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Hamiltonian magnetohydrodynamics: Lagrangian, Eulerian, and dynamically accessible stability—Examples with translation symmetry

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Because different constraints are imposed, stability conditions for dissipationless fluids and magnetofluids may take different forms when derived within the Lagrangian, Eulerian (energy-Casimir), or dynamically accessible frameworks. This is in particular the case when flows are present. These differences are explored explicitly by working out in detail two magnetohydrodynamic examples: convection against gravity in a stratified fluid and translationally invariant perturbations of a rotating magnetized plasma pinch. In this second example, we show in explicit form how to perform the time-dependent relabeling introduced in Andreussi *et al.* [Phys. Plasmas **20**, 092104 (2013)] that makes it possible to reformulate Eulerian equilibria with flows as Lagrangian equilibria in the relabeled variables. The procedures detailed in the present article provide a paradigm that can be applied to more general plasma configurations and in addition extended to more general plasma descriptions where dissipation is absent. *Published by AIP Publishing.*
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I. INTRODUCTION

The early plasma literature on magnetohydrodynamics (MHD) is specked with traces of a general underlying structure: the self-adjointness of the MHD force operator in terms of the displacement ξ of the original energy principle, the Woltjer invariants of helicity and cross helicity and their use in obtaining Beltrami states, and the representation of the magnetic and velocity fields in terms of "Clebsch" potentials being examples. All of these are symptoms of the fact that MHD is a Hamiltonian field theory, whether expressed in Lagrangian variables as shown by Newcomb[1] or in terms of Eulerian variables as shown by Morrison and Greene.[2] General ramifications of the Hamiltonian nature of MHD were elucidated in our series of publications,[3–6] while in the present work, we examine explicitly the stability of stratified plasma and of rotating pinch equilibria within each of the three Lagrangian, Eulerian, and dynamically accessible descriptions.

These particular two examples were chosen because they are at once tractable and significant. They display the difficulties one faces in ascertaining stability within the three approaches and provide a means to compare and contrast the stability results. The paper is designed to serve as a "how-to" guide for application of the three approaches, providing a framework for what one might expect, and delineating the sometimes subtle differences between the approaches. Here and in our previous papers, the scope was limited to MHD, but the same Hamiltonian structure exists for all important dissipation free plasma models, kinetic as well as fluid, and the story we tell for MHD applies to them as well. (See e.g., Ref. 7 for review.) Recently there has been great progress in understanding the Hamiltonian structure of extended MHD,[8–14] the effect of gyroviscosity,[15] and relativistic magnetofluid models.[16,17] In addition, recent work on hybrid kinetic-fluid models[18,19] and gyrokinetics[20,21] now also lie within the purview.

There are many concepts of stability of importance in plasma physics (see Sec. VI of Ref. 22 for a general discussion)—here we will only be concerned with what could be referred to as a formal Lyapunov stability, which has received wide attention in the fluid and plasma literature, both in the Hamiltonian and non-Hamiltonian contexts (see e.g., Refs. 23 and 24 and references therein for the latter). In the Hamiltonian context, the Lyapunov stability we consider provides at least a sufficient condition for stability, implied by the positive-definiteness of a quadratic form obtained from the second variation of an energy-like quantity. This kind of stability is stronger than spectral or eigenvalue stability: for finite-dimensional systems it implies nonlinear stability, i.e., stability to infinitesimal perturbations under the nonlinear evolution of the system. Note, nonlinear stability should not be confused with finite-amplitude stability that explores the extent of the basin of stability, a confusion that oft appears in the plasma literature. For infinite-dimensional systems like MHD, there are technical issues that need to be addressed in order to rigorously claim that formal Lyapunov stability implies nonlinear stability (see e.g., Ref. 25 for an example of a rigorous nonlinear stability analysis), but the formal Lyapunov stability of our interest is a most important ingredient, and it does imply linear stability.

A common practice in the plasma literature, employed e.g., by Chandresekhar,[26] is to manipulate the linear equations of motion in order to obtain a conserved quadratic form that implies stability. Although this procedure shows linear stability, it cannot be used to obtain nonlinear stability and may give a misleading answer. This is evidenced by the

Hamiltonian system, which when linearized has both of the two Hamiltonians for two linear oscillators

$$H_{\pm} = \omega_1(p_1^2 + q_1^2)/2 \pm \omega_2(p_2^2 + q_2^2)/2. \tag{1}$$

Both signs of (1) are conserved by the linear system, yet only one arises from the expansion of the nonlinear Hamiltonian of the system. Nonlinear Hamiltonians that give rise to linear Hamiltonians of the form of H_- can in fact be unstable (see Ref. 27 for an example), and are prototypes for systems with negative energy modes. This example shows why the formal Lyapunov stability, our subject, is stronger than spectral or eigenvalue stability. To reiterate, throughout by stability we will mean a formal Lyapunov stability.

The remainder of the paper is organized as follows: in Sec. II, we review the basic ideas of the three approaches, giving essential formulas so as to make the paper self-contained. Of note is the new material of Sec. II D that summarizes various comparisons between the approaches. This is followed by our convection example of Sec. III and our pinch example of Sec. IV. These sections are organized in parallel with Lagrangian, Eulerian (or so-called energy-Casimr), and dynamically accessible stability treated in order, followed by a subsection on comparison of the results. Finally, we conclude in Sec. V.

II. BASICS

In what follows we will consider the stability of MHD equilibria that are solutions to the following equations:

$$\rho_e \mathbf{v}_e \cdot \nabla \mathbf{v}_e = -\nabla p_e + \mathbf{J}_e \times \mathbf{B}_e + \rho_e \nabla \Phi_e, \tag{2}$$

$$\nabla \times (\mathbf{v}_e \times \mathbf{B}_e) = 0, \tag{3}$$

$$\nabla \cdot (\rho_e \mathbf{v}_e) = 0, \tag{4}$$

$$\mathbf{v}_e \cdot \nabla s_e = 0, \tag{5}$$

for the equilibrium velocity field $\mathbf{v}_e(\mathbf{x})$, magnetic field $\mathbf{B}_e(\mathbf{x})$, current density $4\pi\mathbf{J}_e = \nabla \times \mathbf{B}_e$, density field $\rho_e(\mathbf{x})$, and entropy/mass field $s_e(\mathbf{x})$. Here $\Phi(\mathbf{x}, t)$ represents an external gravitational potential. The pressure field is assumed to be determined by an internal energy function $U(\rho, s)$, where $p = \rho^2 \partial U/\partial \rho$ and the temperature is given by $T = \partial U/\partial s$. For the ideal gas $p = c\rho^\gamma \exp(\lambda s)$, with c, λ constants and $\rho U = p/(\gamma - 1)$. MHD has four thermodynamical variables ρ, s, p, and T. The assumption of local thermodynamic equilibrium implies that knowledge of two of these variables at all points $\mathbf{x}$ is sufficient to determine the other two, once the U appropriate to the fluid under consideration is specified.

For static equilibria with $\mathbf{v}_e \equiv 0$, the only equation to solve is

$$\nabla p_e = \mathbf{J}_e \times \mathbf{B}_e + \rho_e \nabla \Phi_e. \tag{6}$$

Equation (6) is one equation for several unknown quantities; consequently, there is freedom to choose profiles such as those for the current and pressure as we will see in our examples.

If we neglect the gravity force by removing $\nabla \Phi_e$, Eq. (6) leads as usual to the Grad-Shafranov equation, e.g., by noting that $\mathbf{B}_e \cdot \nabla p = 0$ implies pressure as a flux function. However, unlike the barotropic case where p only depends on ρ, in general this does not imply that ρ and s are flux functions, since their combination in $p(\rho, s)$ could cancel out their variation on a flux surface. Thus, as far as the static ideal MHD is concerned, because only p occurs in the equilibrium equation, density and temperature on a flux function can vary while the pressure is constant. The MHD static equilibrium equations give no information/constraints on this variation.

When gravity is included, Eq. (6) still is only one constraining equation for several unknown quantities. In Sec. III, we consider the stratified equilibria both with and without a magnetic field and we will investigate there the role played by entropy.

For stationary equilibria, the full set of Eqs. (2)–(5) must be solved. Because in general there are many possibilities, we will restrict our analysis to the rotating pinch example of Sec. IV, where we describe the equilibrium in detail.

A. Lagrangian formulae

The Hamiltonian for MHD in Lagrangian variables is

$$H[\mathbf{q}, \boldsymbol{\pi}] = \int d^3a \left[\frac{\pi_i \pi^i}{2\rho_0} + \rho_0 U(s_0, \rho_0/\mathcal{J}) + \frac{\partial q_i}{\partial a^k} \frac{\partial q^i}{\partial a^\ell} \frac{B_0^k B_0^\ell}{8\pi \mathcal{J}} + \rho_0 \Phi(q, t) \right], \tag{7}$$

where $(\mathbf{q}, \boldsymbol{\pi})$ are the conjugate fields with $\boldsymbol{q}(\mathbf{a}, t) = (q^1, q^2, q^3)$ denoting the position of a fluid element at time t labeled by $\mathbf{a} = (a^1, a^2, a^3)$ and $\boldsymbol{\pi}(\mathbf{a}, t)$ being its momentum density. In (7) the quantities s_0, ρ_0, and $\mathbf{B}_0$ are fluid element attributes that only depend on the label $\mathbf{a}$, and $\mathcal{J} := \det(\partial q^i/\partial a^j)$. Also, $A_j^i \partial q^j/\partial a^k = \mathcal{J} \delta_k^i$, where A_j^i denotes elements of the cofactor matrix of $\partial q/\partial a$. In a general coordinate system $\pi^i = g^{ij}(\mathbf{q})\, \pi_i$ where g^{ij} is the metric tensor. This Hamiltonian together with the canonical Poisson bracket

$$\{F, G\} = \int d^3a \left(\frac{\delta F}{\delta q^i} \frac{\delta G}{\delta \pi_i} - \frac{\delta G}{\delta q^i} \frac{\delta F}{\delta \pi_i} \right), \tag{8}$$

renders the equations of motion in the form

$$\dot{\pi}_i = \{\pi_i, H\} = -\frac{\delta H}{\delta q^i} \quad \text{and} \quad \dot{q}^i = \{q^i, H\} = \frac{\delta H}{\delta \pi_i}, \tag{9}$$

where "·" denotes time differentiation at constant label $\mathbf{a}$ and $\delta H/\delta q^i$ is the usual functional derivative. The results of these calculations can be found in Appendix A and further details can be found in Refs. 5 and 22.

In Ref. 5 we introduced the general time-dependent relabeling transformation $\mathbf{a} = \mathfrak{A}(\mathbf{b}, t)$, with the inverse $\mathbf{b} = \mathfrak{B}(\mathbf{a}, t)$, which gave rise to the new dynamical variables

$$\boldsymbol{\Pi}(\mathbf{b}, t) = \mathfrak{J}\boldsymbol{\pi}(\mathbf{a}, t), \quad \mathbf{Q}(\mathbf{b}, t) = \mathbf{q}(\mathbf{a}, t), \tag{10}$$

and the new Hamiltonian

$$\widetilde{H}[\mathbf{Q},\mathbf{\Pi}] = H - \int d^3b\ \mathbf{\Pi}\cdot(\mathbf{V}\cdot\nabla_b\mathbf{Q}), \tag{11}$$

$$= \int d^3b \left[\frac{\Pi_i\Pi^i}{2\widetilde{\rho}_0} - \Pi_i V^j\frac{\partial Q^i}{\partial b^j} + \widetilde{\rho}_0\,U\left(\widetilde{s}_0,\widetilde{\rho}_0/\widetilde{\mathcal{J}}\right) + \frac{\partial Q_i}{\partial b^k}\frac{\partial Q^i}{\partial b^\ell}\frac{\widetilde{B}_0^k\widetilde{B}_0^\ell}{8\pi\widetilde{\mathcal{J}}}\right],$$

$$= K + H_f + W, \tag{12}$$

where K is the kinetic energy, H_f is the fictitious term due to the relabeling, and W represents the sum of the internal and magnetic field energies. In the first equality of (12)

$$\mathbf{V}(\mathbf{b},t) := \dot{\mathfrak{B}}\circ\mathfrak{B}^{-1} = \dot{\mathfrak{B}}(\mathfrak{A}(\mathbf{b},t),t), \tag{13}$$

which is the label velocity, $\nabla_b := \partial/\partial\mathbf{b}$, and H is to be written in terms of the new variables. In the second equality, we used $d^3a = \mathfrak{J}\,d^3b$, with $\mathfrak{J} := \det(\partial a^i/\partial b^j)$, $\widetilde{\rho}_0 = \mathfrak{J}\rho_0$, $\widetilde{\mathcal{J}} := \det(\partial Q^i/\partial b^j) = \mathcal{J}\mathfrak{J}$, and $\widetilde{\rho}_0/\widetilde{\mathcal{J}} = \rho_0/\mathcal{J}$, which follows from mass conservation $\rho_0 d^3a = \widetilde{\rho}_0 d^3b$. The relabeled entropy is $\widetilde{s}_0(\mathbf{b},t) = s_0(\mathfrak{A}(\mathbf{b},t))$.

From (9) it is clear that extremization of Hamiltonians gives equilibrium equations. For the Hamiltonian $H[\mathbf{q},\boldsymbol{\pi}]$ of (7) this gives static equilibria, while for $\widetilde{H}[\mathbf{Q},\mathbf{\Pi}]$ of (12) one obtains stationary equilibria. This was the point of introducing the relabeling: it allows us to express stationary equilibria in terms of Lagrangian variables, which would ordinarily be time dependent, as time-independent orbits with the moving labels.

The equilibrium equations are

$$0 = \partial_t\mathbf{Q}_e = \frac{\mathbf{\Pi}_e}{\widetilde{\rho}_0} - \mathbf{V}_e\cdot\nabla_b\mathbf{Q}_e,$$
$$0 = \partial_t\mathbf{\Pi}_e = -\nabla_b\cdot(\mathbf{V}_e\otimes\mathbf{\Pi}_e) + \mathbf{F}_e, \tag{14}$$

where $\mathbf{F}_e$ comes from the W part of the Hamiltonian. From (14) the equilibrium equation follows:

$$\nabla_b\cdot(\widetilde{\rho}_0\,\mathbf{V}_e\mathbf{V}_e\cdot\nabla_b\mathbf{Q}_e) = \mathbf{F}_e. \tag{15}$$

Using $\mathbf{b} = \mathbf{Q}_e(\mathbf{b}) = \mathbf{q}_e(\mathfrak{A}_e(\mathbf{b},t),t) = \mathfrak{B}_e(\mathbf{a},t)$ and the definition of $\mathbf{V}$ of (13), $\mathbf{V}(\mathbf{b},t) = \dot{\mathfrak{B}}_e(\mathfrak{A}_e(\mathbf{b},t),t) = \mathbf{v}_e(\mathbf{b})$, where $\mathbf{v}_e(\mathbf{b})$ denotes an Eulerian equilibrium state, we obtain upon setting $\mathbf{b} = \mathbf{x}$ the usual stationary equilibrium equation

$$\nabla\cdot(\rho_e\mathbf{v}_e\mathbf{v}_e) = \mathbf{F}_e, \tag{16}$$

where $\rho_e(\mathbf{x})$ is the usual equilibrium density. It can be shown that $\mathbf{v}_e\cdot\nabla s_e = 0$, $\nabla\cdot(\rho_e\mathbf{v}_e) = 0$, and $\mathbf{v}_e\cdot\nabla\mathbf{B}_e - \mathbf{B}_e\cdot\nabla\mathbf{v}_e + \mathbf{B}_e\nabla\cdot\mathbf{v}_e = 0$, follow from the Lagrange to Euler map. Further details of this relabeling transformation are given in Ref. 5, while application to our rotating pinch example of Sec. IV is worked out in Appendix B.

For stability, we expand as follows:

$$\mathbf{Q} = \mathbf{Q}_e(\mathbf{b},t) + \boldsymbol{\eta}(\mathbf{b},t), \quad \mathbf{\Pi} = \mathbf{\Pi}_e(\mathbf{b},t) + \boldsymbol{\pi}_\eta(\mathbf{b},t), \tag{17}$$

and calculate the second variation of the Hamiltonian in terms of the relabeled canonically conjugate variables $(\boldsymbol{\eta},\boldsymbol{\pi}_\eta)$ giving

$$\delta^2 H_{\rm la}[Z_e;\boldsymbol{\eta},\boldsymbol{\pi}_\eta] = \frac{1}{2}\int d^3x\left[\frac{1}{\rho_e}|\boldsymbol{\pi}_\eta - \rho_e\mathbf{v}_e\cdot\nabla\boldsymbol{\eta}|^2 + \boldsymbol{\eta}\cdot\mathfrak{V}_e\cdot\boldsymbol{\eta}\right], \tag{18}$$

which depends on the time independent equilibrium quantities $Z_e = (\rho_e, s_e, \mathbf{v}_e, \mathbf{B}_e)$, i.e., the operator $\mathfrak{V}_e$ has no explicit time dependence. (Again, see in Refs. 5 and 22 for details.) The functional

$$\delta^2 W_{\rm la}[Z_e;\boldsymbol{\eta}] := \frac{1}{2}\int d^3x\,\boldsymbol{\eta}\cdot\mathfrak{V}_e\cdot\boldsymbol{\eta}$$
$$= \frac{1}{2}\int d^3x\left[\rho_e(\mathbf{v}_e\cdot\nabla\mathbf{v}_e)\cdot(\boldsymbol{\eta}\cdot\nabla\boldsymbol{\eta}) - \rho_e|\mathbf{v}_e\cdot\nabla\boldsymbol{\eta}|^2\right] + \delta^2 W[\boldsymbol{\eta}], \tag{19}$$

is identical to that obtained by Frieman and Rotenberg,[28] although obtained here in an alternative and more general manner.

The energy $\delta^2 W_{\rm la}$ can be transformed in the more familiar expression of Ref. 29

$$\delta^2 W_{\rm la}[Z_e;\boldsymbol{\eta}] = \frac{1}{2}\int d^3x\left[\rho_e\frac{\partial p_e}{\partial\rho_e}(\nabla\cdot\boldsymbol{\eta})^2 + (\nabla\cdot\boldsymbol{\eta})(\nabla p_e\cdot\boldsymbol{\eta}) + \frac{|\delta\mathbf{B}|^2}{4\pi} + \mathbf{J}_e\times\boldsymbol{\eta}\cdot\delta\mathbf{B} - \nabla\cdot(\rho_e\boldsymbol{\eta})(\boldsymbol{\eta}\cdot\nabla\Phi_e)\right], \tag{20}$$

where $4\pi\mathbf{J}_e = \nabla\times\mathbf{B}_e$ is the equilibrium current and $\delta\mathbf{B} := \nabla\times(\boldsymbol{\eta}\times\mathbf{B}_e)$.

For completeness, we record the first order Eulerian perturbations that are induced by the Lagrangian variation written in terms of the displacement $\boldsymbol{\eta}$

$$\delta\rho_{\rm la} = -\nabla\cdot(\rho_e\boldsymbol{\eta}), \tag{21}$$

$$\delta\mathbf{v}_{\rm la} = \boldsymbol{\pi}_\eta/\rho_e - \boldsymbol{\eta}\cdot\nabla v_e$$
$$= \partial\boldsymbol{\eta}/\partial t + \mathbf{v}_e\cdot\nabla\boldsymbol{\eta} - \boldsymbol{\eta}\cdot\nabla\mathbf{v}_e, \tag{22}$$

$$\delta s_{\rm la} = -\boldsymbol{\eta}\cdot\nabla s_e, \tag{23}$$

$$\delta\mathbf{B}_{\rm la} = -\nabla\times(\mathbf{B}_e\times\boldsymbol{\eta}), \tag{24}$$

where $\delta s_{\rm la}$ can be replaced by the pressure perturbation, $\delta p_{\rm la} = -\gamma p_e\nabla\cdot\boldsymbol{\eta} - \boldsymbol{\eta}\cdot\nabla p_e$, that is often used.

B. Eulerian formulae

The Hamiltonian for MHD in Eulerian variables is

$$H[Z] = \int d^3x\left[\frac{\rho}{2}|\mathbf{v}|^2 + \rho\,U(s,\rho) + \frac{|\mathbf{B}|^2}{8\pi} + \rho\Phi\right], \tag{25}$$

where $Z = (\rho, s, \mathbf{v}, \mathbf{B})$. When (25) is substituted into the noncanonical Poisson bracket $\{F,G\}_{\rm nc}$ of Ref. 2 one obtains the Eulerian equations of motion in the form $\partial Z/\partial t = \{Z,H\}_{\rm nc}$. Because the noncanonical Poisson bracket $\{F,G\}_{\rm nc}$ is degenerate, i.e., there exist functionals C such that $\{F,C\}_{\rm nc} = 0$ for all functionals F, Casimir invariants C exist and equilibria are given by extremization of the energy-Casimir functional $\mathfrak{F} = H + C$. For MHD with no symmetry, the Casimirs are

$$C_s = \int d^3x\, \rho\, \mathcal{S}(s), \tag{26}$$

and the magnetic and cross helicities

$$C_B = \int d^3x\, \mathbf{A}\cdot\mathbf{B}, \quad \text{and} \quad C_v = \int d^3x\, \mathbf{v}\cdot\mathbf{B}, \tag{27}$$

respectively. By manipulation of the MHD equations, the helicities were shown by Woltjer[30–33] to be invariants (C_v requiring the barotropic equation of state) and used by him to predict plasma states. Woltjer's ideas pertaining to magnetic helicity were adapted by Taylor[34,35] to describe the reversed field configurations. The invariant of (26) and Woltjer's helicities were shown to be Casimir invariants in Ref. 36. (See Refs. 37 and 38 for further discussion.)

An important point to note is that knowledge of the Casimirs determines this additional physics, but this knowledge must come from physics outside of the ideal model.

Special attention has been given to the equilibrium states obtained by extremizing the energy subject to the Woltjer invariants, perhaps because these are the states for which Casimirs are at hand. (See Refs. 39 and 40 for discussion of the Casimir deficit problem.) However, we will see in Sec. II C that *all* MHD equilibria are obtainable from the variational principles with directly constrained variations, the dynamically accessible variations, rather than using Lagrange multipliers and helicities, etc.

In the case were translational symmetry is assumed, all variables are assumed to be independent of a coordinate z with

$$\mathbf{B} = B_z\hat{\mathbf{z}} + \nabla\psi\times\hat{\mathbf{z}}, \tag{28}$$

$$\mathbf{M} = M_z\hat{\mathbf{z}} + \nabla\chi\times\hat{\mathbf{z}} + \nabla\Upsilon, \tag{29}$$

where χ, Υ, and ψ are "potentials," $\mathbf{M} = \rho\mathbf{v}$, $M_z = \rho v_z$, and $\hat{\mathbf{z}}$ is the unit vector in the symmetry direction. The Hamiltonian then becomes

$$H_{TS}[Z_s] = \int d^3x \left[\frac{M_z^2}{2\rho} + \frac{|\nabla\chi|^2}{2\rho} + \frac{|\nabla\Upsilon|^2}{2\rho} + \frac{[\Upsilon,\chi]}{\rho} + \frac{|\nabla\psi|^2}{8\pi} + \frac{B_z^2}{8\pi} + \rho U + \rho\Phi\right], \tag{30}$$

where $Z_s = (\rho, s, M_z, \chi, \Upsilon, \psi, B_z)$. With this symmetry assumption, the set of Casimir is expanded and is sufficient to obtain a variational principle for the equilibria considered here. However, because of this symmetry assumption, it is only possible to obtain the stability results restricted to perturbations consistent with this assumption.

In Refs. 3 and 4 the translationally symmetric noncanonical Poisson brackets were obtained for both neutral fluid and MHD dynamics. For the case of a neutral fluid, which we consider in Sec. III B for convection, the Poisson bracket for translationally symmetric flows was given in Ref. 3. This bracket with the Hamiltonian of (30), where the magnetic energy terms involving B_z and ψ are removed, gives the compressible Euler's equations for fluid motion. The translationally symmetric fluid Poisson bracket has the following Casimir invariants:

$$C_1 = \int d^3x\; \rho\, \mathcal{S}(s, v_z, [s, v_z]/\rho, \ldots), \tag{31}$$

$$\begin{aligned} C_2 &= \int d^3x\; (\nabla\mathcal{A}(s)\cdot\nabla\chi + [\Upsilon, \mathcal{A}(s)])/\rho \\ &= \int d^3x\; \mathcal{A}(s)\,\hat{\mathbf{z}}\cdot\nabla\times\mathbf{v}, \end{aligned} \tag{32}$$

where $[f,g] = \hat{\mathbf{z}}\cdot\nabla f\times\nabla g$. The second Casimir applies if v_z depends only on s, which will suit our purpose, i.e., the energy-Casimir variational principle $\delta\mathfrak{F} = 0$ will give our desired equilibria.

For the case of MHD it was shown in Refs. 3 and 4 that the following are the Casimir invariants with translational symmetry:

$$C_s = \int d^3x\, \rho\, \mathcal{J}(s, \psi, [s,\psi]/\rho, [[s,\psi]/\rho,\psi]/\rho, [s,[s,\psi]/\rho]/\rho, \ldots), \tag{33}$$

$$C_{B_z} = \int d^3x\, B_z\mathcal{H}(\psi), \tag{34}$$

$$C_{v_z} = \int d^3x\, \rho v_z\mathcal{G}(\psi), \tag{35}$$

and, if the entropy is assumed to be a flux function, i.e., $[\psi, s] = 0$, then (33) collapses to

$$C_s = \int d^3x\, \rho\, \mathcal{J}(\psi), \tag{36}$$

and there is the additional cross helicity Casimir

$$\begin{aligned} C_v &= \int d^3x \left(v_zB_z\,F'(\psi) + \frac{1}{\rho}\nabla F(\psi)\cdot\nabla\chi + \frac{[\Upsilon, F(\psi)]}{\rho}\right) \\ &= \int d^3x\, \mathbf{v}\cdot\mathbf{B}\, F'(\psi), \end{aligned} \tag{37}$$

where $\mathcal{S}$, $\mathcal{A}$, $\mathcal{J}$, $\mathcal{H}$, $\mathcal{G}$, and $\mathcal{F}$ are arbitrary functions of their arguments ($\mathcal{J}$ distinguished from the Jacobian with the same symbol by context) with prime denoting differentiation with respect to argument.

For both the neutral fluid and MHD equilibria that satisfy $\delta\mathfrak{F} = 0$, a sufficient condition for stability follows if the second variation $\delta^2\mathfrak{F}$ can be shown to be positive definite. For MHD it was shown in Refs. 5 and 6 that $\delta^2\mathfrak{F}$ could be put into the following diagonal form:

$$\delta^2\mathfrak{F}[Z_e;\delta Z_s] = \int d^3x\, [a_1|\delta\mathbf{S}|^2 + a_2(\delta Q)^2 + a_3(\delta R_z)^2 + a_4|\delta\mathbf{R}_\perp|^2 + a_5(\delta\psi)^2], \tag{38}$$

where the variations $(\delta\mathbf{S}, \delta\mathbf{R}, \delta Q, \delta\psi)$ are linear combinations of $(\delta\mathbf{v}, \delta\mathbf{B}, \delta\rho, \delta\psi)$. The coefficients a_i for $i = 1-5$ depend on space through the equilibrium and were first given explicitly in Ref. 5 (and corrected in Ref. 6). Note, for these calculations, the external potential Φ was omitted. (See Refs. 41 and 42 for related work.)

Upon extremizing over all variables except $\delta\psi$ and then back substituting the resulting algebraic relations, (38) becomes

$$\delta^2\mathfrak{F}[Z_e;\delta\psi] = \int d^3x\,[b_1|\nabla\delta\psi|^2 + b_2(\delta\psi)^2 + b_3|\mathbf{e}_\psi\times\nabla\delta\psi|^2], \tag{39}$$

where $\mathbf{e}_\psi = \nabla\psi/|\nabla\psi|$ and

$$b_1 = \frac{1-\mathcal{M}^2}{4\pi}\,\frac{c_s^2-\mathcal{M}^2(c_s^2+c_a^2)}{c_s^2-\mathcal{M}^2(c_s^2+c_a^2)+\dfrac{\mathcal{M}^4}{4\pi\rho}|\nabla\psi|^2}, \tag{40}$$

$$b_2 = \nabla\cdot\left[\frac{\partial}{\partial\psi}\left(\frac{\mathcal{M}^2}{4\pi}\right)\nabla\psi\right] - \frac{\partial^2}{\partial\psi^2}\left(p+\frac{B_z^2}{8\pi}+\frac{\mathcal{M}^2}{4\pi}|\nabla\psi|^2\right), \tag{41}$$

$$b_3 = \frac{1-\mathcal{M}^2}{4\pi} - b_1, \tag{42}$$

where the poloidal Alfvén-Mach number $\mathcal{M}^2 := 4\pi\mathcal{F}^2/\rho < 1$ has been assumed. Here

$$c_a^2 = B^2/(4\pi\rho) \quad \text{and} \quad c_s^2 = \partial p/\partial\rho, \tag{43}$$

are the Alfvén and the sound speed, respectively.

Thus, stability in this MHD context rests on whether or not (39) is definite, and for the neutral fluid equilibria we treat here, which include a gravity force, the same is true for the corresponding functional.

C. Dynamically accessible formulae

Extremizing the Hamiltonian of (25) without constraints gives trivial equilibria. With energy-Casimir, the constraints are incorporated essentially by using Lagrange multipliers. Dynamically accessible variations, as introduced in Ref. 43, restrict the variations to be those generated by the noncanonical Poisson bracket and in this way assures that all kinematical constraints are satisfied. The first order dynamically accessible variations, obtained directly from the noncanonical Poisson bracket of Ref. 2, are the following:

$$\delta\rho_{\rm da} = \nabla\cdot(\rho\mathbf{g}_1), \tag{44}$$

$$\delta\mathbf{v}_{\rm da} = \nabla g_3 + s\nabla g_2 + (\nabla\times\mathbf{v})\times\mathbf{g}_1 + \mathbf{B}\times(\nabla\times\mathbf{g}_4)/\rho, \tag{45}$$

$$\delta s_{\rm da} = \mathbf{g}_1\cdot\nabla s, \tag{46}$$

$$\delta\mathbf{B}_{\rm da} = \nabla\times(\mathbf{B}\times\mathbf{g}_1), \tag{47}$$

where the freedom of the variations is embodied in the arbitrariness of $\mathbf{g}_1$, g_2, g_3, and $\mathbf{g}_4$. Using these in the variation of the Eulerian Hamiltonian gives

$$\begin{aligned}\delta H_{\rm da} &= \int d^3x\,[(v^2/2+(\rho U)_\rho+\Phi)\delta\rho_{\rm da}+\rho\mathbf{v}\cdot\delta\mathbf{v}_{\rm da} \\ &\quad +\rho U_s\,\delta s_{\rm da}+\mathbf{B}\cdot\delta\mathbf{B}_{\rm da}/4\pi], \\ &= \int d^3x\,[\mathbf{g}_1\cdot(\rho\mathbf{v}\times(\nabla\times\mathbf{v})-\rho\nabla v^2/2 \\ &\quad -\rho\nabla h+\rho T\nabla s+\mathbf{J}\times\mathbf{B})-g_2\nabla\cdot(\rho s\mathbf{v}) \\ &\quad -g_3\nabla\cdot(\rho\mathbf{v})+\mathbf{g}_4\cdot\nabla\times(\mathbf{v}\times\mathbf{B})]=0,\end{aligned} \tag{48}$$

whence it is seen that the vanishing of the terms multiplying the independent quantities $\mathbf{g}_1$, g_2, g_3, and $\mathbf{g}_4$ gives precisely the Eulerian equilibrium equations (2)–(5).

Next, stability is assessed by expanding the Hamiltonian to second order using the dynamically accessible constraints to this order (see Refs. 5 and 22 for details), yielding the following expression:

$$\delta^2 H_{\rm da}[Z_e;\mathbf{g}] = \int d^3x\,\rho|\delta\mathbf{v}_{\rm da}-\mathbf{g}_1\cdot\nabla v+\mathbf{v}\cdot\nabla\mathbf{g}_1|^2 + \delta^2 W_{\rm la}[\mathbf{g}_1]\,. \tag{49}$$

If in (49) $\delta\mathbf{v}_{\rm da}$ were independent and arbitrary, we could use it to nullify the first term and then upon setting $\mathbf{g}_1 = -\boldsymbol{\eta}$, we would see that dynamically accessible stability is identical to Lagrangian stability. However, as we will see in Sec. II D, this is not always possible.

D. Comparison formulae

In our calculations of stability, we obtained the quadratic energy expressions of (18), (38), and (49), which can be written in terms of various Eulerian perturbation variables

$$\mathfrak{P} := \{\delta\rho, \delta\mathbf{v}, \delta s, \delta\mathbf{B}\}. \tag{50}$$

In the case of the Lagrangian energy of (18), the set of perturbations $\mathfrak{P}_{\rm la}$ as given by Eqs. (21)–(24) are constrained, while for the energy-Casimir expression of (38), the perturbations $\mathfrak{P}_{\rm ec}$ are entirely unconstrained provided they satisfy the translation symmetry we have assumed. Similarly the perturbations for the energy expression (49), $\mathfrak{P}_{\rm da}$ of (44)–(47), are constrained. In our previous work of Ref. 5, we showed that the three energy expressions are equivalent if restricted to the same perturbations, and we established the inclusions

$$\mathfrak{P}_{\rm da}\subset\mathfrak{P}_{\rm la}\subset\mathfrak{P}_{\rm ec},$$

which led to the conclusions

$$\mathfrak{stab}_{ec}\Rightarrow\mathfrak{stab}_{\rm la}\Rightarrow\mathfrak{stab}_{\rm da},$$

viz., dynamically accessible stability is the most limited because its perturbations are the most constrained, while energy-Casimir stability is the most general, when it exists, for its perturbations are not constrained at all. The choice between the three approaches should be based on the physics of the situation, which determines the relevant constraints that need to be satisfied by the perturbations. Our goal is to explore further the differences between these kinds of stability by exploring, in particular, the differences between Lagrangian and dynamically accessible perturbations.

From (49), it is clear that if $\delta\mathbf{v}_{\mathrm{da}}$ is arbitrary, independently of $\mathbf{g}_1$, then the first term of this expression can be made to vanish. This would reduce $\delta^2 H_{\mathrm{da}}$ to the energy expression obtained for Lagrangian stability, making the two kinds of stability equivalent. Given that there are five components of g_2, g_3 and $\mathbf{g}_4$, in addition to $\mathbf{g}_1$, one might think that this is always possible. However, as pointed out in Ref. 5, this is not always possible and whether or not it depends on the state or equilibrium under consideration. We continue this discussion here.

Consider first a static equilibrium state that has an entropy as a flux function and no equilibrium flow. Thus, for this case, the cross helicity C_v of (27) vanishes. For a dynamically accessible perturbation

$$\delta C_v = \int d^3x\, \delta\mathbf{v}_{\mathrm{da}} \cdot \mathbf{B}_e = \int d^3x\, (\nabla g_3 + s_e \nabla g_2) \cdot \mathbf{B}_e$$
$$= -\int d^3x\, g_2\, \mathbf{B}_e \cdot \nabla s_e = 0, \tag{51}$$

where the last equality assumes g_3 is single-valued and the vanishing of surface terms, as well as s_e being a flux function. The fact that $\delta C_v = 0$ for this case is not a surprise since it is a Casimir, but we do see clearly that if s were not a flux function, then a perturbation $\delta\mathbf{v}_{\mathrm{da}}$ could indeed create cross helicity. Because of the term $\partial\boldsymbol{\eta}/\partial t$ of (22), which can be chosen arbitrarily, it is clear that $\delta\mathbf{v}_{\mathrm{la}}$ can create cross helicity for any equilibrium state, supplying clear evidence that $\delta\mathbf{v}_{\mathrm{da}}$ is not completely general.

Although $\delta\mathbf{v}_{\mathrm{da}}$ is not completely general, it was noted in Ref. 22 that for static equilibria, the first term of (49) becomes

$$\int d^3x\, \rho |\delta\mathbf{v}_{\mathrm{da}}|^2, \tag{52}$$

and this can be made to vanish independent of $\mathbf{g}_1$ by choosing $g_2 = g_3 = 0$ and $\mathbf{g}_4 = 0$. Thus, for static equilibria, the Lagrangian and dynamically accessible approaches must give the same necessary and sufficient conditions for stability, i.e.

$$\mathfrak{stab}_{\mathrm{la}} \iff \mathfrak{stab}_{\mathrm{da}},$$

As another example, consider the variation of the circulation integral $\Gamma = \oint_c \mathbf{v}\cdot d\mathbf{x}$ on a fixed closed contour c for an equilibrium with $\mathbf{v}_e \equiv 0$ and $\mathbf{B}_e \neq 0$. Clearly $\delta\mathbf{v}_{\mathrm{la}}$ can generate any amount of circulation. However, for a dynamically accessible variation

$$\delta\Gamma = \oint_c \delta\mathbf{v}_{\mathrm{da}} \cdot \mathbf{dx}$$
$$= \oint_c s_e \nabla g_2 \cdot \mathbf{dx} + \oint_c (\nabla \times \mathbf{g}_4)\cdot(\mathbf{dx}\times\mathbf{B}_e)/\rho_e, \tag{53}$$

and we can draw two conclusions: in the case where c is a closed magnetic field line $\mathbf{dx} \parallel \mathbf{B}$ and $\delta\Gamma$ becomes

$$\delta\Gamma_B = \oint_c s_e \nabla g_2 \cdot \mathbf{dx} = \oint_c (\nabla(s_e g_2) - g_2 \nabla s_e)\cdot \mathbf{dx}$$
$$= -\oint_c g_2 \nabla s_e \cdot \mathbf{dx}, \tag{54}$$

whence we see clearly that if ∇s_e is everywhere parallel to $\mathbf{B}_e$, then $\delta\Gamma_B = 0$ and otherwise this is not generally true. Alternatively, suppose the contour c lies within a level set of s_e, for which it need not be true that $\mathbf{B}_e\cdot\nabla s_e = 0$ along c. For this case

$$\delta\Gamma_s = \oint_c (\nabla\times\mathbf{g}_4)\cdot(\mathbf{dx}\times\mathbf{B}_e)/\rho_e, \tag{55}$$

which in general does not vanish. If a magnetic field line were to lie within a surface of constant s_e, then in the general case, $\mathbf{B}_e\cdot\nabla s_e = 0$ otherwise surfaces of constant s_e would be highly irregular, i.e., if $\mathbf{B}_e\cdot\nabla s_e \neq 0$, then $\mathbf{B}_e$ cannot lie within a level set of s_e.

We point out that similar arguments can be supplied for cases where $\mathbf{v}_e \neq 0$, e.g., variation of the fluid helicity $\delta C_\omega = 2\int d^3x\, \boldsymbol{\omega}\cdot\delta\mathbf{v}_{\mathrm{da}}$ for an equilibrium with $\mathbf{B}_e \equiv 0$ becomes

$$\delta C_\omega = 2\int d^3x\, \boldsymbol{\omega}_e \cdot (s_e \nabla g_2 + \boldsymbol{\omega}_e \times \mathbf{g}_1)$$
$$= 2\int d^3x\, \boldsymbol{\omega}_e\cdot\nabla g_2\, s_e = -2\int d^3x\, g_2\, \boldsymbol{\omega}_e\cdot\nabla s_e, \tag{56}$$

which vanishes if $\boldsymbol{\omega}_e$ is perpendicular to ∇s_e or if the entropy is everywhere constant.

In summary, the general conclusion is that $\delta\mathbf{v}_{\mathrm{da}}$, unlike $\delta\mathbf{v}_{\mathrm{la}}$, is not completely arbitrary, and the degree of arbitrariness depends on the equilibrium. We also point out that although we are here interested in perturbations away from equilibrium states, for the purpose of assessing stability, the conditions we have described apply to perturbations away from any state, equilibrium or not.

Now we turn to our examples. For the remainder of this paper, we drop the subscript "e" on equilibrium quantities, so as to avoid clutter.

III. CONVECTION

For this first example, we consider the thermal convection in static equilibria, both with and without a magnetic field. This example has been well studied by various approaches, e.g., heuristic arguments that mix Lagrangian and Eulerian ideas were given in Ref. 44 for the neutral fluid. Here our analysis will be done separately in purely Lagrangian and purely Eulerian terms, and it will illustrate the role played by entropy in determining stability.

We suppose the equilibrium has stratification in the $\hat{\mathbf{y}}$-direction due to gravity, i.e., $\Phi = gy$, with ρ and s dependent only on y. Thus the only equation to be solved for the neutral fluid is

$$\frac{dp}{dy} = -\rho\frac{d\Phi}{dy} = -\rho g. \tag{57}$$

If a magnetic field of the form $\mathbf{B} = B(y)\hat{\mathbf{x}}$ is supposed, then the equilibrium equation is the following:

$$\frac{dp}{dy} = -\frac{dB}{dy}\frac{B}{4\pi} - \rho\frac{d\Phi}{dy} = JB - \rho g. \tag{58}$$

For barotropic fluids, s is constant everywhere and is eliminated from the theory, i.e., $U(\rho)$ alone. Thus, (57) (together with $U(\rho)$) determines completely the thermodynamics at all points y by integrating

$$\frac{p_\rho}{\rho}\frac{d\rho}{dy} = -g, \tag{59}$$

giving $\rho(y)$ and consequently $p(y)$. For this special case, no further information is required. However, in the general case where $p(\rho, s)$, (57) is not sufficient and one needs to know more about the fluid, since now we have

$$\frac{p_\rho}{\rho}\frac{d\rho}{dy} + \frac{p_s}{\rho}\frac{ds}{dy} = -g, \tag{60}$$

which is insufficient because we have only one equation for the two unknown quantities ρ and s. Thus, the knowledge of additional physics is required, which could come from boundary or initial conditions, solution of some heat or transport equation with constitutive relations, etc.

Next consider the case of MHD where

$$\frac{dp}{dy} = p_\rho\frac{d\rho}{dy} + p_s\frac{ds}{dy} = JB - \rho g. \tag{61}$$

If gravity is absent, MHD differs from that of the stratified fluid because only the pressure enters and the thermodynamics of ρ and s do not explicitly enter the equilibrium equation. We will consider the case where gravity is present.

Thus, in general, equilibria depend on two kinds of conditions: force balance, as given in our cases of interest by (57) or (58) and thermodynamics. For latter convenience we record here several thermodynamic relations

$$p = \rho^2 U_\rho \quad \text{and} \quad T = U_s, \tag{62}$$

$$c_s^2 = \left.\frac{\partial p}{\partial \rho}\right|_s = \left(\rho^2 U_\rho\right)_\rho = \rho(\rho U)_{\rho\rho}, \tag{63}$$

$$\left.\frac{\partial p}{\partial s}\right|_\rho = -\left.\frac{\partial \rho}{\partial s}\right|_p c_s^2 = \rho^2 U_{\rho s}, \tag{64}$$

where, without confusion, we use subscripts on U to denote partial differentiation with the other thermodynamic variable held constant and the subscript of c_s denotes "sound."

The coefficient of thermal expansion, α, is given by

$$\alpha = -\frac{1}{\rho}\left.\frac{\partial \rho}{\partial T}\right|_p, \tag{65}$$

and for typical fluids

$$\left.\frac{\partial p}{\partial s}\right|_\rho = \frac{\alpha}{\rho} > 0 \quad \text{and} \quad \left.\frac{\partial \rho}{\partial s}\right|_p < 0. \tag{66}$$

If the pressure is given by $p = c\rho^\gamma \exp(\lambda s)$, then $c_s^2 = \gamma p/\rho$, as it is often written.

A. Lagrangian convection

1. Lagrangian convection equilibria

From (9) Lagrangian equilibria must satisfy

$$\dot{\pi}_i = -\frac{\delta H}{\delta q^i} = 0 \quad \text{and} \quad \dot{q}^i = \frac{\delta H}{\delta \pi_i} = 0, \tag{67}$$

whence if follows from (7) that $\pi_i \equiv 0$ and

$$0 = \dot{\pi}_i = -A_i^j \frac{\partial}{\partial a^j}\left(\frac{\rho_0^2}{\mathcal{J}^2}U_\rho + \frac{1}{2\mathcal{J}^2}\frac{\partial q^k}{\partial a^l}\frac{\partial q_k}{\partial a^m}B_0^l B_0^m\right) + B_0^j\frac{\partial}{\partial a^j}\left(\frac{1}{\mathcal{J}}\frac{\partial q_i}{\partial a^l}B_0^l\right) - \rho_0\frac{\partial \Phi}{\partial q^i}, \tag{68}$$

which is the Lagrangian variable form of the static Eulerian equilibrium equation (6). (See e.g., Refs. 1, 22, and 37 for further details.) Because we are investigating equilibria that only depend on the variable y and have magnetic fields of the form $\mathbf{B} = B(y)\hat{\mathbf{x}}$, we only consider the $\hat{\mathbf{y}}$-component of (68), which is the Lagrangian variable form of the static Eulerian equilibrium equation (58).

2. Lagrangian convection stability

The second variation of the energy about this equilibrium is the usual expression given in Ref. 29. For static equilibria, this is obtained by setting $\boldsymbol{\eta} \equiv \xi$ in (20), and we know that the stability of such configurations is determined by this second variation of the potential energy. We will manipulate the energy expressions to facilitate comparison with results obtained in Sec. III B. Cases with and without $B = 0$ are considered.

Case $B = 0$:

By exploiting the equilibrium equation, we obtain

$$\delta^2 W_{\rm la} = \frac{1}{2}\int d^3x \left[\frac{1}{\rho}\left.\frac{\partial p}{\partial \rho}\right|_s \left[(\rho\nabla\cdot\xi)^2 + 2(\nabla\cdot\xi)(\nabla\rho\cdot\xi) + (\nabla\rho\cdot\xi)^2\right] + \frac{1}{\rho}\left.\frac{\partial p}{\partial s}\right|_\rho [\rho\nabla\cdot\xi + \nabla\cdot(\rho\xi)](\nabla s\cdot\xi)\right].$$

In conventional "δW" stability analyses, one would consider conditions for positivity of the above as a quadratic expression in terms of ξ. However, for our present purposes, we rewrite it in terms of

$$\delta\rho_{\rm la} = -\nabla\cdot(\rho\xi) \quad \delta s_{\rm la} = -\nabla s\cdot\xi, \tag{69}$$

which with

$$\frac{1}{\rho}\left.\frac{\partial p}{\partial s}\right|_\rho (\nabla\rho\cdot\xi)(\nabla s\cdot\xi) = \frac{1}{\rho}\left.\frac{\partial p}{\partial s}\right|_\rho \frac{(\nabla\rho\cdot\xi)}{(\nabla s\cdot\xi)}(\delta s_{\rm la})^2,$$

yields

$$\delta^2 W_{\rm la} = \frac{1}{2}\int d^3x \left[\frac{1}{\rho}\left.\frac{\partial p}{\partial \rho}\right|_s (\delta\rho_{\rm la})^2 + \frac{2}{\rho}\left.\frac{\partial p}{\partial s}\right|_\rho \delta\rho_{\rm la}\delta s_{\rm la} - \frac{1}{\rho}\left.\frac{\partial p}{\partial s}\right|_\rho \frac{(\nabla\rho\cdot\xi)}{(\nabla s\cdot\xi)}(\delta s_{\rm la})^2\right]. \tag{70}$$

Now, using (64), we can rearrange this equation as

$$\delta^2 W_{\rm la} = \frac{1}{2}\int d^3x \frac{c_s^2}{\rho}\left[\left(\delta\rho_{\rm la} - \left.\frac{\partial \rho}{\partial s}\right|_p \delta s_{\rm la}\right)^2 + \left.\frac{\partial \rho}{\partial s}\right|_p \left(\frac{(\nabla\rho\cdot\xi)}{(\nabla s\cdot\xi)} - \left.\frac{\partial \rho}{\partial s}\right|_p\right)(\delta s_{\rm la})^2\right]. \tag{71}$$

We will see that (71) is of the same form as that of (98) of Sec. III B, obtained via the energy-Casimir functional, yet here the perturbations $\delta\rho$ and δs are both constrained to depend on $\boldsymbol{\xi}$ according to (69).

Examination of (71) reveals that positivity of the second term is sufficient for positivity of $\delta^2 W_{\rm la}$, viz.

$$\frac{\nabla\rho\cdot\boldsymbol{\xi}}{\nabla s\cdot\boldsymbol{\xi}} < \frac{\partial\rho}{\partial s}\Big|_p. \tag{72}$$

Given that the equilibrium only depends on the variable y, in which the systems are stratified, (72) gives the following sufficient condition for stability

$$\frac{d\rho/dy}{ds/dy} < \frac{\partial\rho}{\partial s}\Big|_p < 0. \tag{73}$$

If the equilibrium is stably stratified, i.e., $d\rho/dy < 0$, then ds/dy must be positive, and we would have a threshold involving the density and entropy scale lengths.

However, let us proceed further. Define

$$\Delta = \frac{\partial\rho}{\partial s}\Big|_p - \frac{d\rho/dy}{ds/dy} = \frac{\partial\rho}{\partial s}\Big|_p - \frac{d\rho}{ds}, \tag{74}$$

where in the second term of the second equality, we have replaced the coordinate y by s, which is possible if ds/dy does not vanish. Observe in the definition of Δ of (74), this second term depends on the equilibrium profiles, while the first term is of a thermodynamic nature. So far, the sufficient condition for stability $\Delta > 0$ does not account for the fact that $d\rho/dy$ and ds/dy are not independent but are related through the equilibrium equation (57). To address this, we first rewrite the expression for Δ using

$$\frac{dp}{dy} = \frac{\partial p}{\partial s}\Big|_\rho \frac{ds}{dy} + \frac{\partial p}{\partial\rho}\Big|_s \frac{d\rho}{dy} = -c_s^2\frac{\partial\rho}{\partial s}\Big|_p \frac{ds}{dy} + c_s^2\frac{d\rho}{dy}, \tag{75}$$

resulting in

$$\Delta = -\frac{1}{c_s^2}\frac{dp/dy}{ds/dy} = -\frac{1}{c_s^2}\frac{dp}{ds}, \tag{76}$$

where use has been made of (63) and (64). Now inserting (57) into (76) yields for the $B=0$ case, the following condition:

$$\Delta = \frac{1}{c_s^2}\frac{\rho g}{ds/dy} > 0, \tag{77}$$

and because $\rho g > 0$, we obtain the compact sufficient condition for stability

$$\frac{ds}{dy} > 0. \tag{78}$$

We will see that an identical condition is obtained in the Eulerian energy-Casimir context (see Eq. (101)).

Now, given that $ds/dy > 0$ we can use (75) to obtain a condition on $d\rho/dy$

$$c_s^2\frac{d\rho}{dy} + \rho g = c_s^2\frac{\partial\rho}{\partial s}\Big|_p \frac{ds}{dy} < 0,$$

which implies

$$\frac{d\rho}{dy} < -\frac{\rho g}{c_s^2} < 0. \tag{79}$$

Upon defining the scale height $L^{-1} = \rho^{-1}|d\rho/dy|$, (79) is seen to be equivalent to $c_s^2 > Lg$. Thus the system is stable to convection if the free fall kinetic energy is smaller than twice the kinetic energy at the sound speed. Or, equivalently, if the free fall speed through a distance L is smaller than $\sqrt{2}c_s$.

The above procedure leading to (78) and (79) was designed for comparison with Sec. III B. However, the conventional "δW" stability analysis proceeds with an extremization over $\boldsymbol{\xi}$ that takes account of any possible stabilization effect due to the first positive definite term of (71). To this end we let

$$\boldsymbol{\xi}(\mathbf{x}) = (\xi_x(y), \xi_y(y), \xi_z(y))e^{i(kz+\ell x)}/2 + c.c., \tag{80}$$

and rewrite (71) as

$$\delta^2 W_{\rm la} = \frac{1}{2}\int_0^\infty dy\Bigg[-\left(\frac{\rho g^2}{c_s^2} + g\frac{d\rho}{dy}\right)|\xi_y|^2 + \rho\, c_s^2\left|\frac{d\xi_y}{dy} + i\ell\xi_x + ik\xi_z - \frac{\rho g}{c_s^2}\xi_y\right|^2\Bigg]. \tag{81}$$

Given any $\xi_y(y)$, one can choose ξ_x and ξ_z that makes the second term vanish. Thus the smallest value of $\delta^2 W_{\rm la}$ is given by

$$\delta^2 W_{\rm la} = -\frac{1}{2}\int_0^\infty dy\left(\frac{\rho g^2}{c_s^2} + g\frac{d\rho}{dy}\right)|\xi_y|^2, \tag{82}$$

which yields (79) as a necessary and sufficient condition for stability. Thus (79) is in fact a counterpart equivalent to $ds/dy > 0$. Another equivalent condition exists in terms of the temperature

$$\frac{dT}{dy} > \frac{gT}{\rho c_p}\frac{\partial\rho}{\partial T}\Big|_p, \tag{83}$$

which follows in a manner similar to (79).

Lastly, for an ideal gas, (79) and (83) become, respectively,

$$\frac{d\rho}{dy} < -\frac{\rho^2 g}{\gamma p} \quad \text{and} \quad \frac{dT}{dy} > -\frac{g}{c_p}.$$

Observe, (73) could be satisfied with $ds/dy < 0$ and $d\rho/dy > 0$. But, the stability condition $ds/dy > 0$, which came from (77), implies $d\rho/dy < 0$. Thus it is not possible to have stability unless the fluid density is stably stratified.

Case $B \neq 0$:

The case with $B \neq 0$ has been studied extensively, e.g., in the early works on interchange instability of Refs. 45–51. For this application, Eq. (20) can be written as follows:

$$\delta^2 W_{\rm la} = \frac{1}{2}\int d^3x\,\Big[\rho\, c_s^2(\nabla\cdot\boldsymbol{\xi})^2 + (\nabla\cdot\boldsymbol{\xi})(\nabla p\cdot\boldsymbol{\xi}) + \frac{|\delta\mathbf{B}|^2}{4\pi} + \mathbf{J}\cdot(\boldsymbol{\xi}\times\delta\mathbf{B}) - g(\boldsymbol{\eta}\cdot\hat{\mathbf{y}})\nabla\cdot(\rho\boldsymbol{\xi})\Big], \tag{84}$$

where again all equilibrium quantities depend only on y, which we use together with (80) to rewrite this as

$$\delta^2 W_{\mathrm{la}} = \frac{1}{2}\int_0^\infty dy\left[\frac{B^2}{4\pi}\left(k^2\left(|\xi_y|^2+|\xi_x|^2\right)+\left|\frac{d\xi_y}{dy}+i\ell\xi_x\right|^2\right)\right.$$
$$+\rho\, c_s^2\left|\frac{d\xi_y}{dy}+i\ell\xi_x+ik\xi_z\right|^2 - g\,\frac{d\rho}{dy}\,|\xi_y|^2$$
$$\left.-2\rho g\,\xi_y\left(\frac{d\xi_y}{dy}+i\ell\xi_x+ik\xi_z\right)\right], \tag{85}$$

where, following Ref. 49, the displacements ξ_y, $i\ell\xi_x$, and $ik\xi_z$ can be taken to be real-valued. By minimizing this functional, the following necessary and sufficient condition for the interchange stability of Tserkovnikov,[48] can be obtained:

$$\frac{d\rho}{dy} < -\frac{\rho g}{c_s^2+c_a^2} < 0, \tag{86}$$

where recall $c_a^2 = B^2/(4\pi\rho)$.

In Ref. 49 Newcomb rearranges (85) and minimizes it in the limit $k\to 0$ by choosing $i\xi_z \to g\xi_y/(kc_s^2)$ for arbitrary ξ_y. With this approach he obtains the more stringent stability condition of (79), the condition for the case without B. Newcomb's singular approach allows displacements that interchange plasma elements containing long segments along the magnetic field lines, relieving local fluid pressures. In Ref. 50, it is shown that this amounts to the plasma being least stable against these long quasi-interchange displacements because the restoring force due to the magnetic field tension vanishes.

B. Eulerian convection

1. Eulerian convection equilibria

Case $B = 0$:

Using the Casimir invariants of (31) and (32), hydrodynamic equilibria with translational symmetry are obtained as extrema of the following energy-Casimir functional:

$$\mathfrak{F} = \int d^3x\left[\frac{1}{2}\rho|\mathbf{v}|^2+\rho U(\rho,s)+\rho\Phi+\rho\mathcal{S}(s)-\mathcal{A}(s)\hat{\mathbf{z}}\cdot\nabla\times\mathbf{v}\right], \tag{87}$$

where $v_z = v_z(s)$. Variation of (87) will automatically yield equations that are cases of (2)–(5) with $B_e \equiv 0$. Because $v_z(s)$, we have $\mathbf{v}\cdot\nabla s = 0$ and $\mathbf{v}\cdot\nabla v_z = 0$. Variation with respect to $\mathbf{v}$ yields

$$\rho\mathbf{v}_\perp = \nabla\mathcal{A}\times\hat{\mathbf{z}}, \tag{88}$$

while variation with respect to ρ and s, respectively, yield

$$\frac{1}{2}|\mathbf{v}|^2+\Phi+\rho U_\rho+U+\mathcal{S} = 0, \tag{89}$$

$$\rho v_z v_z' + \rho U_s + \rho\mathcal{S}' - \mathcal{A}'\hat{\mathbf{z}}\cdot\nabla\times\mathbf{v} = 0. \tag{90}$$

For our case of interest with $\mathbf{v} = 0$, we merely set $\mathcal{A}\equiv 0$, whereupon the first variation

$$\delta\mathfrak{F} = \int d^3x\,[(\rho U_\rho+U+\Phi+\mathcal{S})\delta\rho+\rho(U_s+\mathcal{S}')\delta s], \tag{91}$$

gives rise to

$$\Phi+\rho U_\rho+U+\mathcal{S} = 0, \tag{92}$$

$$U_s+\mathcal{S}' = 0, \tag{93}$$

where for recall for our analyses, we choose $\Phi = gy$.

Case $B \neq 0$:

For case with equilibrium magnetic field, we choose the following special case for the Casimir of (33)

$$C_s = \int d^3x\,\rho\,\mathcal{S}(s,\psi), \tag{94}$$

which with the Hamiltonian

$$H = \int d^3x\left[\frac{1}{2}\rho|\mathbf{v}|^2+\rho U+\frac{|\nabla\psi|^2}{8\pi}+\rho g y\right], \tag{95}$$

gives upon varying $\mathfrak{F} = H + C_s$

$$\frac{\delta\mathfrak{F}}{\delta\mathbf{v}} = \rho\mathbf{v} = 0,$$
$$\frac{\delta\mathfrak{F}}{\delta\psi} = -\Delta\psi+\rho\mathcal{S}_\psi = 0,$$
$$\frac{\delta\mathfrak{F}}{\delta s} = \rho U_s+\rho\mathcal{S}_s = 0,$$
$$\frac{\delta\mathfrak{F}}{\delta\rho} = \rho U_\rho+U+gy+\mathcal{S} = 0,$$

which imply

$$\nabla(\rho U_\rho+U)+\frac{1}{\rho}\nabla^2\psi\,\nabla\psi - U_s\nabla s = -g. \tag{96}$$

Equation (96) gives for our case with stratification in y, the equilibrium equation (58).

2. Eulerian convection stability

Now we examine $\delta^2\mathfrak{F}$ for our two cases and look for conditions that make this quantity positive definite, conditions that will be sufficient conditions for stability.

Case $B = 0$:

The second variation is

$$\delta^2\mathfrak{F} = \int d^3x\,[(\rho U_{\rho\rho}+2U_\rho)(\delta\rho)^2+2(\rho U_{\rho s}+U_s+\mathcal{S}')\delta\rho\,\delta s$$
$$+\rho(U_{ss}+\mathcal{S}'')(\delta s)^2]. \tag{97}$$

By exploiting the equilibrium equations, (97) can be rewritten as

$$\delta^2\mathfrak{F} = \int d^3x\,\frac{c_s^2}{\rho}\left[(\delta\rho)^2-2\frac{\partial\rho}{\partial s}\Big|_p\delta\rho\,\delta s+\frac{\partial\rho}{\partial s}\Big|_p\frac{d\rho}{dy}\frac{dy}{ds}(\delta s)^2\right], \tag{98}$$

where we used (62) and (64), and the derivative of the equilibrium equation $U_s+\mathcal{S}'=0$ with respect to y

$$\mathcal{S}'' + U_{ss} + U_{s\rho}\frac{d\rho}{dy}\frac{dy}{ds} = 0. \tag{99}$$

Next, we use

$$(\delta\rho)^2 - 2\frac{\partial\rho}{\partial s}\Big|_p \delta\rho\delta s = \left(\delta\rho - \frac{\partial\rho}{\partial s}\Big|_p \delta s\right)^2 - \left(\frac{\partial\rho}{\partial s}\Big|_p\right)^2(\delta s)^2, \tag{100}$$

obtaining

$$\delta^2\mathfrak{F} = \int d^3x\,\frac{c_s^2}{\rho}\left[\left(\delta\rho - \frac{\partial\rho}{\partial s}\Big|_p\delta s\right)^2 + \frac{\partial\rho}{\partial s}\Big|_p\left(\frac{d\rho/dy}{ds/dy} - \frac{\partial\rho}{\partial s}\Big|_p\right)(\delta s)^2\right], \tag{101}$$

an expression of the form of (71). Thus, as in Sec. III A 2, stability is again determined by positivity of the quantity Δ of (74) and all of the conditions of that section are reproduced as sufficient stability conditions.

In Eq. (101), unlike the case of (71), $\delta\rho$ and δs are independent so a sharper sufficient condition cannot be pursued by relying on the positivity of the first term, even though in the $\delta^2 W_{\rm la}$ formulation, this did not materialize. Also, the approach here gives $ds/dy > 0$ as a sufficient condition for stability (or equivalently (79)), while the $\delta^2 W_{\rm la}$ formulation shows that this condition is both necessary and sufficient

Case $B \neq 0$:

Now consider the second variation of $\mathfrak{F} = H + C_s$ with H given by (95) and C_s given by (94) with

$$\mathcal{S}(\psi, s) = \mathcal{K}(\psi) + \mathcal{L}(s),$$

which is general enough to describe the equilibria of our interest as given by (58). This leads to

$$\begin{aligned}\delta^2\mathfrak{F} = \int d^3x\,[&(\rho U)_{\rho\rho}(\delta\rho)^2 + 2[(\rho U)_{\rho s} + \mathcal{L}_s]\delta\rho\,\delta s\\ &+\rho(U_{ss} + \mathcal{L}_{ss})(\delta s)^2 + |\nabla\delta\psi|^2 + 2\mathcal{K}_\psi\delta\psi\,\delta\rho\\ &+\rho\mathcal{K}_{\psi\psi}(\delta\psi)^2].\end{aligned} \tag{102}$$

Rewriting (102) in terms of equilibrium quantities and manipulating then gives

$$\begin{aligned}\delta^2\mathfrak{F} = \int d^3x\,&\left[\frac{c_s^2}{\rho}(\delta P)^2 + \frac{p_s}{\rho}\Delta(\delta s)^2 + 2\frac{Jp_s}{\rho c_s^2}\,\delta s\,\delta\psi\right.\\ &\left.+\rho\left(\mathcal{K}_{\psi\psi} - \frac{J^2}{\rho^2 c_s^2}\right)(\delta\psi)^2\right],\end{aligned} \tag{103}$$

where use has been made of the definition of Δ of (74), the current density J, defined by

$$-J = \nabla^2\psi = \rho\mathcal{K}_\psi, \tag{104}$$

the thermodynamic expressions of (62) and the following, which is a consequence of the equilibrium equation

$$U_{ss} + \mathcal{K}_{ss} = -U_{s\rho}\frac{d\rho}{ds}, \tag{105}$$

which implies

$$\begin{aligned}U_{ss} + \mathcal{K}_{ss} - \frac{1}{c_s^2}\frac{p_s^2}{\rho^2} &= -U_{s\rho}\frac{d\rho}{ds} - \frac{1}{c_s^2}\frac{p_s^2}{\rho^2}\\ &= -\frac{p_s}{\rho^2}\left(\frac{d\rho}{ds} + \frac{p_s}{c_s^2}\right) = \frac{p_s}{\rho^2}\Delta.\end{aligned} \tag{106}$$

In addition we have introduced the new variable δP defined by

$$\delta P = \delta\rho + \frac{p_s}{c_s^2}\delta s - \frac{J}{c_s^2}\delta\psi. \tag{107}$$

Next, we collect the terms with δs to obtain

$$\begin{aligned}\delta^2\mathfrak{F} = \int d^3x\,&\left[\frac{c_s^2}{\rho}(\delta P)^2 + |\nabla\delta\psi|^2 + \frac{p_s}{\rho}\Delta\left[\delta s - \frac{J}{c_s^2\Delta}\delta\psi\right]^2\right.\\ &\left.+\rho\left[\mathcal{K}_{\psi\psi} - \frac{J^2}{\rho^2 c_s^2} - \frac{J^2 p_s}{\rho^2 c_s^4\Delta}\right](\delta\psi)^2\right].\end{aligned} \tag{108}$$

If we introduce the variation

$$\delta Q = \delta s - \frac{J}{c_s^2\Delta}\delta\psi, \tag{109}$$

and we use the gradient of (104)

$$\nabla J = \frac{J}{\rho}\nabla\rho - \rho\mathcal{K}_{\psi\psi}\nabla\psi, \tag{110}$$

which for equilibria that depend only on the y coordinate can be written as

$$\rho\mathcal{K}_{\psi\psi} = \frac{J}{\rho}\frac{d\rho}{d\psi} - \frac{dJ}{d\psi} = -\rho\frac{d(J/\rho)}{d\psi}, \tag{111}$$

or

$$\rho\mathcal{K}_{\psi\psi} = \frac{J}{\rho}\frac{d\rho/ds}{d\psi/ds} - \frac{dJ/ds}{d\psi/ds}, \tag{112}$$

then the last term of Eq. (108) can be rewritten as

$$\mathcal{K}_{\psi\psi} - \frac{J^2}{\rho^2 c_s^2} - \frac{J^2 p_s}{\rho^2 c_s^4\Delta} = -\frac{d(J/\rho)}{d\psi} + \frac{J^2}{\rho^2 c_s^2\Delta}\frac{d\rho}{ds}. \tag{113}$$

Then, finally

$$\begin{aligned}\delta^2\mathfrak{F} = \int d^3x\,&\left[\frac{c_s^2}{\rho}(\delta P)^2 + \frac{p_s}{\rho}\Delta(\delta Q)^2 + |\nabla\delta\psi|^2\right.\\ &\left.+\rho\left(-\frac{d(J/\rho)}{d\psi} + \frac{J^2}{\rho^2 c_s^2\Delta}\frac{d\rho}{ds}\right)(\delta\psi)^2\right].\end{aligned} \tag{114}$$

From the energy expression of (114) we can immediately read off the following sufficient conditions for stability:

$$0 < \Delta = -\left(\frac{d\rho}{ds} + \frac{p_s}{c_s^2}\right), \tag{115}$$

$$0 < -\frac{d(J/\rho)/dy}{d\psi/dy} + \frac{J^2}{\rho^2 c_s^2 \Delta}\frac{d\rho/dy}{ds/dy}, \tag{116}$$

where we recall the form of Δ of (115) which is equivalent to that of (74).

In the case with $B = 0$, we had the two free functions, ρ and s and one stability inequality. Thus we were able to obtain separate conditions on the equilibrium profiles of ρ and s for stability. In the present case, we again have one equilibrium equation, but now with three profiles ρ, s, and B and two inequalities. Again we should expect to obtain independent conditions on the profiles ρ, s, and B. However, even the condition of (79), which has a clear physical meaning, is not immediately implementable because c_s depends on y through both ρ and s. Similarly, the inequalities (115) and (116) require the profiles for their determination. In practice one may construct a family of equilibria with profiles that depend on one or more parameters and then seek thresholds in parameter space.

Inequalities (115) and (116) can be written in various ways. For example, using the equilibrium equation (61)

$$\frac{dp}{dy} = c_s^2\left(\frac{d\rho}{dy} - \frac{\partial\rho}{\partial s}\Big|_p \frac{ds}{dy}\right) = -g\rho - (B^2)'/(8\pi), \tag{117}$$

the inequality $\Delta > 0$ can be rewritten as

$$\Delta = -\frac{1}{c_s^2}\frac{dp}{ds} = \frac{g\rho + (B^2)'/2}{c_s^2\, ds/dy} > 0. \tag{118}$$

Consequently, if dp/dy is negative for stability, we must have $ds/dy > 0$ and, conversely, we must have $ds/dy < 0$ if, due to B decreasing sufficiently fast with height, we have $dp/dy > 0$. This is effectively the threshold against the magnetized Rayleigh-Taylor instability. Thus, as for the case with $B = 0$, $dp/ds < 0$ ensures stability. Also note, as in the $B = 0$ case, a critical point arises if for some y we have $dp/dy = 0$ unless at the same point we also have $ds/dy = 0$, in which case one then has to look deeper into the limit.

If $dp/dy < 0$ and $ds/dy > 0$, we obtain from (115) an inequality for $d\rho/dy$ analogous to the inequality (79), in particular, $d\rho/dy$ must be negative because $p_s/c_s^2 > 0$; however, this inequality is different from the "Tserkovnikov" inequality of (86). If $dp/dy > 0$ and $ds/dy < 0$, we obtain a reversed inequality, i.e., $d\rho/dy$ must be positive.

This implies that in the inequality (116), if Δ is positive, the second term is always negative and thus for $B > 0$ we obtain the condition

$$d(J/\rho)/dy < 0, \quad \text{or} \quad dJ/dy < (J/\rho)(d\rho/dy). \tag{119}$$

Consider the two cases of decreasing and increasing magnetic fields: for a magnetic field decreasing with height, $J = -dB/dy > 0$, so

$$d\ln J/dy < d\ln\rho/dy, \tag{120}$$

and if $d\rho/dy < 0$ we can use the inequality obtained before for $d\rho/dy$ and obtain an inequality that involves the second derivative of the magnetic field and the density profile. Similarly, if $J = -dB/dy < 0$

$$d\ln|J|/dy > d\ln\rho/dy, \tag{121}$$

and if $d\rho/dy > 0$ we can use the reverse inequality obtained before for $d\rho/dy$ and again obtain an inequality that involves the second derivative of the magnetic field and the density profile. These cases above do not exhaust all possibilities. It is perhaps best to consider families of equilibria and investigate parameter dependencies as mentioned above.

C. Dynamically accessible convection

1. Dynamically accessible convection equilibria

In Sec. II C, we showed how the general dynamically accessible variations of (44)–(47), when inserted into the first variation of the Hamiltonian (48), give rise to the general MHD equilibrium equations of (2)–(5). Thus, equilibria that are solutions of (58), with or without the magnetic field, are extremal points of this kind of variation, and we can proceed to assess the stability by examination of the energy expression of (49).

2. Dynamically accessible convection stability

For static equilibria, the first term of (49) reduces to the form of (52). As noted in Sec. II D, this term vanishes if $g_3 = g_2 = \mathbf{g}_4 \equiv 0$. Thus, choosing $\mathbf{g}_1$ proportional to $\boldsymbol{\xi}$, the condition for dynamically accessible stability in the case of static equilibria is determined by $\delta^2 W_{\rm la}$, viz., the Lagrangian energy expression. In both the cases with and without a magnetic field, this is the usual "δW" energy, for each case, respectively, and thus dynamically accessible stability in both cases is identical to that of the Lagrangian stability.

D. Convection comparisons

Results for the case with equilibria $B = 0$ can be summarized succinctly: the Lagrangian and dynamically accessible approaches both give the simple necessary and sufficient condition for stability, $ds/dy > 0$, or equivalently the inequality of (79) on $d\rho/dy$, while the Eulerian energy-Casimir approach gives this same result, but only as a sufficient condition for stability and only applicable to the case with the imposed translational symmetry.

For case of equilibria $B \neq 0$, the situation is more complex, although it again must be true, in light of the general discussion of Sec. II D, that the Lagrangian and dynamically accessible approaches must give the same necessary and sufficient condition for stability, viz., that of (86). However, this necessary and sufficient condition is much simpler than the inequalities of (115) and (116) obtained by the energy-Casimir method and, again, these inequalities are only applicable to the case with the imposed translational symmetry and only give sufficient conditions for stability. Moreover, the energy-Casimir inequalities depend on an extra derivative with respect to y of at least one of the equilibrium profiles; e.g., (115) contains a derivative of the current J, which can be eliminated in terms of two derivatives of the pressure p, but cannot easily be eliminated entirely.

If one inserts the Lagrangian variations of (21)–(22), adapted to the convection example, into $\delta^2\mathfrak{F}$ of (102), then

dJ/dy is removed. In the context of our convection example, the relevant connection is provided by $\delta\psi_{\rm la} = \xi\cdot\nabla\psi = \xi_y\psi'$, with prime denoting y-differentiation. Whence, the line-bending term of (102) becomes

$$|\nabla\delta\psi|^2 = (\xi_y'\psi' + \xi_y\psi'')^2 = \xi_y'\psi'^2 + \xi_y^2\psi''^2 + 2\xi_y\xi_y'\psi''\psi', \tag{122}$$

and one finds upon integrating the last term of (122) by parts, a term proportional to J'. This term cancels the J term of $(\delta\psi)^2$ (the same cancellation was shown to occur in the context of the magnetorotational instability in Ref. 52). As noted in Sec. II (cf. Refs. 5 and 22) such a correspondence by constraining the Eulerian variations in general connects energy-Casimir and Lagrangian stability.

IV. ROTATING PINCH

Now we investigate the stability of the azimuthally symmetric rotating pinch, again within the Lagrangian, Eulerian energy-Casimir, and dynamically accessible frameworks. This example is chosen to illustrate two features introduced in Sec. II associated with the inclusion of an equilibrium velocity field: the relabeling transformation that removes time dependence from a Lagrangian state associated with a stationary Eulerian equilibrium and the origin of the difference between Lagrangian and dynamically accessible stability. As in Sec. III, we begin by discussing the plasma equilibrium configurations of interest by solving directly the Eulerian MHD equations (2)–(5) without referring specifically to any of the three frameworks.

We use cylindrical coordinates (r,ϕ,z) and consider plasma equilibrium configurations where all equilibrium quantities (including entropy) depend only on the radial coordinate r

$$\mathbf{B} = B_z(r)\hat{\mathbf{z}} + B_\phi(r)\hat{\phi}, \tag{123}$$

$$\mathbf{v} = v_z(r)\hat{\mathbf{z}} + v_\phi(r)\hat{\phi}, \tag{124}$$

$$\rho = \rho(r), \quad s = s(r), \tag{125}$$

$$B_\phi = \hat{\phi}\cdot\nabla\psi\times\hat{\mathbf{z}} = -\frac{d\psi(r)}{dr}. \tag{126}$$

Equation (125) implies that $p = p_e(r)$. From Eqs. (2)–(5), we obtain the generalized Grad-Shafranov equation for the flux function $\psi(r)$

$$\frac{1}{r}\frac{d}{dr}\left(\frac{1-\mathcal{M}^2}{4\pi}rB_\phi\right) - \frac{1}{\psi_r}\frac{d}{dr}\left(p + \frac{B_z^2}{8\pi}\right) + \frac{d}{dr}\left(\frac{\mathcal{M}^2}{4\pi}B_\phi\right) = 0, \tag{127}$$

where

$$\mathcal{M}(r) = \left[\frac{4\pi\rho(r)v_\phi^2(r)}{B_\phi^2(r)}\right]^{1/2},$$

is the poloidal Alfvén Mach number. Note that $v_z(r)$ does not appear in (127) and in the following it will be set equal to zero.

In (127), we need to assign three free functions. We will assign $B_z(r), B_\phi(r)$, and $v_\phi(r)$ and treat (127) as an equation for $p(r)$ that can be written as

$$\frac{1}{B_\phi}\frac{d}{dr}\left(p + \frac{B_z^2}{8\pi}\right) + \frac{d}{dr}\left(\frac{B_\phi}{4\pi}\right) + \frac{1-\mathcal{M}^2}{4\pi}\frac{B_\phi}{r} = 0. \tag{128}$$

For the sake of simplicity, we will examine the case of an isothermal plasma configuration as it makes the relationship between p and ρ linear, and also makes $\mathcal{M}^2$ linear in p. A further simplification is obtained by taking the current density J_z to be uniform. By defining a dimensionless radial variable r in terms of a characteristic length r_0, the latter assumption leads to $B_\phi = B_0 r$ and (128) becomes

$$\frac{d}{dr^2}\left(\hat{p} + \frac{\hat{B}^2}{2}\right) = -\left[1 - \hat{p}(r)\,w(r)^2/2\right], \tag{129}$$

where we have set

$$\begin{aligned} p(r) &= c_s^2\rho(r) = \hat{p}(r)\,B_0^2/(4\pi),\\ \hat{B}(r) &= B_z/B_0, \quad \text{and}\\ \mathcal{M}^2(r) &= (4\pi p/B_0^2)\,[v_\phi^2/(rc_s)^2] = \hat{p}(r)\,w^2(r),\end{aligned}$$

with $\hat{B}$ being the dimensionless magnetic field, $\hat{p}$ the dimensionless pressure, $w(r) = v_\phi/(rc_s)$ the dimensionless rotation rate, and c_s the sound velocity in the isothermal case.

For a configuration where B_z is uniform and the plasma rotation is rigid with rotation frequency Ω, Eq. (129) takes the elementary form

$$\frac{d\,\hat{p}(r)}{d\,r^2} = -\left[1 - \hat{p}(r)\,w^2/2\right], \tag{130}$$

where $w = \Omega r_0/c_s$. While a uniform B_z field does not alter these equilibrium configurations, it will be shown to affect their stability. Assuming $w^2/2 < 1$, we obtain

$$\hat{p}(r) = \frac{2}{w^2}\left[1 - \left(1 - \frac{w^2}{2}\right)\exp\left(\frac{w^2r^2}{2}\right)\right], \tag{131}$$

where $\hat{p}(0) = 1$, $\hat{p}(\bar{r}) = 0$ for $\bar{r}^2 = -(2/w^2)\ln(1 - w^2/2)$. Equation (131) describes a one-parameter family of equilibria. In the absence of rotation, this configuration reduces to the standard parabolic pinch with $\bar{r} = 1$ and $\hat{p}(r) = 1 - r^2$, while for $w^2 \to 2$ we have $\bar{r} = \infty$ and $\hat{p}(r) \equiv 1$.

A. Lagrangian pinch

1. Lagrangian pinch equilibria

For the rotating pinch the appropriate Hamiltonian is that of (7) with $\Phi \equiv 0$ and, as before, the pinch equilibrium equations should follow from Eq. (9) adapted to the pinch geometry. In particular, with the cylindrical coordinate system with indices $i,j \in \{r,\phi,z\}$, $\mathbf{a} = (a^r, a^\phi, a^z)$, $\mathbf{q} = (q^r, q^\phi, q^z)$, with $|\boldsymbol{\pi}|^2 = g^{ij}(\mathbf{q})\,\pi_i\,\pi_j = \pi^i\pi_i$, and

$$g^{ij}(\mathbf{q}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (q^r)^{-2} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \qquad (132)$$

From (132) we obtain

$$\pi^r = g^{rr}\pi_r = \pi_r,$$
$$\pi^\phi = g^{\phi\phi}\pi_\phi = \pi_\phi/(q^r)^2,$$
$$\pi^z = g^{zz}\pi_z = \pi_z,$$

and similarly $B^r = B_r$, $B^\phi = B_\phi/(q^r)^2$, and $B^z = B_z$.

As shown in Appendix A, the equations of motion in terms of the Lagrangian variables (q^r, q^ϕ, q^z) follow from Eq. (9), and are

$$\dot{q}^i = g^{ij}\frac{\pi_j}{\rho_0} \quad \text{and} \qquad (133)$$

$$\dot{\pi}_i = \delta_i^r \frac{\pi^\phi \pi_\phi}{q^r \rho_0} - \mathcal{J}\frac{\partial}{\partial q^i}\left[\left(\frac{\rho_0}{\mathcal{J}}\right)^2 U_\rho\right] - \delta_i^r \frac{\mathcal{J}}{q^r}\frac{B^\phi B_\phi}{4\pi} + \mathcal{J}B^j\frac{\partial}{\partial q^j}\left(\frac{B_i}{4\pi}\right) - \mathcal{J}\frac{\partial}{\partial q^i}\left(\frac{B^2}{8\pi}\right). \qquad (134)$$

Transforming (133) and (134) to Eulerian variables, we first obtain the intermediate form

$$\begin{aligned} \rho_0 \dot{q}^r &= g^{rr}\pi_r = \pi_r \\ \rho_0 \dot{q}^\phi &= g^{\phi\phi}\pi_\phi = \pi_\phi/r^2 \\ \rho_0 \dot{q}^z &= g^{zz}\pi_z = \pi_z \end{aligned} \qquad (135)$$

and

$$\dot{\pi}_r = \frac{\pi_\phi^2}{r^3\rho_0} - \mathcal{J}\nabla\left(p + \frac{B^2}{8\pi}\right)\cdot\hat{\mathbf{r}} + \mathcal{J}\frac{\mathbf{B}\cdot\nabla\mathbf{B}}{4\pi}\cdot\hat{\mathbf{r}}, \qquad (136)$$

$$\dot{\pi}_\phi = -r\mathcal{J}\nabla\left(p + \frac{B^2}{8\pi}\right)\cdot\hat{\phi} + r\mathcal{J}\frac{\mathbf{B}\cdot\nabla\mathbf{B}}{4\pi}\cdot\hat{\phi}, \qquad (137)$$

$$\dot{\pi}_z = -\mathcal{J}\nabla\left(p + \frac{B^2}{8\pi}\right)\cdot\hat{\mathbf{z}} + \mathcal{J}\frac{\mathbf{B}\cdot\nabla\mathbf{B}}{4\pi}\cdot\hat{\mathbf{z}}, \qquad (138)$$

from which, using

$$\dot{\pi}_r = \rho_0\frac{D}{Dt}v_r(\mathbf{q}, t),$$
$$\dot{\pi}_\phi = \rho_0\frac{D}{Dt}\left(q^r v_\phi(\mathbf{q}, t)\right),$$
$$\dot{\pi}_z = \rho_0\frac{D}{Dt}v_z(\mathbf{q}, t),$$

with $D/Dt = \partial_t + \dot{q}^i\,\partial/\partial q^i$ and $\mathbf{q}.(\mathbf{a}, t) = \mathbf{v}(\mathbf{x}, t)$, we recover the cylindrical components of the Eulerian equation of motion

$$\rho\left(\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) = -\nabla\left(p + \frac{B^2}{8\pi}\right) + \frac{\mathbf{B}\cdot\nabla\mathbf{B}}{4\pi}. \qquad (139)$$

The rotating pinch equilibrium configuration of this section corresponds to

$$\dot{\pi}_r = \dot{\pi}_z = \dot{\pi}_\phi = 0, \quad \dot{q}^r = \dot{q}^z = 0, \quad \dot{q}^\phi = \Omega, \qquad (140)$$

with $\rho_0(b^r) = p(r)/c_s^2$ where $p(r)$ is given by (131). Because $q^\phi = \Omega t + a^\phi$, we see explicitly that stationary Eulerian equilibria correspond to time-dependent Lagrangian trajectories.

Next, we consider the relabeling transformation introduced in Ref. 5 and described in Sec. II

$$\mathbf{a} = \mathfrak{A}(\mathbf{b}, t) \leftrightarrow \mathbf{b} = \mathfrak{B}(\mathbf{a}, t),$$

where $\mathbf{a} = \mathfrak{A}(\mathbf{b}, t)$ is given by

$$a^r = b^r, \quad a^\phi = b^\phi - \Omega(b^r)t, \quad a^z = b^z - V^z(b^r)t, \quad (141)$$

and $\mathbf{b} = \mathfrak{B}(\mathbf{a}, t)$ is given by

$$b^r = a^r, \quad b^\phi = a^\phi + \Omega(a^r)t, \quad b^z = a^z + V^z(a^r)t, \quad (142)$$

with $\mathfrak{J} := |\partial a^i/\partial b^j| = 1$, with

$$\mathbf{V}(\mathbf{b}, t) := \dot{\mathfrak{B}}\circ\mathfrak{B}^{-1} = \dot{\mathfrak{B}}(\mathfrak{A}(\mathbf{b}, t), t), \qquad (143)$$

given by

$$V^r = 0, \quad V^\phi = \Omega(b^r), \quad V^z = V^z(b^r). \qquad (144)$$

By inserting (144) into the transformed Hamiltonian of (12) (see Appendix B) we obtain the "time-relabeled" equations of motion corresponding to (133) and (134) (see (B3) and (B4)). Then in the relabeled variables by explicitly setting $\partial/\partial t = 0$, $Q^i = b^i$ and by assigning the functions B_0^i and $\rho_0 U$ as functions of b^i consistently with the choices made in Sec. IV, these equations yield the equilibrium equations in the relabeled form of (B6)–(B9).

Thus, we have shown that the equilibrium equation of (129) describes the reference state $(\mathbf{Q}_e, \mathbf{\Pi_e})$ that follows from:

$$\frac{\delta\widetilde{H}}{\delta\mathbf{\Pi}} = 0 \quad \text{and} \quad \frac{\delta\widetilde{H}}{\delta\mathbf{Q}} = 0. \qquad (145)$$

Given that our equilibrium corresponds to the vanishing of the first variation of the Hamiltonian $\widetilde{H}$ of (12), we can expand as in (17) to address stability via the energy principle described in Sec. IV A 2.

2. Lagrangian pinch stability

Now, to address the stability, we expand $\widetilde{H}$ by inserting (17) (see also Eq. (27) of Ref. 5), where the reference state is our pinch equilibrium of Sec. IV A 1. This leads to the second variation of the Hamiltonian $\widetilde{H}$ written in terms of the canonically conjugate variables $(\boldsymbol{\eta}, \boldsymbol{\pi}_\eta)$ as given by (18) with $\delta^2 W_{\rm la}[\boldsymbol{\eta}]$ defined by (19) with (20). Due to the arbitrariness of $\boldsymbol{\pi}_\eta$, we can make the first term of (18) vanish, so that a sufficient stability condition for the configuration (14) is given by $\delta^2 W_{\rm la}[\boldsymbol{\eta}] > 0$. We will proceed further by minimizing $\delta^2 W_{\rm la}$ for our pinch example.

In order to be able to compare the Lagrangian stability conditions with those obtained in the Energy-Casimir framework, we restrict our analysis to perturbations $\boldsymbol{\eta}$ that do not depend on z.

Working out terms of (19) with (20) for our example, we obtain in cylindrical curvilinear coordinates

$$\rho(\mathbf{v}_\phi\cdot\nabla\mathbf{v}_\phi)\cdot(\boldsymbol{\eta}\cdot\nabla\boldsymbol{\eta}) = -(\rho v_\phi^2/r)\,[\eta_r\partial_r\eta_r+(\eta_\phi/r)\partial_\phi\eta_r-\eta_\phi^2/r], \tag{146}$$

$$-\rho\,|\mathbf{v}_\phi\cdot\nabla\boldsymbol{\eta}|^2 = -\rho(v_\phi/r)^2\,[(\partial_\phi\eta_r-\eta_\phi)^2 +(\partial_\phi\eta_\phi+\eta_r)^2+(\partial_\phi\eta_z)^2], \tag{147}$$

$$\rho\,\partial p/\partial\rho(\nabla\cdot\boldsymbol{\eta})^2 = \rho\,(c_s^2/r^2)[\partial_r(r\eta_r)+\partial_\phi\eta_\phi]^2, \tag{148}$$

where in (148), the isothermal equation of state $\rho\,\partial p/\partial\rho = p = \rho c_s^2$ has been used

$$(\eta_r\,\partial_r p)\nabla\cdot\boldsymbol{\eta} = [(\eta_r/r)\,\partial_r p]\,[\partial_r(r\eta_r)+\partial_\phi\eta_\phi], \tag{149}$$

$$|\nabla\times(\boldsymbol{\eta}\times\mathbf{B})|^2/(4\pi) = (B_0^2/4\pi)\,[(\partial_r(r\eta_r))^2+(\partial_\phi\eta_r)^2] +(B_0^2/4\pi)\,[\partial_\phi\eta_z-(\hat{B}/r) \times\left(\partial_r(r\eta_r)+\partial_\phi\eta_\phi\right)]^2, \tag{150}$$

$$\mathbf{J}\times\boldsymbol{\eta}\cdot\delta\mathbf{B} = -(B_0^2/2\pi)\,[\eta_r\partial_r(r\eta_r)-\eta_\phi\partial_\phi\eta_r], \tag{151}$$

where the restriction that $\hat{B} = B_z/B_0$ and J_z be independent of r has been used in accordance with the derivation in Sec. IV. In the above, we used the notation $\partial_r := \partial/\partial r$, etc., which we use throughout the present section.

In the following, we will refer explicitly to the rigid rotation equilibrium given by (131) and adopt the dimensionless variables used there. Also, we suppose $\boldsymbol{\eta}\sim\exp(im\phi)$ and consider azimuthally symmetric ($m=0$) and azimuthally asymmetric ($m\neq 0$) perturbations separately.

Case $m=0$:

If $\partial_\phi = 0$ the functional $\delta^2 W_{\rm la}$ depends only on the radial component η_r and its radial derivative

$$\delta^2 W_{\rm la}[\boldsymbol{\eta}] = \pi\int r dr(-w^2\hat{p}\,[\eta_r\partial_r(r\eta_r)]+(\hat{p}/r^2)[\partial_r(r\eta_r)]^2 +(\eta_r/r)\,(\partial_r\hat{p})[\partial_r(r\eta_r)]+[\partial_r(r\eta_r)]^2[1+(\hat{B}/r)^2] -2\eta_r[\partial_r(r\eta_r)])\,; \tag{152}$$

then using the equilibrium (127), this reduces to

$$\delta^2 W_{\rm la}[\boldsymbol{\eta}] = \pi\int r dr(-4\eta_r\partial_r(r\eta_r)+[(\hat{p}+r^2+\hat{B}^2)/r^2]\,[\partial_r(r\eta_r)]^2). \tag{153}$$

The first term of (153) is a divergence and vanishes by integration with the proper boundary conditions, while the second term is positive definite. Thus we conclude that our pinch equilibrium is stable to azimuthally symmetric perturbations.

Case $m\neq 0$:

In this case, besides η_r and η_ϕ, the functional $\delta^2 W_{\rm la}$ depends also on η_z if $\hat{B}\neq 0$. We use the orthogonality of the different m-components and consider the mth component. The resulting expressions, as obtained from (146)–(151), are given in Appendix C.

Case $B_z = 0$:

If $\hat{B}=0$, the displacement η_z along the symmetry axis of the perturbation decouples, and minimization with respect to η_z gives $\eta_z = 0$, provided

$$m^2(1-w^2\hat{p}) > 0 \rightarrow w^2 < 1. \tag{154}$$

Combining (C1)–(C5) and using (130), we can write the integrand of the functional $\delta^2 W_{\rm la}$ in the following matrix form:

$$[\,\eta_\phi^*,\,\eta_r^*,\,\partial_r(r\eta_r^*)\,]\cdot\mathcal{W}\cdot\begin{bmatrix}\eta_\phi\\ \eta_r\\ \partial_r(r\eta_r)\end{bmatrix}, \tag{155}$$

where $\mathcal{W}$ is the 3×3 matrix given by

$$\mathcal{W} = \begin{bmatrix} m^2\hat{p}\,\varsigma/r^2 & im\hat{p}w^2 & -im\hat{p}/r^2\\ -im\hat{p}w^2 & m^2\varpi & 0\\ im\hat{p}/r^2 & 0 & 1+\hat{p}/r^2\end{bmatrix},$$

where for convenience we have defined

$$\varpi := 1-w^2\hat{p} \quad\text{and}\quad \varsigma := 1-w^2r^2. \tag{156}$$

Then, to ascertain stability, we use Sylvester's criterion on the matrix $\mathcal{W}$. This criterion states that a necessary and sufficient criterion for the positive definiteness of a Hermitian matrix is that the leading principal minors be positive. The first principal minor of $\mathcal{W}$ is seen to be positive if

$$1-w^2\bar{r}^2 > 0,\quad\text{i.e.,}\quad w^2 < 2(1-\exp(-1/2)), \tag{157}$$

while the second principal minor of $\mathcal{W}$ is positive if for $m=1$ (which is the worst case)

$$\hat{p}(1-w^2r^2)(1-\hat{p}w^2)-r^2\hat{p}^2w^4 > 0, \tag{158}$$

which implies

$$w^2 \le \frac{1}{r^2+\hat{p}} < \frac{1}{\bar{r}^2}, \tag{159}$$

and coincides with the condition given by (157). Finally, the determinant of $\mathcal{W}$ is positive for the worst case $m=1$ if

$$(r^2+\hat{p})(1-w^2r^2)(1-\hat{p}w^2) -\hat{p}(1-\hat{p}w^2)-\hat{p}r^2w^4(r^2+\hat{p}) > 0, \tag{160}$$

which implies

$$w^2 < \frac{1}{r^2+2\hat{p}}, \tag{161}$$

and yields the stronger condition $w^2 < 1/2$.

Alternatively we can first minimize $\delta^2 W_{\rm la}$ with respect to η_ϕ in order to obtain a quadratic form involving η_r and $\partial_r(r\eta_r)$ only, from which we can derive an Euler-Lagrange equation. Now observe η_ϕ enters $\delta^2 W_{\rm la}$ through a combination of terms that we rewrite as

$$-w^2\hat{p}\,|m\eta_\phi - i\eta_r|^2 + \frac{\hat{p}}{r^2}\,|m\eta_\phi - i\partial_r(r\eta_r)|^2 + w^2\hat{p}\,|\eta_r|^2 - \frac{\hat{p}}{r^2}\,|\partial_r(r\eta_r)|^2. \tag{162}$$

In the absence of rotation, minimization with respect to η_ϕ would lead to the incompressibility condition. Assuming $w^2\bar{r}^2 < 1$, we introduce the new variable $\widetilde{\eta}_\phi = \eta_\phi[1-w^2r^2]^{1/2}$ and rewrite the expression (162) as

$$\frac{\hat{p}}{r^2}\,|m\widetilde{\eta}_\phi + i\alpha\eta_r - i\beta\partial_r(r\eta_r)|^2 + \mathrm{R}, \tag{163}$$

where $\alpha = w^2r^2/(1-w^2r^2)^{1/2}$, $\beta = 1/(1-w^2r^2)^{1/2}$, and

$$\mathrm{R} = -\frac{\hat{p}}{r^2}\left[\alpha^2|\eta_r|^2 + \beta^2|\partial_r(r\eta_r)|^2 - \alpha\beta\left(\eta_r^*\partial_r(r\eta_r) + \eta_r\partial_r(r\eta_r^*)\right)\right].$$

Then minimization with respect to $\widetilde{\eta}_\phi$ gives the following reduced expression for $\delta^2\widetilde{W}_{\mathrm{la}}$:

$$\delta^2\widetilde{W}_{\mathrm{la}} = \pi\int rdr\left\{\left[m^2\varpi - \frac{\hat{p}w^4r^2}{\varsigma}\right]|\eta_r|^2 + \left(1 - \frac{\hat{p}}{r^2}\frac{w^2r^2}{\varsigma}\right)|\partial_r(r\eta_r)|^2 + \frac{\hat{p}}{r^2}\frac{w^2r^2}{\varsigma}\left[\eta_r^*\partial_r(r\eta_r) + \eta_r\partial_r(r\eta_r^*)\right]\right\}, \tag{164}$$

which we can rewrite as

$$\delta^2\widetilde{W}_{\mathrm{la}} = \pi\int rdr\left[\left(1 - \frac{\hat{p}}{r^2}\frac{w^2r^2}{\varsigma}\right)|\partial_r(r\eta_r)|^2 + \left(m^2\varpi - \hat{p}w^4r^2/\varsigma - r\partial_r(\hat{p}w^2/\varsigma)\right)|\eta_r|^2\right], \tag{165}$$

where the contribution of the last term of R has been integrated by parts.

It can be directly verified numerically that for $|m| = 1$, the coefficient of $|\eta_r|^2$ is positive for $w^2 \lesssim 0.62$. Since in this interval also, the coefficient of $|\partial_r(r\eta_r)|^2$ is positive, $w^2 \lesssim 0.62$ provides a less restrictive sufficient stability condition that falls between the values given by (157) and (161). We note that an even less restrictive condition could be identified by solving the Euler-Lagrange equation obtained via variation of $\delta^2\widetilde{W}_{\mathrm{la}}$ of (165) subject to the constraint of $\int rdr\,|r\eta_r|^2$. Such a procedure leads to an eigenvalue equation that can be searched for the lowest eigenvalue.

Case $B_z \neq 0$:

For $\hat{B} \neq 0$, the component η_z is coupled to the other components of the displacement, and instead of (155) we obtain

$$[\,\eta_\phi^* \quad \eta_r^* \quad \partial_r(r\eta_r^*) \quad \eta_z^*\,]\cdot\mathcal{W}\cdot\begin{bmatrix}\eta_\phi\\ \eta_r\\ \partial_r(r\eta_r)\\ \eta_z\end{bmatrix}, \tag{166}$$

where the matrix $\mathcal{W}$ is now the 4×4 matrix

$$\begin{bmatrix} m^2(\hat{\Pi}/r^2 - \hat{p}w^2) & im\hat{p}w^2 & -im\hat{\Pi}/r^2 & -m^2\hat{B}/r\\ -im\hat{p}w^2 & m^2\varpi & 0 & 0\\ im\hat{\Pi}/r^2 & 0 & 1+\hat{\Pi}/r^2 & -im\hat{B}/r\\ -m^2\hat{B}/r & 0 & im\hat{B}/r & m^2\varpi\end{bmatrix},$$

where we recall $\varpi = 1 - \hat{p}w^2$ and $\hat{\Pi} = \hat{p} + \hat{B}^2$. Proceeding as above using Sylvester's criterion now leads to $m=1$ to the four conditions

$$0 < \hat{p}(1-w^2r^2) + \hat{B}^2, \tag{167}$$

$$0 < \frac{\hat{p}+\hat{B}^2}{\hat{p}\left(\hat{p}+\hat{B}^2+r^2\right)} - w^2, \tag{168}$$

$$0 < \frac{\hat{p}+\hat{B}^2}{\hat{p}\left[2\left(\hat{p}+\hat{B}^2\right)+r^2\right]} - w^2, \tag{169}$$

$$0 < 1 - w^2(r^2 + 3\hat{p} + \hat{B}^2) + \hat{p}w^4[r^2 + 2(\hat{p}+\hat{B}^2)]. \tag{170}$$

Note that the first two conditions give threshold values that increase with $\hat{B}$ while the third gives $w^2 < 1/2$ independent of $\hat{B}$, i.e., the effect of B_z would appear to be stabilizing or neutral if we were to neglect the coupling to η_z that appears instead in the fourth condition, where the effect of B_z is destabilizing (for $w^2 < 1/2$).

The inequality (170) can be better cast in the form

$$w^2\hat{b}^2(1-2w^2\hat{p}) < (1-w^2\hat{p})[1-w^2(r^2+2\hat{p})], \tag{171}$$

which, since $1-2w^2\hat{p}$ is positive for $w^2 < 1/2$ and $r < \bar{r}$, can be used to compute the maximum value of $\hat{B}$ that yields a sufficient stability condition when $w^2 < 1/2$. This yields $\hat{B}^2w^2 < 1$ for $w^2 \to 0$ and $\hat{B}^2 < 1/3$ for $w^2 \to 1/2^-$.

Alternatively we can perform separate minimizations with respect to η_z and η_ϕ by defining the new variables

$$\widetilde{\widetilde{\eta}}_z = \eta_z[1-w^2\hat{p}]^{1/2},$$
$$\widetilde{\widetilde{\eta}}_\phi = \eta_\phi\,[1 - w^2[r^2 + \hat{B}^2/(1-w^2\hat{p})]]^{1/2}.$$

Provided $w^2\hat{p} < 1$ and

$$w^2[r^2 + \hat{b}^2/(1-w^2\hat{p})] < 1,$$

i.e., $w^2[\bar{r}^2 + \hat{B}^2] < 1$, minimization with respect to these variables gives after integration by parts the following reduced expression:

$$\delta^2\widetilde{\widetilde{W}}_{\mathrm{la}} = \pi\int rdr\left[\left(1 + \frac{\hat{p}+\hat{B}^2}{r^2} - \frac{\hat{p}\left(1-\hat{B}^2w^2/\varpi\right)^2}{r^2\left[1-w^2\left(r^2+\hat{B}^2/\varpi\right)\right]}\right)\times|\partial_r(r\eta_r)|^2 + \left(m^2\varpi - \frac{\hat{p}w^4r^2}{1-w^2\left[r^2+\hat{B}^2/\varpi\right]} - r\partial_r\left(\frac{\hat{p}w^2\left[1-\hat{B}^2w^2/\varpi\right]}{1-w^2\left[r^2+\hat{B}^2/\varpi\right]}\right)\right)|\eta_r|^2\right]. \tag{172}$$

Note that the minimization with respect to $\widetilde{\eta}_z$ can be shown to have introduced a negative, i.e., destabilizing, contribution to $\delta^2\widetilde{\widetilde{W}}_{\rm la}$. It can be directly verified numerically that for $|m|=1$, the coefficient of $|\eta_r|^2$ is no longer positive for $w^2 \lesssim 0.62$ if $\hat{B}^2 > 0$; e.g., for $\hat{B}^2 = 1$, the coefficient of $|\eta_r|^2$ is positive for $w^2 \lesssim 0.46$ (this value is essentially in agreement with the result that would be obtained from (167)–(170). Since in this latter interval also, the coefficient of $|\partial_r(r\eta_r)|^2$ is positive, $w^2 \lesssim 0.46$ provides a sufficient stability condition for $\hat{B}^2 = 1$. As for the $\hat{B} = 0$ case, a less restrictive condition could be identified by solving the Euler-Lagrange equation derived by variation with the normalization constraint $\int r dr\, |r\eta_r|^2$.

B. Eulerian pinch

1. Eulerian pinch equilibria

In Ref. 4, which was reviewed in Sec. II B, both the equilibrium and the perturbations were assumed to be helically symmetric. In the present section, we have assumed the equilibrium to be both translationally symmetric along z and azimuthally symmetric along ϕ, while we considered perturbations that have only translational symmetry along z. Then the full configuration is symmetric under translations along z.

Now we consider the first variation of the energy-Casimir functional $\mathfrak{F}[Z] = H_{TS}[Z] + \sum C[Z]$ (see Sec. II B and Eq. (1) of Ref. 6) with a translational and rotational symmetry, which leads to the equilibrium equation

$$\frac{1}{4\pi r}\frac{d}{dr}\left[\left(1-\frac{4\pi\mathcal{F}^2}{\rho}\right) r\frac{d\psi}{dr}\right] = \rho T\mathcal{S}' - \rho\mathcal{J}' - B_z\mathcal{H}' - \rho v_z\mathcal{G}' - (v_\phi B_\phi + v_z B_z)\mathcal{F}', \tag{173}$$

where now a prime denotes differentiation with respect to the flux function ψ, and specific equilibrium solutions are defined by the choice of the Casimir functions $\mathcal{F}$, $\mathcal{H}$, $\mathcal{J}$, $\mathcal{G}$, and $\mathcal{S}$ as functions of ψ. Using the definition of these Casimirs (see Sec. II B) in terms of the plasma variables, this choice allows us to bring (173) into the form of (127) and to assign the dependence on ψ of the free functions in this equation.

For the isothermal case, the internal energy is $U = c_s^2 \ln(\rho/\rho_0)$ to within a constant and the relevant combination of Casimirs is

$$\mathcal{F}B_\phi = \rho v_\phi, \tag{174}$$

$$\mathcal{F}B_z + \rho\mathcal{G} = \rho v_z, \tag{175}$$

$$\mathcal{H} + \mathcal{F}v_z = \frac{B_z}{4\pi}, \tag{176}$$

$$\mathcal{J} + v_z\mathcal{G} = v_z^2/2 + v_\phi^2/2 + c_s^2\ln(\rho/\rho_0). \tag{177}$$

The rigid rotating pinch solution that we have chosen, has B_z constant and is invariant along z, as given by (130), is obtained by choosing

$$\mathcal{F}(\hat{\psi}) = \frac{B_0}{2\pi\Omega r_0}\left[1-\left(1-\frac{w^2}{2}\right)\exp\left(-w^2\hat{\psi}\right)\right], \tag{178}$$

$$\mathcal{G}(\hat{\psi}) = -\Omega r_0 \hat{B}, \tag{179}$$

$$\mathcal{H}(\hat{\psi}) = \frac{B_0}{4\pi}\hat{B}, \tag{180}$$

$$\mathcal{J}(\hat{\psi}) = -c_s^2[w^2\hat{\psi} - \ln[1-(1-w^2/2)\exp(-w^2\hat{\psi})]], \tag{181}$$

from which by solving the generalized Grad-Shafranov equation, we obtain $\hat{\psi} = -r^2/2$ (or $B_\phi = B_0 r$) and where, in accordance with (130), the dimensionless variables $\hat{\psi} = \psi/(r_0 B_0)$, $\hat{B}$, and w are used and r is the scaled radius.

2. Eulerian pinch stability

Proceeding as described in Sec. II B, a sufficient stability condition is obtained by considering the second variation of $\mathfrak{F}[Z]$, viz., Eq. (39).

Starting from (39)–(42), we restrict the coefficients b_1, b_2, and b_3 to depend only on r, because our pinch equilibrium configuration is both azimuthally and translationally symmetric. For b_2 defined by (41), we obtain

$$b_2 = \frac{1}{r}\frac{d}{dr}\left[\frac{\partial}{\partial\psi}\left(\frac{\mathcal{M}^2}{4\pi}\right) r\psi_r\right] - \frac{\partial}{\partial\psi^2}\left(p + \frac{B_z^2}{8\pi} + \frac{\mathcal{M}^2}{4\pi}B_\phi^2\right) \tag{182}$$

and, using

$$\frac{df}{dr} = \frac{\partial f}{\partial r} + \frac{\partial f}{\partial\psi}\psi_r + \frac{\partial f}{\partial\psi_r}\frac{d\psi_r}{dr}, \tag{183}$$

and $\rho = \rho(\psi, \psi_r)$, as implicitly given by the Bernoulli functional $\mathcal{J}$, b_2 becomes

$$b_2 = \frac{\partial}{\partial\psi}\left(\frac{\mathcal{M}^2}{4\pi}\left[1 + \frac{1}{\rho}\frac{\partial\rho}{\partial\psi_r}B_\phi\right]\right)\frac{d\psi_r}{dr} - \frac{B_\phi}{r}\frac{\partial}{\partial\psi}\left(\frac{\mathcal{M}^2}{4\pi}\right) - \frac{\partial^2}{\partial\psi^2}\left(p + \frac{B_z^2}{8\pi}\right). \tag{184}$$

Finally, using the equilibrium of (128), we obtain

$$b_2 = -\frac{1}{r^2}\frac{1-\mathcal{M}^2}{4\pi} + \frac{1}{rB_\phi}\frac{d}{dr}\left(b_1 r\frac{dB_\phi}{dr}\right). \tag{185}$$

Before proceeding, let us consider some special limits. If the plasma is static, i.e., $v_\phi = 0$, we obtain $b_1 = 1/4\pi$, $b_3 = 0$, and

$$b_2 = -\frac{1}{B_\phi}\frac{d}{dr}\left[\frac{1}{B_\phi}\frac{d}{dr}\left(p + \frac{B_z^2}{8\pi}\right)\right]. \tag{186}$$

If $B_z = 0$, we obtain

$$b_1 = \frac{1}{4\pi} - \frac{1}{4\pi}\frac{\mathcal{M}^2}{1-\bar{\mathcal{M}}^2}, \tag{187}$$

$$b_3 = \frac{1}{4\pi}\frac{\mathcal{M}^2\bar{\mathcal{M}}^2}{1-\bar{\mathcal{M}}^2}, \tag{188}$$

where $\bar{\mathcal{M}}^2 = v_\phi^2/c_s^2$ is the gas dynamic Mach number, and

$$b_2 = \frac{1}{rB_\phi}\frac{d}{dr}\left[\frac{\mathcal{M}^2}{4\pi}\left(B_\phi - \frac{1}{1-\bar{\mathcal{M}}^2} r\frac{dB_\phi}{dr}\right)\right] - \frac{1}{B_\phi}\frac{d}{dr}\left(\frac{1}{B_\phi}\frac{dp}{dr}\right). \tag{189}$$

Now we return to our analysis of $\delta^2\mathfrak{F}$ of (39) for the pinch case at hand. For $\mathcal{M}^2 < 1$, a sufficient stability condition is provided by $b_1 > 0, \quad b_1 + b_3 > 0$ and $b_2 > 0$. Since $4\pi(b_1 + b_3) = 1 - \mathcal{M}^2(r) = 1 - w^2\hat{p}$, we find that $b_1 + b_3 > 0$ if $w < 1$ independent of B_z.

Using (130) in Eqs. (40), (41), and (42) we find

$$4\pi b_3 = \frac{w^4\hat{p}\left(1 - w^2\hat{p}\right)r^2}{(1 - w^2\hat{p})(1 - w^2r^2) - w^2\hat{B}^2} \tag{190}$$

and thus

$$\begin{aligned}4\pi b_1 &= -4\pi b_3 + 1 - \mathcal{M}^2\\ &= \left(1 - w^2\hat{p}\right)\left[1 - \frac{w^4\hat{p}\, r^2}{\left(1 - w^2\hat{p}\right)\left(1 - w^2r^2\right) - w^2\hat{B}^2}\right].\end{aligned} \tag{191}$$

Note that $\partial b_1/\partial\hat{B}^2 < 0$ and $b_1 > 0$ so $1 - w^2(\hat{p} + r^2 + \hat{B}^2) > 0$, which reduces (in agreement with the conditions listed above (172)) to

$$w^2(\bar{r}^2 + \hat{B}^2) < 1\,.$$

From $4\pi r^2 b_2 = -4\pi\, b_3 + 4\pi\, r\, db_1/dr$, we obtain

$$4\pi b_2 = -\frac{w^4\hat{p}\left(1 - w^2\hat{p}\right)}{(1 - w^2\hat{p})(1 - w^2r^2) - w^2\hat{B}^2} - 2\frac{d}{dr^2}\left[w^2\hat{p} + \frac{w^4\hat{p}\left(1 - w^2\hat{p}\right)r^2}{(1 - w^2\hat{p})(1 - w^2r^2) - w^2\hat{B}^2}\right]. \tag{192}$$

Note that the value of b_2 decreases with increasing $\hat{B}^2$ and that $b_2 > 0$ implies

$$w^2 < \frac{3 + \hat{B}^2 - \left(1 + 4\hat{B}^2 + \hat{B}^4\right)^{1/2}}{4 + \hat{B}^2}, \tag{193}$$

i.e., $w^2 < 1/2 - (3/8)\hat{B}^2$ for small $\hat{B}^2$, and $w^2 < 1/\hat{B}^2$ for large $\hat{B}^2$. To obtain (193), we have exploited the fact that b_2 starts to become negative at $r^2 = 0$.

For $\hat{B}^2 = 1$, we find $w^2 \lesssim 0.31$, which is more restrictive than the condition $w^2 \lesssim 0.46$ found in the Lagrangian framework below (172). This result is consistent with the expectation (see Ref. 5) that energy-Casimir stability conditions are more restrictive than the Lagrangian stability conditions.

The Euler-Lagrange equation associated with the extrema of (39) subject to the normalization constraint of constant $\int d^3x\,(\delta\psi)^2$ is

$$\nabla\cdot\left[b_1 I + b_3\left(I - \mathbf{e}_\psi\mathbf{e}_\psi\right)\right]\cdot\nabla\delta\psi - (b_2 - \lambda)\delta\psi = 0, \tag{194}$$

where λ is the Lagrange multiplier, I is the identity tensor, and $(I - \mathbf{e}_\psi\mathbf{e}_\psi)$ is the projector on the tangent plane to the ψ-surfaces. Writing $\delta\psi$ as

$$\delta\psi = \delta\hat{\psi}(r)\,\exp(im\phi), \tag{195}$$

with m the azimuthal wave number, (194) becomes

$$\frac{1}{r}\frac{d}{dr}\left[r\, b_1\frac{d\,\delta\hat{\psi}(r)}{dr}\right] - \left[\frac{m^2}{r^2}\left(b_1 + b_3\right) + (b_2 - \lambda)\right]\delta\hat{\psi}(r) = 0. \tag{196}$$

Note that b_3 becomes irrelevant for stability in the case of azimuthally symmetric perturbations.

In terms of w, $\hat{p}(r)$ and $\hat{B}$, and our shorthand $\varpi = 1 - w^2\hat{p}$, (196) takes the form

$$\begin{aligned}&\frac{1}{r}\frac{d}{dr}\left[r\,\varpi\left(1 - \frac{w^4\hat{p}\, r^2}{\varpi\left(1 - w^2r^2\right) - w^2\hat{B}^2}\right)\frac{d\,\delta\hat{\psi}}{dr}\right]\\ &-\left[\varpi\,\frac{m^2}{r^2} - \frac{\lambda}{4\pi} + \frac{w^4\hat{p}\,\varpi}{\varpi(1 - w^2r^2) - w^2\hat{B}^2}\right.\\ &\left.+2\frac{d}{d\,r^2}\left(w^2\hat{p} + \frac{w^4\hat{p}\,\varpi r^2}{\varpi(1 - w^2r^2) - w^2\hat{B}^2}\right)\right]\delta\hat{\psi} = 0.\end{aligned} \tag{197}$$

Searching for the lowest eigenvalue of the Lagrange multiplier λ as a function of w in the range

$$\frac{3 + \hat{B}^2 - \left(1 + 4\hat{B}^2 + \hat{B}^4\right)^{1/2}}{4 + \hat{B}^2} < w^2 < \frac{1}{\bar{r}^2 + \hat{B}^2}, \tag{198}$$

would yield a more accurate sufficient stability condition that could be compared with the one obtained by solving the constrained Euler-Lagrange equation derived from the functional (172). We leave it here and continue on to discuss the dynamically accessible stability.

C. Dynamically accessible pinch

1. Dynamically accessible pinch equilibria

As discussed in Sec. II C, with the dynamically accessible approach one considers the constrained variations of Eqs. (44)–(47). Upon evaluating these expressions on the pinch equilibrium of this section, expressed by (123)–(126), it is straightforward to show that $\delta H_{\rm da}$ of (48) vanishes. For example, vanishing of the coefficients of g_2 and g_3 give immediately that $\rho(r)s(r)v_\phi(r)$ and $r\rho(r)v_\phi(r)$ are constant. Evaluation of the coefficients of $\mathbf{g}_1$ and $\mathbf{g}_4$ are more tedious, but must vanish since we have shown in general that (48) gives all equilibria.

2. Dynamically accessible pinch stability

Given that $\delta H_{\rm da} = 0$, we can proceed to examine $\delta^2 H_{\rm da}$ of (49) with the variations of (44)–(47) evaluated on our rotating pinch equilibrium. Rather than starting from scratch, we will appeal to our results already obtained in Ref. 5.

For a translationally symmetric equilibrium along the z-direction, the stability condition derived from dynamically accessible variations may or may not coincide with that obtained in terms of the Lagrangian variations.[5,53] Starting from Eq. (103) of Ref. 5 with $\mathbf{h} = \mathbf{e}_z$, $k = 1$, the crucial quantity for translationally symmetric equilibria is

$$\Gamma = \begin{bmatrix} \langle 2\mathbf{B} \cdot (\mathbf{v} \cdot \nabla \mathbf{g}_1) \rangle \\ \langle \rho \mathbf{v}_\perp \cdot \nabla g_{1z} \rangle \end{bmatrix}, \tag{199}$$

where $\langle\,\rangle = \int_\psi d^2x/|\nabla\psi|$ denotes the surface integral over a flux surface. If the expression of (199) vanishes, the two kinds of stability coincide.

The first stabilizing term in $\delta^2 H_{\rm da}$ of 49, which can be eliminated in $\delta^2 H_{\rm la}$ by minimizing over Lagrangian variations, here becomes

$$\Delta = \int d^3x\, \rho |\mathbf{X}|^2, \tag{200}$$

where

$$\begin{aligned} \mathbf{X} :&= \nabla g_3 + \frac{\sigma}{\rho}\nabla g_2 + \mathbf{v} \times (\nabla \times \mathbf{g}_1) \\ &+ 2(\mathbf{v}\cdot\nabla)\mathbf{g}_1 + \frac{1}{\rho}\mathbf{B} \times (\nabla \times \mathbf{g}_4), \end{aligned} \tag{201}$$

and this term is minimum for

$$\mathbf{X}_{\rm min} = (\Xi_1/\rho)\,\mathbf{B} + \Xi_2\,\mathbf{e}_z, \tag{202}$$

where $\Xi = \mathbb{A}^{-1}\Gamma$, i.e.

$$\begin{bmatrix} \Xi_1 \\ \Xi_2 \end{bmatrix} = \begin{bmatrix} \langle |\mathbf{B}|^2/\rho \rangle & \langle B_z \rangle \\ \langle B_z \rangle & 1 \end{bmatrix}^{-1} \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix}. \tag{203}$$

For our rotating pinch example, we obtain

$$\mathbb{A} = 4\pi h \begin{bmatrix} (B_0(r^2 + b^2))/\rho & b \\ b & 1/B_0 \end{bmatrix}, \tag{204}$$

where $\pm h$ is the height of the plasma column in the $\pm z$-directions; ideally $h \to \infty$ but it cancels and does not appear in the result. Finally

$$\Gamma = \begin{bmatrix} \langle 2\mathbf{B} \cdot (\mathbf{v} \cdot \nabla \mathbf{g}_1) \rangle \\ \langle \rho \mathbf{v}_\perp \cdot \nabla g_{1z} \rangle \end{bmatrix} = \begin{bmatrix} rVB_0 \langle g_{1r} \rangle \\ 0 \end{bmatrix}. \tag{205}$$

It can be noted on general grounds that $\langle g_{1r} \rangle$ vanishes identically for perturbations that average to zero after integration over the azimuthal angle (i.e., that do not contain an $m = 0$ component). Since in Sec. IV A 2, we have shown that for our rotating pinch example azimuthally symmetric perturbations of our rotating pinch equilibrium are stable to Lagrangian perturbations, thus the restriction to dynamically accessible perturbations does not modify the stability condition. However, for general equilibria this is not true.

D. Pinch comparisons

Let us now summarize and compare our three stability approaches for the rotating pinch equilibria. In order to compare the Lagrangian and the dynamically accessible stability conditions with those obtained in the energy-Casimir framework, it is necessary to restrict our analysis to perturbations $\boldsymbol{\eta}$ that do not depend on z. This excludes "sausage" or kink type instabilities. The results of the stability analysis for such perturbations can be expressed as stability bounds on the normalized rotation frequency w. These bounds are modified by the presence of an equilibrium magnetic field along the symmetry direction, B_z, that couples the component η_z to the other components of the displacement leading in general to stricter bounds.

For the equilibrium under examination, the Lagrangian and the dynamically accessible approaches lead to equivalent conditions. Although the constraints obeyed by the dynamically accessible perturbations in the presence of flows lead to an additional stabilizing term that cannot be made to vanish for azimuthally symmetric perturbations, this term does not modify the stability analysis since azimuthally symmetric perturbations are found to be stable even within the Lagrangian framework. For more general equilibria than the ones considered here, this need not be the case.

The minimization of $\delta^2 W_{\rm la}$ of (19) for our pinch case reduced to the study of the 3×3 matrix of (155) (the 4×4 matrix for $B_z \neq 0$ of (166)) for $|m| = 1$ perturbations. Two different methods can be used: a necessary and sufficient condition for the positivity of this matrix is provided by the Sylvester criterion which yields $w^2 < 1/2$ for $B_z = 0$ and $w^2 B_z^2 < 1$ for $B_z \neq 0$ and $w^2 \to 0$. A partial minimization procedure with respect to η_ϕ (to η_z and η_ϕ for $B_z \neq 0$) leads to less restrictive conditions: $w^2 \lesssim 0.62$ for $B_z = 0$ and $w^2 \lesssim 0.46$ choosing, e.g., $B_z^2 = 1$.

Extremization of the energy-Casimir functional over all variables except $\delta\psi$ leads to sufficient stability bounds on w^2 that, similar to the Lagrangian case, become stricter as B_z^2 increases. As predicted in Ref. 5 and recalled in Sec. II, these bounds are in general more restrictive than those found within the Lagrangian framework, as shown, for e.g., by considering again $B_z^2 = 1$, in which case we find $w^2 \lesssim 0.31$. Sharper stability conditions could be obtained by solving the Euler-Lagrange equation associated with this reduced energy-Casimir functional subject to a normalization constraint on $\delta\psi$.

V. CONCLUSIONS

To summarize, we have investigated the MHD stability in the Lagrangian, Eulerian, and dynamically accessible approaches. In Sec. II, we reviewed the general properties, in particular, the time-dependent relabeling idea introduced in Ref. 5 that gives Eulerian stationary equilibria as a static state in terms of a relabeled Lagrangian variable. New details on the general comparison of the three approaches were given in Sec. II D. Then we proceeded to our two examples, the convection problem of Sec. III and the rotating pinch of Sec. IV, with comparison of the stability results for the three methods given in Secs. IV D and III D, respectively. Of note, is the explicit incorporation of the time-dependent relabeling for the rotating pinch, which to our knowledge is the first time this has been done.

As noted previously, the methods described here for the three approaches are of general utility—they apply to all important plasma models, kinetic as well as fluid, when dissipation is neglected. In fact, some time ago in Refs. 27 and 43, the approaches were compared for the Vlasov and guiding-center kinetic equations (see also Refs. 54–57), including a dynamically accessibly calculation in this kinetic context akin to the one done here and in Refs. 5 and 53 for MHD. Given the large amount of recent progress on extended magnetofluid models,[8–13,15–17] hybrid kinetic-fluid models,[18,19] and gyrokinetics[20,21] a great many stability calculations like the ones of this paper are now possible. For example, the techniques that have been used in the context of Hamiltonian reconnection[58–60] can be further adapted to explore this effect in more general models.

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APPENDIX A: LAGRANGIAN EQUATIONS OF MOTION AND ROTATING PINCH EQUILIBRIA

In order to obtain the MHD equations of motion from the Hamiltonian of (7), as described in Sec. II A, we split H into two terms $H = H_F + H_B$ where H_F is the sum of the fluid kinetic and internal energies, and H_B is the magnetic field energy given by

$$H_B = \int d^3a\, \frac{\partial q_i}{\partial a^j}\frac{\partial q^i}{\partial a^k}\frac{B_0^j B_0^k}{8\pi\mathcal{J}}. \tag{A1}$$

The functional derivative of H_F is given by (see Ref. 22 for details)

$$\frac{\delta H_F}{\delta q^i} = \frac{\pi_n\,\pi_m}{2\rho_0}\frac{\partial g^{nm}}{\partial q^i} + \frac{\partial}{\partial a^m}\left[\left(\frac{\rho_0}{\mathcal{J}}\right)^2 U_\rho\, \frac{\partial \mathcal{J}}{\partial q^j_{,m}}\right]. \tag{A2}$$

Using

$$\frac{\partial \mathcal{J}}{\partial q^i_{,m}} = A_i^m = \epsilon_{ijk}\epsilon^{mnl}\frac{1}{2}\frac{\partial q^j}{\partial a^n}\frac{\partial q^k}{\partial a^l} \tag{A3}$$

and

$$\frac{\partial A_i^m}{\partial a^m} = \frac{\partial}{\partial a^m}\epsilon_{ijk}\epsilon^{mnl}\frac{1}{2}\frac{\partial q^j}{\partial a^n}\frac{\partial q^k}{\partial a^l} = 0, \tag{A4}$$

we can rewrite Eq. (A2) as

$$\frac{\delta H_F}{\delta q^i} = \frac{\pi_n\,\pi_m}{2\rho_0}\frac{\partial g^{nm}}{\partial q^i} + A_i^m\frac{\partial}{\partial a^m}\left[\left(\frac{\rho_0}{\mathcal{J}}\right)^2 U_\rho\right]. \tag{A5}$$

Similarly for (A1) we obtain

$$\frac{\delta H_B}{\delta q^i} = \frac{\partial g_{lm}}{\partial q^i}\frac{\partial q^l}{\partial a^j}\frac{\partial q^m}{\partial a^k}\frac{B_0^j B_0^k}{8\pi\mathcal{J}} - \frac{\partial}{\partial a^j}\left(g_{im}\frac{\partial q^m}{\partial a^k}\frac{B_0^j B_0^k}{4\pi\mathcal{J}}\right) + \frac{\partial}{\partial a^t}\left(g_{lm}\frac{\partial q^l}{\partial a^j}\frac{\partial q^m}{\partial a^k}\frac{B_0^j B_0^k}{8\pi\mathcal{J}^2}\frac{\partial \mathcal{J}}{\partial q^i_{,t}}\right), \tag{A6}$$

and the Lagrangian equations of motion are given by

$$\dot{\pi}_i = -\frac{\delta H}{\delta q^i} = -\frac{\delta H_F}{\delta q^i} - \frac{\delta H_B}{\delta q^i}, \tag{A7}$$

with (A2) and (A6), and

$$\dot{q}^i = -\frac{\delta H}{\delta \pi_i} = \frac{\pi^i}{\rho_0} = g^{ij}\frac{\pi_j}{\rho_0}. \tag{A8}$$

Note that the first terms of (A2) and (A6) give the effect of non-Cartesian coordinates.

To obtain from (A7) and (A8) the Eulerian form of the equations of motion it is convenient to recall that the cofactor matrix A_k^i satisfies the identity

$$\delta_j^i\mathcal{J} = \frac{\partial q^k}{\partial a^j}A_k^i$$

and consequently

$$\frac{\partial}{\partial q^k} = \frac{\partial a^i}{\partial q^k}\frac{\partial}{\partial a^i} = \frac{A_k^i}{\mathcal{J}}\frac{\partial}{\partial a^i},$$

where $\partial/\partial q^k$ becomes ∇ in the Eulerian description. Using $p = \rho^2 U_\rho$, the second term of (A5) becomes the pressure force, and using the flux conservation expression

$$B^i = \frac{\partial q^i}{\partial a^k}\frac{B_0^k}{\mathcal{J}}, \tag{A9}$$

the last two terms of (A6) become

$$-\mathcal{J}B^j\frac{\partial}{\partial q^j}\left(\frac{B_i}{4\pi}\right) + \mathcal{J}\frac{\partial}{\partial q^i}\left(\frac{B^2}{8\pi}\right), \tag{A10}$$

where we used the divergence equation $\partial B_0^j/\partial a^j = 0$.

To facilitate our calculation of the rotating pinch equilibrium (cf. Appendix B), consider the cylindrical pinch geometry where the metric is given by (132). Evidently

$$\frac{\partial g^{nm}}{\partial q^i} = -\delta_\phi^n\,\delta_\phi^m\,\delta_i^r\,\frac{2}{(q^r)^3}, \tag{A11}$$

and consequently

$$\frac{\pi_n\,\pi_m}{2\rho_0}\frac{\partial g^{nm}}{\partial q^i} = -\delta_i^r\,\frac{\pi^\phi\pi_\phi}{q^r\rho_0}, \tag{A12}$$

and the first term of Eq. (A6) is

$$\delta_i^r\,\frac{g_{\phi\phi}}{q^r}\frac{\partial q^\phi}{\partial a^j}\frac{\partial q^\phi}{\partial a^k}\frac{B_0^j B_0^k}{4\pi\mathcal{J}} = \delta_i^r\,\frac{\mathcal{J}}{q^r}\frac{B^\phi B_\phi}{4\pi}. \tag{A13}$$

Expressions (A12) and (A13) are of use for our equilibrium calculation.

APPENDIX B: RELABELING TRANSFORMATION FOR THE PINCH

The canonical transformation induced by the time-dependent relabeling is generated by the functional

$$F[\mathbf{q},\mathbf{\Pi},t] = \int d^3a \int d^3b\ \mathbf{q}\cdot\mathbf{\Pi}\,\delta(\mathbf{a}-\mathfrak{A}(\mathbf{b},t)),$$

and yields (see Eq. (9) of Ref. 5) the new Hamiltonian of (11) according to

$$\widetilde{H}[\mathbf{Q},\mathbf{\Pi}] = H + \frac{\partial F}{\partial t},$$

with $\mathbf{V}(\mathbf{b},t)\to V^\phi(b,t) = b^r\Omega(b^r)$ for the relabeling defined by Eq. (141).

With an integration by parts involving the time derivatives of the delta functions, we obtain

$$\begin{aligned}\frac{\partial F}{\partial t} = \int d^3b \int da^r da^\phi da^z \delta(a^r-\mathfrak{A}^r)\delta(a^\phi-\mathfrak{A}^\phi)\delta(a^z-\mathfrak{A}^z)\\ \times\left[\partial_t\mathfrak{A}^r\frac{\partial}{\partial a^r}(\mathbf{q}\cdot\mathbf{\Pi}) + \partial_t\mathfrak{A}^\phi\frac{\partial}{\partial a^\phi}(\mathbf{q}\cdot\mathbf{\Pi})\right.\\ \left.+\partial_t\mathfrak{A}^z\frac{\partial}{\partial a^z}(\mathbf{q}\cdot\mathbf{\Pi})\right],\end{aligned}$$

where ∂_t denotes time derivative at a constant label b. Using $\mathbf{Q}(\mathfrak{B}(\mathbf{a},t),t) = \mathbf{q}(\mathbf{a},t)$, the first term in the bracket $\times[\ \ ]$ above becomes

$$\begin{aligned}\partial_t\mathfrak{A}^r\frac{\partial}{\partial a^r}(\mathbf{q}\cdot\mathbf{\Pi}) = \Pi_r\partial_t\mathfrak{A}^r\frac{\partial Q^r}{\partial b^i}\frac{\partial\mathfrak{B}^i}{\partial a^r}\\ +\Pi_\phi\partial_t\mathfrak{A}^r\frac{\partial Q^\phi}{\partial b^i}\frac{\partial\mathfrak{B}^i}{\partial a^r} + \Pi_z\partial_t\mathfrak{A}^r\frac{\partial Q^z}{\partial b^i}\frac{\partial\mathfrak{B}^i}{\partial a^r}.\end{aligned}\tag{B1}$$

Similar expressions follow for the other two terms. Collecting all the terms proportional to Π_r, we obtain

$$\begin{aligned}\left[\partial_t\mathfrak{A}^r\frac{\partial\mathfrak{B}^i}{\partial a^r} + \partial_t\mathfrak{A}^\phi\frac{\partial\mathfrak{B}^i}{\partial a^\phi} + \partial_t\mathfrak{A}^z\frac{\partial\mathfrak{B}^i}{\partial a^z}\right]\Pi_r\frac{\partial}{\partial b^i}Q^r\\ = -\Pi_r\cdot\dot{\mathfrak{B}}^i\frac{\partial}{\partial b^i}Q^r,\end{aligned}\tag{B2}$$

where we used the identity

$$\dot{\mathfrak{B}}^i + \frac{\partial\mathfrak{B}^i}{\partial a^r}\partial_t\mathfrak{A}^r + \frac{\partial\mathfrak{B}^i}{\partial a^\phi}\partial_t\mathfrak{A}^\phi + \frac{\partial\mathfrak{B}^i}{\partial a^z}\partial_t\mathfrak{A}^z = 0.$$

Finally, employing (13)

$$V^r = \dot{\mathfrak{B}}^r,\quad V^\phi = \dot{\mathfrak{B}}^\phi,\quad V^z = \dot{\mathfrak{B}}^z,$$

we obtain $\partial F/\partial t = -\int d^3b\ [(\mathbf{V}\cdot\nabla_b\mathbf{Q})_r\Pi_r]$.

With this additional term in the Hamiltonian (12), (133) and (134) become

$$\partial_t Q^i = \frac{\delta\widetilde{H}}{\delta\Pi_i} = g^{ij}\frac{\Pi_j}{\widetilde{\rho}_0} - V^k\frac{\partial Q^i}{\partial b^k},\tag{B3}$$

and

$$\begin{aligned}\partial_t\Pi_i = -\frac{\delta\widetilde{H}}{\delta Q^i} = \delta_i^r\frac{\Pi_\phi\Pi^\phi}{Q^r\widetilde{\rho}_0} - \widetilde{J}\frac{\partial}{\partial Q^i}\left[\left(\frac{\widetilde{\rho}_0}{\widetilde{J}}\right)^2U_\rho\right]\\ -\delta_i^r\frac{\widetilde{J}}{Q^r}\frac{\widetilde{B}_\phi\widetilde{B}^\phi}{4\pi} + \widetilde{J}\widetilde{B}^j\frac{\partial}{\partial Q^j}\left(\frac{\widetilde{B}_i}{4\pi}\right)\\ -\widetilde{J}\frac{\partial}{\partial Q^i}\left(\frac{\widetilde{B}^2}{8\pi}\right) - \frac{\partial}{\partial b^k}(V^k\Pi_i).\end{aligned}\tag{B4}$$

By assuming

$$\widetilde{B}_0^r(\mathbf{b},t) = 0,\quad \widetilde{B}_0^z(\mathbf{b},t) = 0,$$

and

$$\widetilde{B}_0^\phi(\mathbf{b},t) = \mathfrak{J}B_0^\phi(\mathfrak{A}(\mathbf{b},t)) = B_0b^r,$$

relabeled equilibria are obtained by setting $\partial_tQ^i = 0$, $\partial_t\Pi_i = 0$, and $Q^i = b^i$ in Eqs. (B3) and (B4), which yields

$$\Pi_r = \widetilde{\rho}_0V^r,\quad \Pi_\phi = (b^r)^2\widetilde{\rho}_0V^\phi,\quad \Pi_z = \widetilde{\rho}_0V^z,\tag{B5}$$

and

$$\begin{aligned}0 = \frac{\Pi^\phi\Pi_\phi}{b^r\widetilde{\rho}_0} - \frac{\partial}{\partial b^r}\left(\widetilde{\rho}_0^2U_\rho\right) - \frac{\widetilde{B}_0^\phi\widetilde{B}_{0\phi}}{4\pi b^r} - \frac{\partial}{\partial b^r}\left(\frac{\widetilde{B}_0^\phi\widetilde{B}_{0\phi}}{8\pi}\right)\\ -\frac{\partial}{\partial b^k}(V^k\Pi_r),\end{aligned}\tag{B6}$$

$$0 = -\frac{\partial}{\partial b^\phi}\left(\widetilde{\rho}_0^2U_\rho\right) - \frac{\partial}{\partial b^k}(V^k\Pi_\phi),\tag{B7}$$

$$0 = -\frac{\partial}{\partial b^z}\left(\widetilde{\rho}_0^2U_\rho\right) - \frac{\partial}{\partial b^k}(V^k\Pi_z),\tag{B8}$$

where we used the fact that $\widetilde{J} = 1$.

If we consider only equilibria with both axial and translational symmetries, i.e., $\partial/\partial b^\phi = 0$ and $\partial/\partial b^z$, then by substituting (B5) into (B7) and (B8), we obtain

$$\frac{\partial}{\partial b^r}\left(\frac{\Pi_\phi\Pi_r}{\widetilde{\rho}_0}\right) = 0\quad\text{and}\quad\frac{\partial}{\partial b^r}\left(\frac{\Pi_z\Pi_r}{\widetilde{\rho}_0}\right) = 0,\tag{B9}$$

which have the trivial solution $\Pi_r = 0$. If we assume a uniform temperature $\widetilde{T}_0$ and an initial density field $\widetilde{\rho}_0 = \widetilde{\rho}_0(b^r)$, such that the pressure $p(\widetilde{\rho}_0) = \widetilde{\rho}_0^2U_\rho(\widetilde{T}_0,\widetilde{\rho}_0)$ is the one given in (131), Eq. (B6) can be solved for Π_ϕ and, consequently, written in terms of the relabeling velocity $V^\phi = \Pi_\phi/((b^r)^2\widetilde{\rho}_0)$ in agreement with Sec. IV.

APPENDIX C: PINCH DETAILS

Here we record some formulas needed for the stability development of Sec. IV A 2. We use * to denote the complex conjugate and *c.c.* to denote the complex conjugate of the preceding term. From Eqs. (146)–(151) we obtain for the mth component of these equations, the following five terms:

$$\rho[(\mathbf{v}_\phi\cdot\nabla\mathbf{v}_\phi)\cdot(\boldsymbol{\eta}\cdot\nabla\boldsymbol{\eta})-|(\mathbf{v}_\phi\cdot\nabla\boldsymbol{\eta})|^2]_{|m|}$$
$$\to -w^2\hat{p}\,[\,(\eta_r^*\,\partial_r\eta_r+c.c)/2-i\,3m/\,(\eta_\phi^*\eta_r-c.c.)/2$$
$$+m^2(|\eta_\phi|^2+|\eta_r|^2+|\eta_z|^2)], \tag{C1}$$

$$p|(\nabla\cdot\boldsymbol{\eta})^2|_{|m|}\to(\hat{p}/r^2)[\,|\partial_r(r\eta_r)|^2+m^2|\eta_\phi|^2$$
$$-i\,m[\eta_\phi^*\partial_r(r\eta_r)-c.c.]\,], \tag{C2}$$

$$[(\eta_r\,\partial_r p)(\nabla\cdot\boldsymbol{\eta})]_{|m|}\to[(\eta_r^*/r)\,(\partial_r\hat{p})[\partial_r(r\eta_r)+im\eta_\phi]+c.c.]/2, \tag{C3}$$

$$|\nabla\times(\boldsymbol{\eta}\times\mathbf{B})|^2_{|m|}\to|(\partial_r(r\eta_r)|^2+m^2(|\eta_r|^2+|\eta_z^2)]$$
$$+(\hat{B}^2/r^2)\,[\,|(\partial_r(r\eta_r)|^2+m^2|\eta_\phi|^2$$
$$-i\,m(\eta_\phi\partial_r(r\eta_r^*)-c.c.)\,]$$
$$+(\hat{B}/r)\,[\,i\,m(\eta_z^*\partial_r(r\eta_r)-c.c.)$$
$$-m^2\,(\eta_z^*\eta_\phi+c.c.)], \tag{C4}$$

$$\mathbf{J}\times\boldsymbol{\eta}\cdot\delta\mathbf{B}\to-[\eta_r^*\partial_r(r\eta_r)+c.c.+i\,m(\eta_\phi^*\eta_r-c.c.)]\,. \tag{C5}$$

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