Hierarchical structure of noncanonical Hamiltonian systems

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Abstract

Topological constraints play a key role in the self-organizing processes that create structures in macro systems. In fact, if all possible degrees of freedom are actualized on equal footing without constraint, the state of 'equipartition' may bear no specific structure. Fluid turbulence is a typical example—while turbulent mixing seems to increase entropy, a variety of sustained vortical structures can emerge. In Hamiltonian formalism, some topological constraints are represented by *Casimir invariants* (for example, helicities of a fluid or a plasma), and then, the effective phase space is reduced to the Casimir leaves. However, a general constraint is not necessarily *integrable*, which precludes the existence of an appropriate Casimir invariant; the circulation is an example of such an invariant. In this work, we formulate a systematic method to embed a Hamiltonian system in an extended phase space; we introduce *phantom fields* and extend the Poisson algebra. A phantom field defines a new Casimir invariant, a *cross helicity*, which represents a topological constraint that is not integrable in the original phase space. Changing the perspective, a singularity of the extended system may be viewed as a subsystem on which the phantom fields (though they are *actual fields*, when viewed from the extended system) vanish, i.e., the original system. This hierarchical relation of degenerate Poisson manifolds enables us to see the 'interior' of a singularity as a sub Poisson manifold. The theory can be applied to describe bifurcations and instabilities in a wide class of general Hamiltonian systems (Yoshida and Morrison 2014 *Fluid Dyn. Res.* **46** 031412).

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1. Introduction

Concepts of stability, transport, mixing, and turbulence are intimately related to invariant sets of governing dynamical systems. In particular, constants of motion foliate the phase space with invariant sets, reducing the dimension of a system, and in two-degree of freedom Hamiltonian systems even a single invariant set can provide a barrier to transport as occurs for magnetic field lines of confined plasma configurations (e.g. [1]). For the two-dimension Eulerian fluid equations, an infinite-dimensional dynamical system, the existence of the enstrophy is associated with the inverse cascade to large scale structures and explains features of turbulent spectra (e.g. [2]). Indeed the search for invariants and their consequences has been the subject of research for literally centuries.

Historically, the use of Lyapunov functions expanded upon the work of Lagrange and Dirichlet for determining stability in Hamiltonian systems. In the context of plasma and fluid dynamics many works appeared based on the idea of using constants of motion to find sufficient conditions for stability. For example, Woltjer [3] used his helicity together with energy to determine stability in magnetohydrodynamics, while Oberman and Kruskal [4] determined stability using a generalized entropy and Gardner [5] used his measure preserving rearrangement idea in the context of kinetic theory. Later, these ideas were used by Arnold [6] in fluid mechanics to determine stability of shear flows. It was not until the 1980s that the invariants in a variety of stability arguments, which were previously obtained by ad hoc means, were seen to be a consequence of the degenerate Poisson brackets of the general Hamiltonian structure of continuum matter models in the

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Eulerian variable description (e.g. [7, 8]). At this time constants of motion associated with Poisson bracket degeneracy were dubbed *Casimir invariants*, and all helicity like variables in fluids and plasmas are of this kind. Therefore, topological of fluid mechanics and plasma physics are in reality a consequence of the particular noncanonical Hamiltonian structure of models in these fields. It is the study of such topological invariants and their association to noncanonical Poisson brackets that is the subject of this paper.

The topological constraints of helicity-like Casimir invariants are often deemed to be the key element of self-organizing mechanisms that create *structures* in macro systems, the large scale vortices of the inverse cascade being an example. In fact, if all possible degrees of freedom are actualized on equal footing without constraint, one would expect the state of 'equipartition' to occur and there to be no specific structure. Fluid turbulence is a typical example—while turbulent mixing seems to increase entropy, a variety of sustained vortical structures can emerge. By suppression of microscopic degrees of freedom, a macro system can behave very differently from the simple sum of microscopic elements (or, the direct product of the microscopic phase spaces). In a more geometrical language, macro system dynamics may be identified as lying in a submanifold (often called a *leaf*) embedded in the total phase space spanning the microscopic degrees of freedom, where the reduction of the dimension is due to topological constraints.

If a topological constraint can be represented by a scalar invariant, we may formulate a variational principle augmenting the energy to find some but not all equilibrium points [8] (see [9] for examples pertinent to the magnetic helicity constraint) or the entropy to find statistical equilibria [10] (see [11] for an example pertinent to adiabatic invariants) using Lagrange multipliers. We may also formulate a perturbation theory about the topological constraint by including the scalar invariant into the Hamiltonian while supplying a conjugate variable [12]. As noted above, here we study topological constraints of Casimirs in the framework of Hamiltonian systems theory. Because of these invariants the state vector is constrained to move on a submanifold that is an intersection of the level-sets of the Casimir invariants (which we call Casimir leaves). Although Casimir invariants are first integrals of the Hamiltonian formulation, they are not pertinent to symmetries of a specific Hamiltonian; instead, Casimir invariants are attributes of the underlying Poisson bracket algebra, in particular, its degeneracy.

In general there are not enough Casimir invariants to obtain all equilibria from an augmented variational principle. This is because the nullity of the noncanonical Poisson bracket that determines Casimir invariants may have subtle singularities. The 'integrability' of the determining equation for Casimir invariants is the main mathematical issue we explore here. A general constraint is not necessarily integrable, which precludes the existence of an appropriate Casimir invariant; the circulation is an example of such an invariant [13]. In order to overcome the Casimir deficit, in this work we formulate a systematic method for embedding a Hamiltonian system into an extended phase space by introducing *phantom fields* that extend the Poisson algebra. A special case of this embedding was treated in our earlier work [13] where the extension of the Poisson algebra was of the semi-direct product form; such fields were called mock fields, a special case of phantom fields. A phantom field defines a new Casimir invariant, a *cross helicity*, which represents a topological constraint that was not integrable in the original phase space (see Morrison [14] for an early precursor to this idea).

We may reverse the point of view, whereby a singularity in the determining equation for Casimir invariants gives rise to a *singular Casimir invariant* [15, 16]. Viewed from the extended phase space, the original Poisson manifold is embedded as the singularity of the extended system. One can probe the singularity (which contains an infinite number of degrees of freedom) with the help of singular Casimir invariants.

In this paper, we elucidate the physical meaning of the singular Casimir invariants by invoking some explicit examples, while the rigorous mathematical framework will be described elsewhere. Specifically, in section 2 we review noncanonical Hamiltonian structure, describe the Casimir determining equations, and explain the Casimir deficit problem. In section 3 we describe our embedding, giving rise to a hierarchy of vortex-like Poisson structures, in which we can explore topological constraints and submanifold singularity. Section 4 contains as an example the nonlinear ion acoustic wave and associated soliton leaves. Finally, we conclude in section 5.

2. Noncanonical Hamiltonian systems (degenerate Poisson algebras)

2.1. Singularity of the Poisson operator

A noncanonical Hamiltonian system may be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}z = \mathcal{J}(z)\partial_z H(z), \tag{1}$$

where z is the state vector, a member of the phase space X (here a Hilbert space endowed with an inner product $\langle\,,\,\rangle$; in the later discussion, X will be a function space, and then we will denote the state vector by u), $H(z)$ is the Hamiltonian (here a real-valued functional on X), ∂_z is the gradient in X, and $\mathcal{J}$ is the Poisson operator. We allow $\mathcal{J}$ to be a function of z on X, and write it as $\mathcal{J}(z)$. We assume that the bilinear product

$$\{F, G\} = \langle \partial_z F(z), \mathcal{J}\partial_z G(z)\rangle$$

is antisymmetric and satisfies the Jacobi identity; then $\{,\}$ is a Poisson bracket. By a *noncanonical Hamiltonian system* we mean a Poisson algebra $C_{\{,\}}^{\infty}(X)$, together with a Hamiltonian function.

A *canonical* Hamiltonian system is endowed with a *symplectic* Poisson operator where

$$\mathcal{J}_c = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

However, our interest is in noncanonical systems endowed with Poisson operators $\mathcal{J}$ that are inhomogeneous and degenerate (i.e., $\mathrm{Ker}(\mathcal{J}(z))$ contains nonzero elements, and its dimension may change depending on the position in X). Since $\mathcal{J}$ is antisymmetric, $\mathrm{Ker}(\mathcal{J}(z)) = \mathrm{Coker}(\mathcal{J}(z))$, and hence, every orbit is topologically constrained on the orthogonal complement of $\mathrm{Ker}(\mathcal{J}(z))$.

A functional $C(z)$ such that $\{C, G\} = 0$ for all G is called a Casimir invariant (or an element of the *center* of the Poisson algebra). If $\mathrm{Ker}(\mathcal{J}) = \{0\}$, the case for a canonical Hamiltonian system, then there is only a trivial invariant $C=$ constant in the center. Evidently, a Casimir invariant $C(z)$ is a solution to the differential equation

$$\mathcal{J}(z)\partial_z C(z) = 0. \qquad (2)$$

When the phase space X has a finite dimension, say n, (2) is a first-order partial differential equation. If $\mathrm{Ker}(\mathcal{J}(z))$ has a constant dimension ν in an open set $X_\nu \subseteq X$, and $n - \nu$ is an even number, we can integrate (2) in X_ν to obtain ν independent solutions, i.e., $\mathrm{Ker}(\mathcal{J}(z))$ is locally spanned by the gradients of ν Casimir invariants (Lie–Darboux theorem). The intersection of all Casimir leaves (the level-sets of Casimir invariants) is the effective phase space, on which $\mathcal{J}(z)$ reduces to a symplectic Poisson operator.

However, the general (global) integrability of (2) is a mathematical challenge. In the next subsection, we see a simple example that epitomizes the problem.

2.2. Example of singularity and nonintegrable center

Let us consider a two-dimensional system $z = {}^t(x, y) \in X = \mathbb{R}^2$ with

$$\mathcal{J} = x\mathcal{J}_c.$$

The plane $x = 0$ is a singularity at which $\mathrm{Rank}(\mathcal{J})$ drops by two. The kernel of $\mathcal{J}$ consists of two eigenvectors:

$$\mathrm{Ker}(\mathcal{J}) = \{\alpha_x \boldsymbol{\nu}_x + \alpha_y \boldsymbol{\nu}_y; \ \alpha_x, \alpha_y \in \mathbb{R}\},$$
$$\begin{cases} \boldsymbol{\nu}_x = \delta(x)\boldsymbol{e}_x, \\ \boldsymbol{\nu}_y = \delta(x)\boldsymbol{e}_y, \end{cases} \qquad (3)$$

where $\delta(x)$ is the Dirac distribution. The first component $\boldsymbol{\nu}_x$ of the kernel is *integrable* and produces a singular Casimir invariant:

$$\boldsymbol{\nu}_x = \partial_z C_{\mathrm{ex}}(z), \quad C_{\mathrm{ex}}(z) = Y(x),$$

where $Y(x)$ is the Heaviside step function (with the gap filled). We call C_{ex} an *exterior Casimir invariant*. However, the second component $\boldsymbol{\nu}_y$ is not a closed one-form, thus it is not representable as a gradient of some scalar function, i.e., we cannot integrate $\boldsymbol{\nu}_y$ to produce a Casimir invariant. Characterization of this peculiar element of $\mathrm{Ker}(\mathcal{J})$ is one of the main issues of this work.

2.3. The Casimir deficit problem

The problem and our strategy of study are summarized as follows:

(i) To see the essence of the problem, let us assume that X is a cotangent bundle T^*M of a finite-dimensional manifold M. Suppose that $\boldsymbol{w}$ is a nonzero element of $\mathrm{Ker}(\mathcal{J})$. For $\boldsymbol{w} \in T^*M$ to be written as $\boldsymbol{w} = \partial_z C \ (=\mathrm{d}C)$ with a Casimir invariant (zero-form) C, $\boldsymbol{w}$ must be an exact one-from (or, for local integrability, it must be a closed one-form). This is a rather strong condition, for one can easily construct a counter example that violates the integrability, e.g., the example given in section 2.2. Our strategy for improving the integrability of (2) is to embed the Poisson manifold into higher-dimensional spaces; by adding extra components to $\boldsymbol{w}$, we may make it exact in a higher-dimension space.

(ii) The point where the rank of $\mathcal{J}(z)$ changes is a singularity of (2), from which singular (hyperfunction) solutions are generated [16]. However, the hyperfunction Casimir invariants fall short of spanning $\mathrm{Ker}(\mathcal{J})$; (see section 2.2). This 'Casimir deficit' problem suggests that the interior of the singularity cannot be well described by hyperfunctions (which are, in fact, the cohomology class of the sheaf of holomorphic functions). We may yet describe the interior of the singularity as a subsystem (or, a Poisson submanifold), on which a nonintegrable element of $\mathrm{Ker}(\mathcal{J})$ may be integrable to define a Casimir invariant of a 'reduced' Poisson operator.

(iii) Because models of fluids and plasmas are formulated on infinite-dimensional phase spaces, we have to develop an infinite-dimensional theory. For these systems (2) is a functional differential equation, and a singularity may cause an infinite-dimensional rank change. The reader is referred to Yoshida *et al* [15] for an example of a singular Casimir invariant generated by singularities in a function space.

3. Hierarchy of degenerate Poisson manifolds and phantom fields

Given our perspective of seeing a singularity of a Poisson operator as a subsystem in which a different dynamics may reside, we may say that singularities define a hierarchy of Hamiltonian systems. To this end we extend by introducing *phantom fields* that helps us to 'integrate' the kernel of the Poisson operator and connect the hierarchy of Hamiltonian systems. To understand what is meant by this we start with an illustrative example.

3.1. Example: a hierarchy of vortex dynamics systems

In table 1 we compare the Hamiltonian formalisms of well-known examples of two-dimensional vortex dynamics systems [7, 13, 17, 18]. We denote by $\omega = -\Delta \varphi$ the vorticity with Δ being the Laplacian and $\varphi \in H_0^1(\Omega) \cap H^2(\Omega)$ for the two-dimensional Eulerian velocity field $\boldsymbol{V} = {}^t(\partial_y \varphi, -\partial_x \varphi)$.

Table 1. Hierarchy of two-dimensional vortex systems. Here $[a, b] = \partial_y a \partial_x b - \partial_x a \partial_y b$.

System	State vector	Poisson operator
(I)	ω	$\mathcal{J}_{\mathrm{I}} = [\omega, \bigcirc]$
(II)	$\begin{pmatrix} \omega \\ \psi \end{pmatrix}$	$\mathcal{J}_{\mathrm{II}} = \begin{pmatrix} [\omega, \bigcirc] & [\psi, \bigcirc] \\ [\psi, \bigcirc] & 0 \end{pmatrix}$
(III)	$\begin{pmatrix} \omega \\ \psi \\ \check{\psi} \end{pmatrix}$	$\mathcal{J}_{\mathrm{III}} = \begin{pmatrix} [\omega, \bigcirc] & [\psi, \bigcirc] & [\check{\psi}, \bigcirc] \\ [\psi, \bigcirc] & 0 & 0 \\ [\check{\psi}, \bigcirc] & 0 & 0 \end{pmatrix}$

Given a Hamiltonian

$$H_{\mathrm{E}}(\omega) = -\frac{1}{2} \int \omega \left(\Delta^{-1} \omega \right) \mathrm{d}^2 x,$$

the system (I) is the vorticity equation for Eulerian flow

$$\partial_t \omega + \mathbf{V} \cdot \nabla \omega = 0.$$

The Casimir invariants of the system (I) are

$$C_0 = \int f(\omega) \mathrm{d}^2 x,$$

where f is an arbitrary smooth function.

If ψ is the Gauss potential of a magnetic field, i.e., $\mathbf{B} = {}^t(\partial_y \psi, -\partial_x \psi)$, and the Hamiltonian is

$$H_{\mathrm{RMHD}}(\omega, \psi) = -\frac{1}{2} \int [\omega \left(\Delta^{-1} \omega \right) + \psi \left(\Delta \psi \right)] \, \mathrm{d}^2 x,$$

the system (II) is the reduced MHD system

$$\partial_t \omega + \mathbf{V} \cdot \nabla \omega = [\Delta \psi, \psi],$$
$$\partial_t \psi + \mathbf{V} \cdot \nabla \psi = 0.$$

In the system (II), C_0 is no longer a constant of motion, it being replaced by

$$C_1 = \int \omega g(\psi) \mathrm{d}^2 x,$$
$$C_2 = \int f(\psi) \mathrm{d}^2 x.$$

However, if the Hamiltonian H is independent of ψ, the dynamics of ω is unaffected by ψ, while both ω and ψ obey the same evolution equation. Then, we call ψ a *phantom field* (previously called mock fields in [13]), since it can be chosen arbitrarily without changing the dynamics of the *actual field* ω. With the special choice $\psi = \omega$, both C_1 and C_2 evaluate as C_0, i.e., C_0 is subsumed by C_1, the so-called cross helicity [19], and C_2 as their special value (indeed, C_1 and C_2 carry more information about system (I); in section 3.2, we will see how the phantom field probes the topological constraints). The constancy of C_0 is, then, due to the symmetry $\partial_\psi H = 0$. A modification of the Hamiltonian to involve ψ destroys the constancy of C_0; the electromagnetic interaction would be a physical example of such a modification.

We can extend the phase space further to obtain a system (III) by adding another field $\check{\psi}$ that obeys the same evolution equation as ψ. In the reduced MHD system, $\check{\psi}$ is a phantom field, i.e., it does not have a direct physical meaning;

however, in the original RMHD context such a field physically correspond to the pressure in the high-beta MHD model [17]. The Casimir invariants of this further extended system are

$$C_2 = \int f(\psi) \mathrm{d}^2 x,$$
$$C_3 = \int h(\psi \check{\psi}) \mathrm{d}^2 x,$$
$$C_4 = \int \check{f}(\check{\psi}) \, \mathrm{d}^2 x.$$

For this system, C_1 is replaced with the new Casimir invariants C_3 and C_4.

3.2. Integrability of topological constraints

An interesting consequence of extending the system from (I) to (II) is found in the *integrability* of the $\mathrm{Ker}(\mathcal{J}_{\mathrm{I}})$. In (I),

$$\mathrm{Ker}(\mathcal{J}_{\mathrm{I}}(\omega)) = \{\psi; \; [\omega, \psi] = 0\},$$

which implies that ψ and ω are related, invoking a certain scalar $\zeta(x, y)$, by

$$\psi = \eta(\zeta), \quad \omega = \xi(\zeta). \tag{4}$$

As far as ξ is a monotonic function, we may write $\psi = \eta(\xi^{-1}(\omega))$, which we can integrate to obtain the Casimir invariant $C_0(\omega)$ with $f(\omega)$ such that $f'(\omega) = \eta(\xi^{-1}(\omega))$. Other elements of $\mathrm{Ker}(\mathcal{J}_{\mathrm{I}}(\omega))$ that are given by nonmonotonic ξ are not integrable and thus are unable to yield Casimir invariants. Yet, we can integrate such elements as $C_1(\omega, \psi)$ in the extended space of (II). In fact, every member of $\mathrm{Ker}(\mathcal{J}_{\mathrm{I}}(\omega))$ can be represented as $\partial_\omega C_1 = g(\psi)$ by choosing ψ in $\mathrm{Ker}(\mathcal{J}_{\mathrm{I}}(\omega))$.

Similarly, in the system (II), we encounter the deficit of the Casimir invariant $C_2 = \int f(\psi) \mathrm{d}^2 x$ in covering all elements ${}^t(0, \chi) \in \mathrm{Ker}(\mathcal{J}_{\mathrm{II}}(\omega, \psi))$ such that $[\psi, \chi] = 0$. With the help of a phantom field $\check{\psi}$, we can integrate every element of $\mathrm{Ker}(\mathcal{J}_{\mathrm{II}}(\omega, \psi))$ as C_3.

3.3. Submanifold of singularity

In the preceding section, we extended the phase space to improve the integrability of the topological constraints (kernel elements). Here we reverse the perspective, and see the 'singularity' as a submanifold of a larger system. We will introduce the new notions of *exterior Casimir invariants* and *interior Casimir invariants*; the former being a hyperfunction that identifies the singularity as a cohomology, while the latter turns out to be a Casimir invariant of the subsystem, which is invisible in the larger (extended) system.

Let us start with the system (II). Apparently, the rank of $\mathcal{J}_{\mathrm{II}}$ drops (by infinite dimension) at the submanifold $\psi = 0$. This singularity is a leaf of the singular Casimir invariant (which we call an exterior Casimir invariant)

$$C_{\mathrm{ex}} = Y(\|\psi\|^2),$$

where $\|\psi\|^2 = \int |\psi|^2 \, \mathrm{d}x^2$. Notice that C_{ex} is a special singular form of the Casimir invariant $C_2(\psi)$. This hyperfunction Casimir invariant has only one leaf $\psi = 0$ (i.e., the equation

$C_{\text{ex}} = c$ has a solution, iff $c \in (0, 1)$, and then the level-set is the singularity $\psi = 0$), in marked contrast to other regular Casimir invariants that densely foliate the phase space.

At the singularity $\psi = 0$, $C_{\text{in}} = \int f(\omega) \mathrm{d}x^2$ satisfies (denoting the state vector by $u = {}^t(\omega, \psi)$)

$$\mathcal{J}_{\text{II}} \partial_u C_{\text{in}}|_{\psi=0} = 0.$$

Notice that C_{in} is nothing but the Casimir invariant C_0 of the system (I), which, however, is not a Casimir invariant of the system (II).

From these observations, we draw the following conclusion: a singularity of a Hamiltonian system (where Rank($\mathcal{J}$) drops) defines a submanifold, which can be a sub Hamiltonian system.[3] Then, the Casimir invariants of the subsystem convert into the *interior Casimir invariants* of the larger system. Simultaneously, the submanifold is identified as a singular leaf of the *exterior Casimir invariant* which is the hyperfunction of the cross helicity describing the coupling of the submanifold and the phantom fields. The pair of the interior and exterior Casimir invariants constitute the *singular Casimir invariants*.

4. Another example: solitons on the vortex-free singular leaf

The ion acoustic wave is an example that may have a completely integrable structure described by the KdV (or KP) equation. Here we show that the KdV (or KP) hierarchy is an internal structure on a singularity embedded in a general three-dimensional parent model. The symmetry is connected to an internal Casimir element.

4.1. Hamiltonian structure of ion acoustic waves

Let us start with the general three-dimensional formulation:

$$\partial_t \rho = -\nabla \cdot (\boldsymbol{V}\rho), \tag{5}$$

$$\partial_t \boldsymbol{V} = -(\nabla \times \boldsymbol{V}) \times \boldsymbol{V} - \nabla(\phi + V^2/2), \tag{6}$$

$$-\Delta \phi = \rho - \mathrm{e}^{\phi}, \tag{7}$$

where ρ is the ion density, $\boldsymbol{V}$ is the fluid velocity, and ϕ is the electric potential (which replaces the enthalpy of usual fluid models). Instead of an equation of state, we use the nonlinear Poisson equation (7) to relate ρ and ϕ. We formally solve (7) for ϕ as

$$\phi = \Phi(\rho). \tag{8}$$

We may cast the system (5)–(7) into Hamiltonian form [7]. For the state vector $u = {}^t(\rho, \boldsymbol{V})$, we define

$$H = \int_{\Omega} \left[\frac{\rho V^2}{2} + \frac{|\nabla \Phi|^2}{2} + (\Phi - 1)\mathrm{e}^{\Phi} \right] \mathrm{d}^3 x, \tag{9}$$

$$\mathcal{J} = \begin{pmatrix} 0 & -\nabla \cdot \\ -\nabla & -\rho^{-1}(\nabla \times \boldsymbol{V}) \times \end{pmatrix}. \tag{10}$$

We assume that u is a smooth function on $\Omega = T^3$ (i.e., periodic in every direction). The phase space is $X = L^2(\Omega)$, and the domain on which the Poisson operator $\mathcal{J}$ is defined is a dense subset of X. By the definition, we obtain

$$\partial_u H = \begin{pmatrix} V^2/2 + \phi \\ \rho \boldsymbol{V} \end{pmatrix};$$

hence, the Hamiltonian system $\partial_t u = \mathcal{J}\partial_u H$ reproduces (5)–(7).

The Poisson operator (10) has Casimir elements

$$C_1 = \int_{\Omega} \rho \, \mathrm{d}^3 x, \tag{11}$$

$$C_2 = \frac{1}{2} \int_{\Omega} (\nabla \times \boldsymbol{V}) \cdot \boldsymbol{V} \, \mathrm{d}^3 x, \tag{12}$$

which, respectively, measure the total particle number and the helicity.

4.2. Structures on a leaf of a singular Casimir invariant

When $\nabla \times \boldsymbol{V} \equiv 0$ (i.e., $\boldsymbol{V}$ is irrotational everywhere in Ω)[4], the rank of $\mathcal{J}$ drops (by an infinite dimension), thus, the submanifold $\Gamma \subset X$ on which $\|\nabla \times \boldsymbol{V}\| = 0$ is a singularity of $\mathcal{J}$. By Kelvin's circulation theorem, an irrotational flow remains irrotational for all time, i.e., an irrotational flow resides on the leaf Γ. The determining Casimir invariant of Γ is a singular functional

$$C_{\text{ex}} = Y(\|\nabla \times \boldsymbol{V}\|^2). \tag{13}$$

The leaf of C_{ex} exists only at the singularity $\|\nabla \times \boldsymbol{V}\| = 0$, unlike those of regular Casimir elements which fill the phase space. Such a leaf Γ is a submanifold (in fact, a linear subspace) included in the leaf of $C_2 = 0$ (the zero helicity may occur even when the vorticity is finite, thus Γ is a subset of the leaf of $C_2 = 0$).

On the singular Casimir leaf Γ (the set of points satisfying $C_{\text{ex}} \in [0, 1)$), remarkable structures emerge, which are characterized by an interior Casimir element:

$$C_{\text{in}} = \int_{\Omega} \boldsymbol{B} \cdot \boldsymbol{V} \, \mathrm{d}^3 x, \tag{14}$$

where $\boldsymbol{B}$ is an arbitrary divergence-free vector. The simplest choice is a fixed unit vector. With $\boldsymbol{B} = \boldsymbol{e}_x$, $C_{\text{in}} = \int_{\Omega} V_x \, \mathrm{d}^3 x$ is the momentum in the direction of $\boldsymbol{e}_x$, which plays an important role in characterizing the ion acoustic soliton propagating along the x-axis.

Let us delineate how the *integrable structure* of ion acoustic waves is embedded in the phase space X of general three-dimensional vortical dynamics. We consider a case where the fields included in H are spatially one-dimensional, i.e., $\boldsymbol{V} = V(x, t)\boldsymbol{e}_x$ and $\rho = \rho(x, t)$. Evidently, $\nabla \times \boldsymbol{V} \equiv 0$, thus such orbits are constrained on Γ. Upon setting

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[3] In general, the singular submanifold may not bear a Hamiltonian structure. For example, if the singularity is a one-dimensional manifold, it cannot be a Hamiltonian system.

[4] Somewhat subtler singularity occurs when $\nabla \times \boldsymbol{V}$ vanishes locally in Ω; see [15].

$\nabla \times \boldsymbol{V} \equiv 0$ this bracket becomes

$$\{F, G\} = -\int (F_\rho \nabla \cdot G_V - G_\rho \nabla \cdot F_V) \mathrm{d}^3 x.$$

Reducing to 1D it becomes

$$\{F, G\} = -\int (F_\rho \partial_x G_V - G_\rho \partial_x F_V) \mathrm{d}x.$$

Finally, upon setting $\rho - \rho_0 = V = w$ gives

$$\{F, G\}_G = -\int (F_w \partial_x G_w - G_w \partial_x F_w) \mathrm{d}x = 2 \int (G_w \partial_x F_w) \mathrm{d}x,$$

which is the Gardner bracket for KdV. Evidently, $\int w \, \mathrm{d}^3 x$ is a Casimir invariant of $\{,\}_G$, which is nothing but the interior Casimir invariant C_{in} with $\boldsymbol{B} = \boldsymbol{e}_x$, and is the first invariant of the KdV hierarchy.

4.3. Introduction of a Phantom and Observation from an Extended Phase Space

As a more nontrivial extension of $\boldsymbol{B}$ defining the interior Casimir element (14), we may regard $\boldsymbol{B}$ as a phantom field (two-form) co-moving with the fluid, i.e., a vorticity-like field obeying

$$\partial_t \boldsymbol{B} = \nabla \times (\boldsymbol{V} \times \boldsymbol{B}). \tag{15}$$

By extending $\mathcal{J}$ as

$$\tilde{\mathcal{J}} = \begin{pmatrix} 0 & -\nabla \cdot & 0 \\ -\nabla & -\rho^{-1}(\nabla \times \boldsymbol{V})\times & -\rho^{-1}(\nabla \times \bigcirc) \times \boldsymbol{B} \\ 0 & \nabla \times (\bigcirc \times \rho^{-1}\boldsymbol{B}) & 0 \end{pmatrix}, \tag{16}$$

we may consider an extended dynamics of $\tilde{u} = {}^t(\rho, \boldsymbol{V}, \boldsymbol{B}) \in \tilde{X}$; as far as $\boldsymbol{B}$ is not included in the Hamiltonian, the dynamics of $u = {}^t(\rho, \boldsymbol{V})$ is unaffected by the phantom field $\boldsymbol{B}$.

We find that (16) is the Poisson operator of magnetohydrodynamics when $\boldsymbol{B}$ is the magnetic field [13, 20]. The Casimir invariants are the total mass $C_1 = \int \rho \mathrm{d}^3 x$, the cross helicity $C_X = \int \boldsymbol{B} \cdot \boldsymbol{V} \, \mathrm{d}^3 x$, and the magnetic helicity $C_M = \int \boldsymbol{B} \cdot \boldsymbol{A} \, \mathrm{d}^3 x$ (where $\boldsymbol{A}$ is the vector potential; $\nabla \times \boldsymbol{A} = \boldsymbol{B}$). On the singular leaf Γ, the cross helicity C_X reads as the interior Casimir invariant (14). Notice that $\nu = {}^t(0, \boldsymbol{B}) \in \mathrm{Ker}(\mathcal{J}|_\Gamma)$ ($\nabla \cdot \boldsymbol{B} = 0$) is not integrable in X (like ν_y of the example given in section 2.2), but it can be integrated as the cross helicity C_X in the extended phase space $\tilde{X}$ (as well as C_{in} on Γ).

5. Conclusion

By embedding a Poisson manifold of a noncanonical Hamiltonian system into a higher-dimensional phase space, we showed how to describe topological structures within a simple picture. In section 3, we invoked two-dimensional vortex systems to explain the systematic method for constructing a hierarchy of Poisson manifolds; we introduced *phantom fields* and extended the Poisson algebra so that the phantom fields are Lie-dragged by the flow vector. A

phantom field was seen to define a new Casimir invariant, a *cross helicity*, which represents topological constraints that are not integrable in the original smaller phase space; the *circulation* being such an example [13].

Representing a topological constraint by a scalar (Casimir invariant) brings about a number of advantages, because we may deal with it in a Hamiltonian (or action) formulation. Multiplying a Casimir invariant by a constant (Lagrange multiplier) and adding it to the Hamiltonian, we may define an *energy-Casimir function* and apply it to the study of stability as a Lyapunov function. Or, we can view a Casimir invariant as a momentum (or an action variable) and supply a conjugate coordinate (or an angle) variable to make a canonical pair. By including the new conjugate variable into the Hamiltonian, we can 'unfreeze' the Casimir invariant, and remove the topological constraint. For example, the tearing-mode instability [21, 22] emerges when the topological constraint on the helical magnetic flux (or the circulation of the magnetic field along a 'resonant' helical loop) is removed by a finite parallel electric field [12, 23].

Alternatively, we saw that a smaller system can be viewed as a singular Casimir leaf of a larger system, such a leaf being a singularity of the Poisson operator at which the rank changes. The corresponding Casimir invariant was shown to be a hyperfunction generated by the singularity. In section 4, we described the example of ion acoustic waves, where the singular Casimir leaf is the subspace of irrotational flows in which the solitons reside. The theory of singular Casimir invariants and phantom fields can be applied to a very large class of fluid and plasma dynamics (see [13]), and in future publications we will explore other examples.

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