ik}}{\partial\partial_x S_j}\right]$$

$$+ \frac{1}{\rho}\partial_x\left(\frac{G_k}{\rho}\right)\partial_x\left(\frac{H_l}{\rho}\right)\frac{F_i}{\rho}\left[\beta_{kj}\frac{\partial\beta_{li}}{\partial S_j} - \beta_{lj}\frac{\partial\alpha_{ik}}{\partial\partial_x S_j} - \beta_{kj}\frac{\partial\alpha_{li}}{\partial\partial_x S_j}\right]\Bigg\}\,\mathrm{d}x. \qquad \text{(C.7)}$$

Choosing $F = \rho S_i$, $G = \int\rho S_k\,\mathrm{d}x$ and $H = \int\rho S_l x\,\mathrm{d}x$ leads to

$$-\partial_x\left[\frac{1}{\rho}\left(\beta_{lj}\frac{\partial\beta_{ik}}{\partial S_j} - \beta_{ij}\frac{\partial\alpha_{kl}}{\partial\partial_x S_j} - \beta_{lj}\frac{\partial\alpha_{ik}}{\partial\partial_x S_j}\right)\right] = 0,$$

which has to be satisfied for any ρ, and therefore

$$\beta_{lj}\frac{\partial\beta_{ik}}{\partial S_j} - \beta_{ij}\frac{\partial\alpha_{kl}}{\partial\partial_x S_j} - \beta_{lj}\frac{\partial\alpha_{ik}}{\partial\partial_x S_j} = 0, \tag{C.8}$$

for all (i, l, k). With this additional constraint, equation (C.7) is always satisfied, which proves that equations (C.5), (C.6), and (C.8) are necessary and sufficient conditions for bracket (11) to satisfy the Jacobi identity.

By introducing the expressions of α and β given by equations (C.1) and (C.2) into equations (C.5), (C.6), and (C.8), we end up with the following constraints:

$$\Gamma_{iklm} = \Gamma_{ilkm}, \tag{C.9}$$

$$\Delta_{ikl} = \Delta_{lki}, \tag{C.10}$$

where

$$\begin{aligned}\Gamma_{iklm} = {} & \delta_m^k\left[(1-i-l)S_{i+l-2} + jS_{j-1}\frac{\partial S_{i+l-1}}{\partial S_j}\right] \\ & - \delta_m^{k-1}\left[(i+l)S_{i+l-1} - (j+1)S_j\frac{\partial S_{i+l-1}}{\partial S_j}\right] - \frac{\partial S_{i+l-1}}{\partial S_j}\frac{\partial S_{k+j-1}}{\partial S_m},\end{aligned}$$

and

$$\begin{aligned}\Delta_{ikl} = {} & \frac{\partial S_{i+k-1}}{\partial S_j}\left[(l+j)S_{l+j-1} - j(l+1)S_l Sj - 1 - l(j+1)S_j S_{l-1}\right] \\ & + l(i+k)S_{l-1}S_{i+k-1} + (l+1)(i+k-1)S_l S_{i+k-2}.\end{aligned}$$

References

[1] Shadwick B A, Tarkenton G M and Esarey E H 2004 Hamiltonian description of low-temperature relativistic plasmas *Phys. Rev. Lett.* **93** 175002

[2] Shadwick B A, Tarkenton G M, Esarey E and Schroeder C B 2005 Fluid and Vlasov models of low-temperature, collisionless, relativistic plasma interactions *Phys. Plasmas* **12** 056710

[3] Goswami P, Passot T and Sulem P L 2005 A Landau fluid model for warm collisionless plasmas *Phys. Plasmas* **12** 102109

[4] Shadwick B A, Tarkenton G M, Esarey E H and Lee F M 2012 Hamiltonian reductions for modeling relativistic laser-plasma interactions *Commun. Nonlinear Sci. Numer. Simulat.* **17** 2153–60

[5] Braginskii S I 1965 Transport processes in a plasma *Rev. Plasma Phys.* **1** 205–311

[6] Ott E and Sudan R N 1969 Nonlinear theory of ion acoustic waves with Landau damping *Phys. Fluids* **12** 2388–94

[7] Hazeltine R D, Kotschenreuther M and Morrison P J 1985 A four-field model for Tokamak plasma dynamics *Phys. Fluids* **28** 2466–77

[8] Hammett G W and Perkins F W 1990 Fluid moment models for Landau damping with application to the ion-temperature-gradient instability *Phys. Rev. Lett.* **64** 3019–22

[9] Hammett G W, Beer M A, Dorland W, Cowley S C and Smith S A 1993 Developments in the gyrofluid approach to tokamak turbulence simulations *Plasma Phys. Control. Fusion* **35** 973–85

[10] Sugama H, Watanabe T-H and Horton W 2003 Comparison between kinetic and fluid simulations of slab ion temperature gradient driven turbulence *Phys. Plasmas* **10** 726–36

[11] Passot T and Sulem P L 2004 A fluid description for Landau damping of dispersive MHD waves *Nonlinear Proc. Geophys.* **11** 245–58

[12] Sarazin Y, Dif-Pradalier G, Zarzoso D, Garbet X, Ghendrih P and Grandgirard V 2009 Entropy production and collisionless fluid closure *Plasma Phys. Control. Fusion* **51** 115003
[13] Morrison P J and Greene J M 1980 Noncanonical Hamiltonian density formulation of hydrodynamics and ideal magnetohydrodynamics *Phys. Rev. Lett.* **45** 790–3
[14] Morrison P J 1982 Poisson brackets for fluids and plasmas *AIP Conf. Proc.* **88** 13–46
[15] Morrison P J 1998 Hamiltonian description of the ideal fluid *Rev. Mod. Phys.* **70** 467–520
[16] Marsden J E and Ratiu T S 1998 *Introduction to Mechanics and Symmetry* (New York: Springer)
[17] de Guillebon L and Chandre C 2012 Hamiltonian structure of reduced fluid models for plasmas obtained from a kinetic description *Phys. Lett.* A **376** 3172–6
[18] Tronci C, Tassi E, Camporeale E and Morrison P J 2014 Hybrid Vlasov-MHD models: Hamiltonian vs. non-Hamiltonian *Plasma Phys. Control. Fusion* **56** 095008
[19] Morrison P J 1980 The Maxwell–Vlasov equations as a continuous Hamiltonian system *Phys. Lett.* A **80** 383–6
[20] Jin S and Li X 2003 Multi-phase computations of the semiclassical limit of the Schrödinger equation and related problems: Whitham versus Wigner *Physica* D **182** 46–85
[21] Gosse L, Jin S and Li X 2003 On two moment systems for computing multiphase semiclassical limits of the Schrödinger equation *Math. Models Methods Appl. Sci.* **13** 1689–723
[22] Chalons E, Kah D and Massot M 2012 Beyond pressureless gas dynamics: quadrature-based velocity moment models *Commun. Math. Sci.* **10** 1241–72
[23] Chandre C, de Guillebon L, Back A, Tassi E and Morrison P J 2013 On the use of projectors for Hamiltonian systems and their relationship with Dirac brackets *J. Phys. A: Math. Theor.* **46** 125203
[24] Kupershmidt B A and Manin J I 1978 Equations of long waves with a free surface: II. Hamiltonian structure and higher equations *Funct. Anal. Appl.* **12** 20–9 (English transl.)
Kupershmidt B A and Manin J I 1978 *Funktsional. Anal. i Prilozhen.* **12** 25–37 (in Russian)
[25] Gibbons J 1981 Collisionless Boltzmann equations and integrable moment equations *Physica* D **3** 503–11
[26] Gibbons J, Holm D D and Tronci C 2008 Vlasov moments, integrable systems and singular solutions *Phys. Lett.* A **372** 1024–33
[27] Gibbings J C 2011 *Dimensional Analysis* (Berlin: Springer)
[28] Polyanin A D and Zaitsev V F 2002 *Handbook of Exact Solutions for Ordinary Differential Equations* 2nd edn (London: Chapman & Hall; Boca Raton, FL: CRC)
[29] Perin M, Chandre C, Morrison P J and Tassi E 2014 Higher-order Hamiltonian fluid reduction of Vlasov equation *Ann. Phys.* **348** 50–63
[30] Labit B, Furno I, Fasoli A, Diallo A, Müller S H, Plyushchev G, Podestà M and Poli F M 2007 Universal statistical properties of drift-interchange turbulence in torpex plasmas *Phys. Rev. Lett.* **98** 255002
[31] Krommes J A 2008 The remarkable similarity between the scaling of kurtosis with squared skewness for TORPEX density fluctuations and sea-surface temperature fluctuations *Phys. Plasmas* **15** 030703
[32] Sandberg I, Benkadda S, Garbet X, Ropokis G, Hizanidis K and del Castillo-Negrete D 2009 Universal probability distribution function for bursty transport in plasma turbulence *Phys. Rev. Lett.* **103** 165001
[33] Guszejnov D, Lazányi N, Bencze A and Zoletnik S 2013 On the effect of intermittency of turbulence on the parabolic relation between skewness and kurtosis in magnetized plasmas *Phys. Plasmas* **20** 112305
[34] Fox R O 2009 High-order quadrature-based moment methods for kinetic equations *J. Comput. Phys.* **228** 7771–91
[35] Yuan C and Fox R O 2011 Conditional quadrature method of moments for kinetic equations *J. Comput. Phys.* **230** 8216–46
[36] Cheng Y and Rossmanith J A 2014 A class of quadrature-based moment closure methods with application to the Vlasov–Poisson–Fokker–Planck system in the high-field limit *J. Comput. Appl. Math.* **262** 384–98
[37] Morrison P J and Hagstrom G I 2014 Continuum Hamiltonian Hopf bifurcation I *Nonlinear Physical Systems Spectral Analysis, Stability and Bifurcations* ed O Kirillov and D Pelinovsky (New York: Wiley)
[38] Thiffeault J L and Morrison P J 2000 Classification and casimir invariants of Lie–Poisson brackets *Physica* D **136** 205–44