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Hamiltonian gyrokinetic Vlasov–Maxwell system

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ABSTRACT

A new formulation of electromagnetic gyrokinetics that possesses Hamiltonian form is constructed. The new formulation replaces Poisson-like equations by hyperbolic equations for the electromagnetic field with the speed of light slowed to that of the gyrokinetic vacuum, thereby significantly reducing computational cost. An energy principle is derived using the field-theoretic noncanonical Poisson bracket formulation of the theory. The energy principle is used to prove stability of the thermal equilibrium state in a uniform background magnetic field.

1. Introduction

Electromagnetic gyrokinetic theory (EMGT) is a model used to describe the turbulent transport of particles and heat induced by fluctuating electric and magnetic fields in strongly magnetized plasmas. EMGT is, in many ways, a more utilitarian tool than the more-fundamental Vlasov–Maxwell kinetic theory (VMKT). However, VMKT enjoys two important advantages over existing formulations of EMGT. (I) When simulated on a computer, the VMKT field solve is local; advancing the electromagnetic field in time at a given grid point only requires communication with nearby grid points [1]. (II) There is an energy principle for assessing the stability of Vlasov–Maxwell equilibria [2] (also see [3–7] for similar energy principles in other contexts.) In contrast, modern EMGT simulations require global Poisson-like field solves at each time step. This prevents EMGT simulations from scaling as favorably [8] as VMKT simulations when the number of processing cores is increased at fixed problem size. Likewise, the free energy of perturbations to EMGT equilibria is unknown. Thus, the basic tool for studying the stability of EMG equilibria by way of an energy principle is unavailable. The purpose of this Letter is to describe a new formulation of electromagnetic gyrokinetics that enjoys properties (I) and (II). The new formulation, which we will refer to as the gyrokinetic Vlasov–Maxwell (GVM) system, enjoys a local field solve and has an energy principle, while retaining the traditional advantages of gyrokinetic theory.

2. The new formulation

The gyrokinetic Vlasov–Maxwell equations are given by

$$\frac{\partial f_s}{\partial t} = -L_{V_s^{\rm gy}} f_s \tag{1a}$$

$$\frac{1}{c}\frac{\partial \boldsymbol{D}}{\partial t} = \nabla\times\boldsymbol{H} - \frac{4\pi}{c}\boldsymbol{J}_{\rm gy} \tag{1b}$$

$$\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla\times\boldsymbol{E} \tag{1c}$$

$$\nabla\cdot\boldsymbol{D} = 4\pi\rho_{\rm gy} \tag{1d}$$

$$\nabla\cdot\boldsymbol{B} = 0. \tag{1e}$$

f_s is the gyrocenter volume form of species s, $V_s^{\rm gy}$ is the gyrocenter phase space velocity, $L_{V_s^{\rm gy}}$ denotes the Lie derivative along the gyrocenter phase space velocity, $\boldsymbol{J}_{\rm gy}$ is the gyrocenter current density, $\rho_{\rm gy}$ is the gyrocenter charge density, $\boldsymbol{E}$, $\boldsymbol{B}$ are the fluctuating electric and magnetic fields, and $\boldsymbol{D}$, $\boldsymbol{H}$ are the auxiliary electric and magnetic fields. The volume form f_s is defined by requiring that the number of particles of species s in a region of phase space U be given by $\int_U f_s$. The gyrocenter phase space velocity is specified by the time-dependent tensor form of Hamilton's equations,

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$$\mathrm{i}_{V_s^{\mathrm{gy}}}\omega_s^{\mathrm{gy}} = \mathbf{d}K_s - e_s \boldsymbol{E}\cdot d\boldsymbol{X}, \tag{2}$$

where ω_s^{gy} is the gyrocenter symplectic form, K_s is the gyrocenter kinetic energy, and $d\boldsymbol{X}$ denotes the vector line element in the space of gyrocenter positions. The gyrocenter symplectic form is the sum of the guiding center symplectic form [9,10] and the fluctuating magnetic flux,

$$\omega_s^{\mathrm{gy}} = \omega_s^{\mathrm{gc}} - \frac{e_s}{c}\boldsymbol{B}\cdot dS, \tag{3}$$

where dS is the surface element in the space of gyrocenter positions. The gyrocenter kinetic energy is a functional of the fluctuating electric and magnetic fields, and is related to the gyrocenter Hamiltonian by $H_s^{\mathrm{gy}} = K_s + e_s\phi$ (an explicit expression for K_s will be given near the end of this Letter.) The auxiliary fields $\boldsymbol{D}$, $\boldsymbol{H}$ are related to $\boldsymbol{E}$, $\boldsymbol{B}$ by using relations that emerge from the Hamiltonian theory developed in [11], i.e., the constitutive relations are given by

$$\boldsymbol{D} = \boldsymbol{E} - 4\pi\frac{\delta\mathcal{K}}{\delta\boldsymbol{E}} \tag{4}$$

$$\boldsymbol{H} = \boldsymbol{B} + 4\pi\frac{\delta\mathcal{K}}{\delta\boldsymbol{B}}, \tag{5}$$

where $\mathcal{K}(f,\boldsymbol{E},\boldsymbol{B}) = \sum_s\int f_s K_s(\boldsymbol{E},\boldsymbol{B})$.

Following [11] the system above constitutes an infinite-dimensional Hamiltonian system with dynamical variables f, $\boldsymbol{D}$, and $\boldsymbol{B}$, and Hamiltonian functional given by

$$\mathcal{H}(f,\boldsymbol{D},\boldsymbol{B}) = \mathcal{K}(f,\hat{\boldsymbol{E}},\boldsymbol{B}) + \int \hat{\boldsymbol{P}}\cdot\hat{\boldsymbol{E}}\,d^3\boldsymbol{X} + \frac{1}{8\pi}\int\left(\hat{\boldsymbol{E}}\cdot\hat{\boldsymbol{E}} + \boldsymbol{B}\cdot\boldsymbol{B}\right)d^3\boldsymbol{X}, \tag{6}$$

where $\hat{\boldsymbol{E}} = \hat{\boldsymbol{E}}(f,\boldsymbol{D},\boldsymbol{B})$ is the electric field operator defined implicitly by the equation

$$\boldsymbol{D} = \hat{\boldsymbol{E}}(f,\boldsymbol{D},\boldsymbol{B}) - 4\pi\frac{\delta\mathcal{K}}{\delta\boldsymbol{E}}(f,\hat{\boldsymbol{E}}(f,\boldsymbol{D},\boldsymbol{B}),\boldsymbol{B}), \tag{7}$$

and $\hat{\boldsymbol{P}} = \hat{\boldsymbol{P}}(f,\boldsymbol{D},\boldsymbol{B})$ is the gyrocenter polarization operator given by

$$\hat{\boldsymbol{P}}(f,\boldsymbol{D},\boldsymbol{B}) = \frac{1}{4\pi}(\boldsymbol{D} - \hat{\boldsymbol{E}}(f,\boldsymbol{D},\boldsymbol{B})). \tag{8}$$

The noncanonical Poisson bracket is given by

$$[\mathcal{F},\mathcal{G}] = \sum_{s=1}^{N_s}\int \mathcal{B}_s^{\mathrm{gy}}\left(\mathbf{d}\frac{\delta\mathcal{F}}{\delta f_s} - 4\pi e_s\frac{\delta\mathcal{F}}{\delta\boldsymbol{D}}\cdot d\boldsymbol{X}, \mathbf{d}\frac{\delta\mathcal{G}}{\delta f_s} - 4\pi e_s\frac{\delta\mathcal{G}}{\delta\boldsymbol{D}}\cdot d\boldsymbol{X}\right) f_s + 4\pi c\int\left(\frac{\delta\mathcal{F}}{\delta\boldsymbol{D}}\cdot\nabla\times\frac{\delta\mathcal{G}}{\delta\boldsymbol{B}} - \frac{\delta\mathcal{G}}{\delta\boldsymbol{D}}\cdot\nabla\times\frac{\delta\mathcal{F}}{\delta\boldsymbol{B}}\right)d^3\boldsymbol{X}. \tag{9}$$

Here $\mathcal{B}_s^{\mathrm{gy}}$ is the gyrocenter Poisson tensor, which is defined as follows. If z^a is a coordinate system on the gyrocenter phase space and α,β are 1-forms on the same space, $\mathcal{B}_s^{\mathrm{gy}}(\alpha,\beta) = \alpha_a\beta_b\{z^a,z^b\}_s^{\mathrm{gy}}$, where $\{\cdot,\cdot\}_s^{\mathrm{gy}}$ is the gyrocenter Poisson bracket. Note that a Poisson bracket for electrostatic gyrokinetics was given in Ref. [12]. The complexity of that bracket should be contrasted with the relative simplicity of the bracket given here for electromagnetic gyrokinetics. This bracket, which has a form akin to that of [11], is to our knowledge the first demonstration of Hamiltonian structure for *any* electromagnetic gyrokinetic theory.

3. Origins and comparisons

We arrived at this electromagnetic gyrokinetic system by modifying the standard variational derivation of electromagnetic gyrokinetics [13–16,12]. In the standard approach, a gyrokinetic system Lagrangian is constructed by adding a gauge-dependent [17] net gyrocenter Lagrangian to a non-relativistic limit (known as the Darwin limit) of the free Maxwell field Lagrangian. Applying appropriate variations to the system Lagrangian then produces the standard equations of EMGT. Roughly speaking, adopting a gyrocenter Lagrangian instead of a particle Lagrangian amounts to dropping terms from the particle equations of motion. Likewise, adopting the Darwin approximation amounts to dropping terms from Maxwell's equations. We modified this approach by adding a manifestly gauge invariant net gyrocenter Lagrangian [18] to the full free Maxwell field Lagrangian to produce the system Lagrangian. Thus, in the modified approach, fewer terms are dropped from Maxwell's equations. While dropping these terms as in the standard approach would be justified (using the assumption of non-relativistic particles), doing so is not necessary. We therefore conclude that the GVM equations are no less accurate than standard EMGT.

4. Computational benefits

The usual argument for invoking the Darwin approximation in EMGT is that doing so eliminates light waves. This may seem to be an especially compelling argument from a computational point of view. After all, the presence of traveling waves with phase velocity c leads to a very restrictive CFL condition for explicit integration schemes. Therefore, avoiding the Darwin approximation as we have done may appear objectionable in a practical sense.

On the other hand, this numerical argument supporting the Darwin approximation is not as strong as it appears. As is evident from the form of the GVM equations given above, avoiding the Darwin approximation does not lead to Maxwell's equations, but Maxwell's equations in a polarized and magnetized medium. Therefore, the light waves supported by these equations do not travel at the speed of light in vacuum.

It is well known [19] that the dielectric constant resulting from gyrocenter polarization is large, which implies that the speed of light is much smaller than c in a gyrokinetic plasma (this is consistent with the notion of a so-called "gyrokinetic vacuum.") Using the long-wavelength limit of the gyrokinetic dielectric function, $\omega_{pi}^2/\omega_{ci}^2$, as a rough approximation, we find that light waves in the GVM equations propagate at the Alfvén speed. Thus, the CFL constraint imposed by light waves in the GVM equations is not nearly as strict as the usual argument might suggest.[1]

An even stronger case can be made for the computational viability of this new formulation of electromagnetic gyrokinetics. We first make the following simple observation. A familiar calculation shows that if $\nabla\cdot\boldsymbol{D} = 4\pi\rho_{\mathrm{gy}}$ and $\nabla\cdot\boldsymbol{B} = 0$ at $t = 0$, then these equations will also be satisfied for all subsequent times. This means that the evolution of the magnetic field and the auxiliary electric field is completely determined by the Ampère equation and the Faraday equation. Interestingly, it can be shown that this property arises as a direct consequence of employing a gauge-invariant gyrocenter Lagrangian; the quantity $\nabla\cdot\boldsymbol{D} - 4\pi\rho_{\mathrm{gy}}$ is the conserved quantity associated with gauge symmetry by Noether's theorem.

[1] Strictly speaking, it is only light waves that travel perpendicular to the magnetic field that experience a reduced propagation speed. Those that travel along the magnetic field lines may still travel near the speed of light in vacuum. However, the numerical grids appropriate for gyrokinetic simulations are significantly elongated along the field lines, which substantially reduces the parallel CFL condition.

Now suppose the Ampère and Faraday equations were used to advance $\boldsymbol{D}$ and $\boldsymbol{B}$ in time on a computer. Employing a simple explicit scheme, the following steps would have to be taken at each time step. (1) Using the constitutive relations, compute $\boldsymbol{E}$ and $\boldsymbol{H}$ from the known values of $\boldsymbol{D}$ and $\boldsymbol{B}$. (2) Compute $\nabla\times\boldsymbol{H}$ and $\nabla\times\boldsymbol{E}$. (3) Using a finite difference approximation for the partial time derivative, solve for the new $\boldsymbol{D}$ and $\boldsymbol{B}$.

Steps (2) and (3) clearly require only local operations, and so represent nearly embarrassingly parallel computations. Again invoking the long wavelength limit, step (1) can also be seen to be local. In this limit, there is a simple algebraic relationship between $\boldsymbol{D}$ and $\boldsymbol{E}$ (see Ref. [20], for example) that can be inverted analytically. Thus, the entire field solve step in an explicit time marching scheme for the GVM equations is nearly embarrassingly parallel. Such a field solve is preferable to the nonlocal Poisson-like solves necessary in conventional EMGT, especially when performing parallel simulations with very few particles per processing core.

5. Theoretical benefits

We will now turn from numerical benefits offered by the GVM equations in order to discuss their analytical benefits. First, we mention the system's conservative properties. An immediate consequence of the GVM Poisson bracket structure is conservation of the Hamiltonian functional (this follows from antisymmetry of the bracket.) It is also not difficult to show that there is a conserved momentum functional for each rotation or translation symmetry of the background magnetic field. Finally, there is a large family of conserved functionals given by the Poisson bracket's Casimirs. These are functionals C that Poisson commute with every other functional, i.e. $\forall\mathcal{F}$, $[C,\mathcal{F}]=0$. Systems of gyrokinetic equations (electromagnetic or electrostatic) with exact energy and momentum conservation laws can also be derived using the standard variational approach [21,13–15,18,12]. Indeed, this was the main motivation for developing the standard variational formulations of gyrokinetics. However, variational approaches do not readily produce the Casimir invariants (nor has it been shown that the usual variational formulations of EMGT possess Poisson brackets and Casimir invariants at all.)

Many of the GVM bracket's Casimirs are given as follows. Let

$$\Omega_s=-\frac{1}{3!}\omega_s^{\mathrm{gy}}\wedge\omega_s^{\mathrm{gy}}\wedge\omega_s^{\mathrm{gy}} \tag{10}$$

be the Liouville volume form defined by the gyrocenter symplectic form and introduce the gyrocenter distribution *function*, F_s, where

$$f_s=F_s\Omega_s, \tag{11}$$

then

$$C_h=\sum_{s=1}^{N_s}\int_{TQ}h_s(F_s)\,\Omega_s \tag{12}$$

is a Casimir for each function of a single real variable h_s. Moreover, any functional of $\nabla\cdot\boldsymbol{D}-4\pi\rho_{\mathrm{gy}}$ is a Casimir, which is one way of seeing that Eq. (1d) is satisfied in the Hamiltonian formulation of the GVM equations.

Another advantage the Poisson bracket formulation of the GVM equations provides, which a variational formulation does not, is immediate access to the theory of dynamically accessible variations [2] (see also [5,7]). Suppose we perturb a GVM equilibrium by switching on a small time-dependent term in the Hamiltonian, i.e. $\mathcal{H}\to\mathcal{H}+\delta\mathcal{H}_t$, where $\delta\mathcal{H}_t$ is a time-dependent functional that is non-zero only in a brief interval of time after $t=0$. Using the Poisson bracket, we can give an energy principle for assessing the stability of this perturbation in the limit where the kick caused by switching on $\delta\mathcal{H}_t$ is infinitesimal.

In this limit, and accounting for the fact that the perturbation is generated by altering the Hamiltonian, we find that the perturbed distribution function, auxiliary electric field, and magnetic field must have the form

$$\delta f_s=-L_{\xi_s}f_s \tag{13}$$

$$\delta\boldsymbol{D}=-4\pi\,\boldsymbol{J}(\xi,f)+4\pi c\,\nabla\times\boldsymbol{\beta} \tag{14}$$

$$\delta\boldsymbol{B}=-4\pi c\,\nabla\times\boldsymbol{\alpha}, \tag{15}$$

where $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ are arbitrary vector fields on configuration space, the phase space fluid displacement vector ξ_s is determined by Hamilton's equations,

$$\mathrm{i}_{\xi_s}\omega_s^{\mathrm{gy}}=\mathbf{d}\chi_s+4\pi e_s\boldsymbol{\alpha}\cdot d\boldsymbol{X}, \tag{16}$$

with χ_s an arbitrary function on gyrocenter phase space, and $\boldsymbol{J}(\xi,f)$ is the gyrocenter current density generated by fiducial gyrocenters with phase space velocity ξ_s and distribution f_s. Appealing to the general theory of dynamically accessible variations (see e.g. [5]), our perturbation will be stable if the *free energy functional* $\delta^2F(\boldsymbol{\alpha},\boldsymbol{\beta},\chi)$ is positive whenever δf_s, $\delta\boldsymbol{D}$, and $\delta\boldsymbol{B}$ are not each zero. The free energy functional is defined by

$$\delta^2F(\boldsymbol{\alpha},\boldsymbol{\beta},\chi)=\frac{1}{2}[[\mathcal{H},S],S], \tag{17}$$

where the functional $S=\sum_s\int\chi_s f_s+\int\boldsymbol{\alpha}\cdot\boldsymbol{D}\,d^3\boldsymbol{X}+\int\boldsymbol{\beta}\cdot\boldsymbol{B}\,d^3\boldsymbol{X}$. Physically, δ^2F is the second-order change in the energy functional $\mathcal{H}$ produced by our perturbation. In fact, δ^2F functions as the (conserved) Hamiltonian of the linearized GVM equations.

We find that δ^2F can be written in the form

$$\begin{aligned}\delta^2F=\sum_s\int\Bigg(&\frac{1}{2}\omega_s^{\mathrm{gy}}(V_s^{\mathrm{gy}},\xi_s)\delta f_s+\delta K_s\,\delta f_s\\&+\frac{e_s}{2c}\delta\boldsymbol{B}\cdot(V_s^{\mathrm{gy}})_{\boldsymbol{X}}\times(\xi_s)_{\boldsymbol{X}}f_s\Bigg)\\&+\frac{1}{8\pi}\int\Big(\delta\boldsymbol{D}\cdot\delta\boldsymbol{E}+\delta\boldsymbol{B}\cdot\delta\boldsymbol{H}\Big)d^3\boldsymbol{X}.\end{aligned} \tag{18}$$

Here $\boldsymbol{X}$ in a subscript denotes the $\boldsymbol{X}$-component of a velocity field on phase space. The variations δK_s, $\delta\boldsymbol{E}$, and $\delta\boldsymbol{H}$ are given by

$$\delta K_s=\frac{\delta K_s}{\delta\boldsymbol{E}}[\delta\boldsymbol{E}]+\frac{\delta K_s}{\delta\boldsymbol{B}}[\delta\boldsymbol{B}] \tag{19}$$

$$\delta\boldsymbol{E}=\varepsilon^{-1}[\delta\boldsymbol{D}]+\eta[\delta\boldsymbol{B}] \tag{20}$$

$$\delta\boldsymbol{H}=\eta^{\dagger}[\delta\boldsymbol{D}]+\mu^{-1}[\delta\boldsymbol{B}], \tag{21}$$

where the linear operators ε, μ, and η are given by (cf. [11])

$$\varepsilon=1-4\pi\frac{\delta^2\mathcal{K}}{\delta\boldsymbol{E}\delta\boldsymbol{E}} \tag{22}$$

$$\mu^{-1}=1+4\pi\frac{\delta^2\mathcal{K}}{\delta\boldsymbol{B}\delta\boldsymbol{B}}+(4\pi)^2\frac{\delta^2\mathcal{K}}{\delta\boldsymbol{E}\delta\boldsymbol{B}}\varepsilon^{-1}\frac{\delta^2\mathcal{K}}{\delta\boldsymbol{B}\delta\boldsymbol{E}} \tag{23}$$

$$\eta=4\pi\varepsilon^{-1}\frac{\delta^2\mathcal{K}}{\delta\boldsymbol{B}\delta\boldsymbol{E}}. \tag{24}$$

In principle, an energy principle for electrostatic gyrokinetics analogous to this one could be derived using the Poisson bracket given in Ref. [12]. However, the authors of that reference deemed the electrostatic gyrokinetic Poisson bracket too complicated to be practically useful, and so did not attempt deriving an expression for δ^2F.

We have used this expression for $\delta^2 F$ to prove that, in the long wavelength limit, the thermal equilibrium state in a uniform background magnetic field is stable. In this case, the gyrocenter kinetic energy is given by

$$K = \frac{1}{2}mv_\parallel^2 + \omega_c J - \frac{1}{2}mc^2\left(\frac{v_\parallel}{c}\frac{\boldsymbol{B}_\perp}{B_o} + \frac{\boldsymbol{E}\times\hat{b}}{B_o}\right)^2, \tag{25}$$

where J is the gyroaction, ω_c is the signed gyrofrequency, $\boldsymbol{B}_\perp = \boldsymbol{B} - \hat{b}\hat{b}\cdot\boldsymbol{B}$, and B_o is the magnitude of the background magnetic field. This expression agrees with that given by Krommes in Ref. [22] in the absence of magnetic fluctuations. The linear response functions $\varepsilon^{-1}, \mu^{-1}, \eta$ are therefore given by the constant matrices

$$\varepsilon = 1 + \frac{4\pi c^2}{v_A^2}(1 - \hat{b}\hat{b}) \tag{26}$$

$$\mu^{-1} = 1 - 4\pi\beta(1 - \hat{b}\hat{b}) \tag{27}$$

$$\eta = 0, \tag{28}$$

where $\beta = \sum_s \frac{m_s n_s \langle v_\parallel^2\rangle_s}{B_o^2}$ is the plasma β and $\langle\cdot\rangle_s$ denotes the velocity space average. Using these expressions and the assumption of thermal equilibrium, a straightforward, but tedious calculation leads to the following form for $\delta^2 F$,

$$\begin{aligned}\delta^2 F = {} & \sum_s \int \frac{1}{2T}\left(L_{\xi_s}H_{os} - T\frac{\delta B_\parallel}{B_o}\right)^2 f_s \\ & + \frac{1}{8\pi}\int \delta\boldsymbol{D}\cdot\varepsilon^{-1}\cdot\delta\boldsymbol{D}\, d^3\boldsymbol{X} \\ & + \frac{1}{8\pi}\int \delta\boldsymbol{B}_\perp\cdot\mu^{-1}\cdot\delta\boldsymbol{B}_\perp d^3\boldsymbol{X} \\ & + \frac{1}{8\pi}\int (1 - 4\pi nT/B_o^2)\delta B_\parallel^2\, d^3\boldsymbol{X},\end{aligned} \tag{29}$$

where $n = \sum_s n_s$ is the total gyrocenter number density. As long as $4\pi\beta$ and $4\pi nT/B_o^2$ are each less than 1, a condition that is generally satisfied, $\delta^2 F$ is manifestly non-negative, which implies linear stability.

6. Concluding remarks

The Hamiltonian formulation of the GVM system given in this Letter is completely determined by two key quantities, the gyrocenter kinetic energy K_s, and the guiding center symplectic form $\omega_s^{\rm gc}$. Suppressing species labels, the gyrocenter kinetic energy is given explicitly to second order in the amplitude of the fluctuating fields, ϵ_δ, by

$$\begin{aligned}K(E,B) = {} & \mathcal{H}^{\rm gc} - \epsilon_\delta\langle\ell\rangle + \epsilon_\delta^2\mathcal{B}^{\rm gy}(\langle\delta\Xi\rangle, \mathbf{d}\langle\ell\rangle) \\ & + \frac{1}{2}\epsilon_\delta^2\left\langle\mathcal{B}_s^{\rm gy}\left(L_R[\delta\tilde{\Xi} - \mathbf{d}I(\tilde{\ell})], [\delta\tilde{\Xi} - \mathbf{d}I(\tilde{\ell})]\right)\right\rangle,\end{aligned} \tag{30}$$

where R is the infinitesimal generator of gyrophase rotations times the local gyrofrequency, I is the inverse of the Lie derivative L_R, angle brackets denote gyroangle averaging, and $\tilde{Q} = Q - \langle Q\rangle$. In standard guiding center coordinates, $L_R = \omega_c\frac{\partial}{\partial\theta}$, where θ is the gyrophase, which means I amounts to an antiderivative in gyrophase. It can be shown that the second-order gyrocenter kinetic energy has the same general form as Eq. (129) in Ref. [16]. The relevant correspondences between our symbols and those of Ref. [16] are $\ell \leftrightarrow -K_1$, $\mathcal{B}^{\rm gyab} \leftrightarrow J_0^{ab}$, $\delta\Xi \leftrightarrow \Delta\Gamma$, and $L_R\delta\Xi \leftrightarrow L_R(\bar{\Gamma}_1 + \Gamma_1)$.

From this expression, it is clear that the gyrocenter kinetic energy is determined by the three quantities $\mathcal{H}^{\rm gc}$, ℓ, and $\delta\Xi$. $\mathcal{H}^{\rm gc}$ denotes the guiding center Hamiltonian truncated at some desired order in ρ/L. The function ℓ and the 1-form $\delta\Xi$ are defined in terms of any choice of the guiding center Lie generators as follows. Decompose the guiding center transformation $\tau_{\rm gc} : TQ \to TQ$ as $\tau_{\rm gc} = \tau_2\circ\tau_1$, where

$$\tau_1 = \exp(G_1) \tag{31}$$

$$\tau_2 = \cdots\circ\exp(G_3)\circ\exp(G_2) \equiv \exp(\bar{G}_2), \tag{32}$$

and the G_k are the guiding center Lie generators. The leading-order guiding center transformation, τ_1, must be handled carefully in gyrokinetics because the fluctuating fields are allowed to have short perpendicular wave lengths. The 1-form

$$\delta\Xi = -\frac{e}{c}(\exp(-L_{\bar{G}_2})\mathrm{i}_{G_1}U(L_{G_1}) + \mathrm{i}_{\bar{G}_2}U(L_{\bar{G}_2}))\boldsymbol{B}\cdot dS, \tag{33}$$

where the function $U(x) = e^{-x/2}\sinh(x/2)/(x/2)$, represents the perturbation to the guiding center Lagrange 1-form produced by the fluctuating electromagnetic fields. The function

$$\delta\mathcal{H} = e(\exp(-L_{\bar{G}_2})\mathrm{i}_{G_1}U(L_{G_1}) + \mathrm{i}_{\bar{G}_2}U(L_{\bar{G}_2}))\boldsymbol{E}\cdot dX \tag{34}$$

represents the perturbation to the guiding center kinetic energy caused by the same fields. The function

$$\ell = \delta\Xi(V_o^{\rm gy}) - \delta H, \tag{35}$$

where $V_o^{\rm gy}$ is the unperturbed gyrocenter phase space velocity.

The Hamiltonian structure of the GVM equations reproduces that of the Vlasov–Maxwell system [23–25] under the substitutions

$$K \to \frac{1}{2}m\boldsymbol{v}^2 \tag{36}$$

$$\omega^{\rm gc} \to m\,\mathbf{d}x^i\wedge\mathbf{d}v_i. \tag{37}$$

It is also interesting to compare $[\cdot,\cdot]$ to the bracket given in Ref. [11]. The only significant difference comes from the manner in which the inductive electric field is built into the kinetic equation.

Finally, we note two possible directions for future research. (1) It may be useful to identify a Poisson bracket for electromagnetic gyrokinetics in the Darwin approximation, i.e. standard EMGT. The gyrokinetic Vlasov–Darwin equations are sometimes also referred to as the gyrokinetic Vlasov–Poisson–Ampère equations [13]. A Hamiltonian formulation of the non-gyrokinetic Vlasov–Darwin equations has already been given in Ref. [26]. (2) It seems likely that the bracket and Hamiltonian given in this Letter will provide the Hamiltonian structure for the oscillation center Vlasov–Maxwell equations with appropriate substitutions for K and $\omega^{\rm gc}$. If this were true, then the benefits that our bracket brings to electromagnetic gyrokinetics could be extended to certain kinds of laser-plasma interactions.

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