

Translationally symmetric extended MHD via Hamiltonian reduction: Energy-Casimir equilibria

D. A. Kaltsas, , G. N. Throumoulopoulos, and , and P. J. Morrison

Citation: [Physics of Plasmas](#) **24**, 092504 (2017); doi: 10.1063/1.4986013

View online: <http://dx.doi.org/10.1063/1.4986013>

View Table of Contents: <http://aip.scitation.org/toc/php/24/9>

Published by the [American Institute of Physics](#)

**COMPLETELY
REDESIGNED!**



**PHYSICS
TODAY**

Physics Today Buyer's Guide
Search with a purpose.

Translationally symmetric extended MHD via Hamiltonian reduction: Energy-Casimir equilibria

D. A. Kaltsas,^{1,a)} G. N. Throumoulopoulos,^{1,b)} and P. J. Morrison^{2,c)}

¹*Department of Physics, University of Ioannina, GR 451 10 Ioannina, Greece*

²*Department of Physics and Institute for Fusion Studies, University of Texas, Austin, Texas 78712, USA*

(Received 31 May 2017; accepted 23 July 2017; published online 11 August 2017)

The Hamiltonian structure of ideal translationally symmetric extended MHD (XMHD) is obtained by employing a method of Hamiltonian reduction on the three-dimensional noncanonical Poisson bracket of XMHD. The existence of the continuous spatial translation symmetry allows the introduction of Clebsch-like forms for the magnetic and velocity fields. Upon employing the chain rule for functional derivatives, the 3D Poisson bracket is reduced to its symmetric counterpart. The sets of symmetric Hall, Inertial, and extended MHD Casimir invariants are identified, and used to obtain energy-Casimir variational principles for generalized XMHD equilibrium equations with arbitrary macroscopic flows. The obtained set of generalized equations is cast into Grad-Shafranov-Bernoulli (GSB) type, and special cases are investigated: static plasmas, equilibria with longitudinal flows only, and Hall MHD equilibria, where the electron inertia is neglected. The barotropic Hall MHD equilibrium equations are derived as a limiting case of the XMHD GSB system, and a numerically computed equilibrium configuration is presented that shows the separation of ion-flow from electromagnetic surfaces. [<http://dx.doi.org/10.1063/1.4986013>]

I. INTRODUCTION

By extended MHD (XMHD), we mean the one-fluid model obtained by reduction of the standard two-fluid plasma model, when the quasineutrality assumption is imposed and expansion in the smallness of electron mass is performed (e.g., Ref. 1). The resulting model has a generalized Ohm's law that contains Hall drift and electron inertia physics, and it was proven in Ref. 2 that energy conservation for ideal XMHD, the version treated in this paper, requires the addition of a commonly neglected term in the momentum equation related to the electron inertia.

Despite the complexity of XMHD, it was shown in Refs. 3 and 4 to possess a Hamiltonian structure. Moreover, in Ref. 4, remarkable connections with the Hamiltonian structure of other models were established, viz., Hall MHD (HMHD) (e.g., Ref. 5) Inertial MHD (IMHD) (e.g., Refs. 2 and 6) and the usual ideal MHD (highlighted in Ref. 7). In addition, the derivation of XMHD and its Hamiltonian structure from its underlying Lagrangian variable action functional was reported in Ref. 8. Recently, the Hamiltonian structure of two-dimensional incompressible XMHD was derived in Ref. 9, a reduced XMHD (RXMHD) model that was used to study Hamiltonian reconnection due to the Hall and electron inertial terms. The Hamiltonian structure of a similar collisionless fluid reconnection model was established earlier in Ref. 10, and a general treatment of reduced Hamiltonian models was given in Ref. 11.

Detailed consequences of the original noncanonical Hamiltonian structure of Morrison and Greene,¹² were explored in a series of papers,^{14–18} including various

variational principles for equilibria, and their use in ascertaining stability via energy principles that incorporate different constraints. Given that XMHD is a Hamiltonian theory and that the investigations of Refs. 14–18 are generic to Hamiltonian theories, all of the considerations of these and other works can be worked out for XMHD. This is the main motivation for conducting this study in the framework of non-canonical Hamiltonian mechanics, i.e., since XMHD is a Hamiltonian theory, the existence of the aforementioned variational principles provides us with a joint tool for the derivation of equilibrium equations and stability criteria. This study is focused on equilibria, but may serve as a starting point for a stability analysis as well. Also, the Hamiltonian formalism is helpful in order to analyze and describe the geometrical structure of the dynamics, e.g., the existence of the so-called Casimir invariants that affect the topological structure of the phase-space, by constraining the dynamics to evolve within specific regions. Lastly, the Hamiltonian description may provide the means for the construction of conservative algorithms for numerical analyses.¹⁹ All these indicate that a Hamiltonian description, whenever possible, is preferable.

The present paper considers the case of translationally symmetric compressible plasmas with an emphasis on equilibrium analyses. We derive the Hamiltonian structure of this translationally symmetric model by applying a method of Hamiltonian reduction, which was used in Ref. 14, on the parent three-dimensional (3D) model. Specifically, we employ a chain rule reduction on the functional derivatives of the noncanonical Poisson bracket of XMHD in order to obtain a bracket expressed in terms of Clebsch-like variables that globally describe the velocity and the magnetic field. This reduction leads to the identification of translationally symmetric Casimir invariants, which due to the spatial symmetry, form infinite families of generalized helicities.

^{a)}dkaltsas@cc.uoi.gr

^{b)}gthroum@cc.uoi.gr

^{c)}morrison@physics.utexas.edu

Exploiting these invariants along with the Hamiltonian functional, written in terms of the aforementioned variables, we formulate an Energy-Casimir variational principle that leads to generalized equilibrium equations describing translationally symmetric XMHD equilibria with flows, a set of equations that we cast into the form of a Grad-Shafranov-Bernoulli (GSB) system, which makes the equilibrium study of the model tractable.

In comparison to MHD, equilibrium and stability calculations for XMHD are considerably more complex. This is because XMHD contains additional physics, viz., XMHD includes the two-fluid phenomena of Hall drift and electron inertia, arising from the individual fluid dynamics of ions and electrons, while maintaining quasineutrality. This gives rise to a plethora of new effects as evidenced by the complexity of the linear modes present in XMHD (see e.g., Refs. 1 and 20). Even in a reduced two-dimensional case, linear and nonlinear physics are significantly modified, and the phenomenon of collisionless reconnection emerges.⁹ Even the simpler case of HMHD, which we will address, contains significant complexities not included in MHD.

The present study is organized as follows: in Sec. II, we review the Hamiltonian field theory of XMHD, along with some basic aspects and features of noncanonical Hamiltonian mechanics. Before proceeding to the Hamiltonian reduction, we present as a preliminary application, the 3D Energy-Casimir variational principle for deriving equilibrium conditions. In Sec. III, we introduce appropriate representations for the magnetic and velocity fields, which ensure that they respect translation symmetry and additionally render the magnetic field divergence free. Using this representation, the Hamiltonian and the XMHD Poisson bracket are reduced to their translationally symmetric counterparts. The Casimir invariants of the symmetric Poisson bracket are computed, and their HMHD and IMHD analogues are presented. In Sec. IV, we establish the symmetric variational principle, from which we derive generalized equilibrium equations. Special cases of equilibria are discussed and studied in detail as applications. In Sec. V, we conclude and discuss extensions of the present study.

II. NONCANONICAL HAMILTONIAN STRUCTURE OF XMHD

A. Hamiltonian formulation

The dynamic equations of the XMHD model, written in the standard Alfvén units, are the following:

$$\partial_t \rho = -\nabla \cdot (\rho \mathbf{v}), \quad (1)$$

$$\begin{aligned} \partial_t \mathbf{v} = & \mathbf{v} \times \nabla \times \mathbf{v} - \nabla \left(h + \frac{v^2}{2} \right) + \rho^{-1} \mathbf{J} \times \mathbf{B}^* \\ & - d_e^2 \nabla \left(\frac{|\nabla \times \mathbf{B}|^2}{2\rho^2} \right), \end{aligned} \quad (2)$$

$$\begin{aligned} \partial_t \mathbf{B}^* = & \nabla \times (\mathbf{v} \times \mathbf{B}^*) - d_i \nabla \times \left[\frac{\mathbf{J} \times \mathbf{B}^*}{\rho} \right] \\ & + d_e^2 \nabla \times \left[\frac{\mathbf{J} \times (\nabla \times \mathbf{v})}{\rho} \right], \end{aligned} \quad (3)$$

where

$$\mathbf{J} = \nabla \times \mathbf{B}, \quad (4)$$

$$\mathbf{B}^* = \mathbf{B} + d_e^2 \nabla \times \left(\frac{\nabla \times \mathbf{B}}{\rho} \right). \quad (5)$$

Here, a barotropic equation of state has been assumed, which means the enthalpy h is related to pressure by $\nabla h = \rho^{-1} \nabla p$, and the parameters d_i and d_e are the normalized ion and electron skin depths, respectively, with $d_s = c/(\omega_{ps} L)$ and $s = i, e$.

As already mentioned in Sec. I, the Hamiltonian structure of the XMHD model of Eqs. (1)–(3) was obtained in Refs. 3 and 4. More precisely, it was shown that the equations of motion can be reproduced by using a Hamiltonian “apparatus” (see Ref. 13 for a comprehensive review) that consists of the Hamiltonian function

$$\begin{aligned} \mathcal{H} = & \frac{1}{2} \int_V d^3x \left[\rho v^2 + 2\rho U(\rho) + B^2 + d_e^2 \frac{|\nabla \times \mathbf{B}|^2}{\rho} \right] \\ = & \frac{1}{2} \int_V d^3x \left[\rho v^2 + 2\rho U(\rho) + \mathbf{B} \cdot \mathbf{B}^* \right], \end{aligned} \quad (6)$$

where $V \subseteq \mathbb{R}^3$, and the noncanonical Poisson bracket

$$\begin{aligned} \{F, G\} = & \int_V d^3x \{ G_\rho \nabla \cdot F_{\mathbf{v}} - F_\rho \nabla \cdot G_{\mathbf{v}} \\ & + \rho^{-1} (\nabla \times \mathbf{v}) \cdot (F_{\mathbf{v}} \times G_{\mathbf{v}}) \\ & + \rho^{-1} \mathbf{B}^* \cdot [F_{\mathbf{v}} \times (\nabla \times G_{\mathbf{B}^*}) - G_{\mathbf{v}} \times (\nabla \times F_{\mathbf{B}^*})] \\ & - d_i \rho^{-1} \mathbf{B}^* \cdot [(\nabla \times F_{\mathbf{B}^*}) \times (\nabla \times G_{\mathbf{B}^*})] \\ & + d_e^2 \rho^{-1} (\nabla \times \mathbf{v}) \cdot [(\nabla \times F_{\mathbf{B}^*}) \times (\nabla \times G_{\mathbf{B}^*})] \}, \end{aligned} \quad (7)$$

where $F_u := \delta F / \delta u$ denotes the functional derivative of F with respect to the dynamic variable u . The bracket of (7) generalizes the original MHD bracket of Ref. 12 by replacing \mathbf{B} by \mathbf{B}^* , and the addition of the terms involving d_e and d_i .

It is evident that the Poisson bracket is antisymmetric, and in Ref. 3, the authors proved by a tedious calculation (simplified in Ref. 4) that it satisfies the Jacobi identity. In view of (6) and (7), the equations of motion can be cast into the following Hamiltonian form:

$$\partial_t \mathbf{u} = \{ \mathbf{u}, \mathcal{H} \}, \quad (8)$$

with $\mathbf{u} = (\rho, \mathbf{v}, \mathbf{B}^*)$. Equation (8) can also be written as

$$\partial_t u^i = J^{ij} \frac{\delta \mathcal{H}}{\delta u^j}, \quad (9)$$

where J^{ij} is the Poisson operator that defines the Poisson bracket according to $\{F, G\} := \langle F_u^i, J^{ij} G_u^j \rangle$ with $\langle \cdot, \cdot \rangle$ being a pairing defined on the phase(function)-space. A characteristic feature of Poisson brackets of form (7) is that they have nontrivial kernels, i.e., there exist functionals \mathcal{C} that satisfy

$$\{F, \mathcal{C}\} = 0, \quad \forall F. \quad (10)$$

Such functionals \mathcal{C} , called Casimirs, are global invariants of the dynamical evolution. From (10), it is evident that the equations of motion are unaffected by the addition of the Casimir invariants to the Hamiltonian \mathcal{H} . Therefore, if we define a family of Hamiltonians, $\mathfrak{H} = \mathcal{H} - \sum_i \mathcal{C}_i$, one can freely write

$$\partial_t \mathbf{u} = \{\mathbf{u}, \mathfrak{H}\}, \quad (11)$$

instead of (8) without changing the resultant equations of motion. It is clear that stationary states are solutions of equations $\{\mathbf{u}, \mathfrak{H}\} = 0$. Hence, from Eq. (9), we understand that $\partial_t \mathbf{u} = 0$ follows from vanishing of the first variation of the generalized Hamiltonian functional \mathfrak{H} (the Energy-Casimir functional), i.e., equilibrium states satisfy the condition

$$\delta \mathfrak{H} = \delta \left(\mathcal{H} - \sum_i \mathcal{C}_i \right) = 0. \quad (12)$$

As pointed out in Refs. 13 and 21, in general, not all equilibria emerge from such variations because of singularities in the Poisson bracket operator.

Regarding the dynamical evolution, the Casimirs play a topological role in the structure of the phase space, since the motion takes place on phase-space surfaces that are the Casimir level sets, commonly called symplectic leaves. Assigning initial values to the Casimirs is equivalent to the choice of a symplectic leaf, i.e., a particular sub-space of the phase space on which the motion is restricted. The intersection with the energy level sets confines the trajectory of the dynamical evolution, and the Energy-Casimir extremal points correspond to equilibrium states. We also note that the stability of equilibrium points depends on the behavior of the second variation of the Energy-Casimir functional.

The Casimir invariants for 3D barotropic XMHD have been calculated in Refs. 3 and 4. The total mass, as it is expected, is conserved, and there also exist two generalized Helicity-like invariants

$$\mathcal{C}_1 = \int_V d^3x \rho, \quad (13)$$

$$\mathcal{C}_{2,3} = \int_V d^3x (\mathbf{A}^* + \lambda_{\pm}^{-1} \mathbf{v}) \cdot (\mathbf{B}^* + \lambda_{\pm}^{-1} \nabla \times \mathbf{v}), \quad (14)$$

with $\mathbf{B}^* = \nabla \times \mathbf{A}^*$ and λ_{\pm} being the two roots of the quadratic equation $1 - d_i \lambda - d_e^2 \lambda^2 = 0$, that is $\lambda_{\pm} = (-d_i \pm \sqrt{d_i^2 + 4d_e^2}) / (2d_e^2)$. The two invariants in (14) have forms similar to the canonical self-helicities of the two-fluid model, which are composites of the fluid and magnetic momenta. However, they are not identical to the canonical helicities of the 2-fluid model since in the framework of XMHD, quasineutrality and smallness of the electron to ion mass ratio are assumed. The XMHD ‘‘canonical momenta’’ are proportional to $\mathbf{v} + \lambda_{\pm} \mathbf{A}$. Regarding the physical interpretation of these invariants, in comparison to the ordinary MHD magnetic helicity, a measure of the twist and the linkage of magnetic flux tubes, these generalized helicities can be seen to measure

the twist and linkage of the flux tubes of the generalized fields $\mathbf{B} + \lambda_{\pm}^{-1} \nabla \times \mathbf{v}$ (or generalized vorticities). The two parameters λ_{\pm} account for the differential motion of electrons and ions departing from magnetic field lines.

B. 3D energy-Casimir variational principle

The Energy-Casimir variational principle, employing the general 3D expressions for the energy and Casimir invariants, leads to equilibrium conditions satisfied by the magnetic and velocity fields. In the framework of single-fluid MHD, the magnetic fields that are solutions of (12) satisfy the so-called Beltrami condition: $\nabla \times \mathbf{B} = \kappa \mathbf{B}$, with the fluid velocity being parallel to \mathbf{B} . In the context of XMHD, due to the form of the Hamiltonian and Casimir functionals, the magnetic and velocity fields satisfy more complicated, coupled conditions that allow more complex field configurations. These conditions can be derived from the vanishing of the first variation of the Energy-Casimir functional \mathfrak{H} , i.e., by requiring the vanishing of the coefficients of the arbitrary variations of ρ , \mathbf{v} , and \mathbf{B}^*

$$\delta \mathfrak{H} = \delta \int_V d^3x \left\{ \rho \frac{v^2}{2} + \rho U(\rho) + \frac{\mathbf{B} \cdot \mathbf{B}^*}{2} - \alpha \rho - \beta_+ (\mathbf{A}^* + \lambda_+^{-1} \mathbf{v}) \cdot (\mathbf{B}^* + \lambda_+^{-1} \nabla \times \mathbf{v}) - \beta_- (\mathbf{A}^* + \lambda_-^{-1} \mathbf{v}) \cdot (\mathbf{B}^* + \lambda_-^{-1} \nabla \times \mathbf{v}) \right\} = 0, \quad (15)$$

with $\mathbf{B} = \nabla \times \mathbf{A}$ as usual and $\mathbf{A}^* = \mathbf{A} + d_e^2 \nabla \times \mathbf{B} / \rho$. The parameters α and β_{\pm} are Lagrangian multipliers with values related to the total mass and total generalized helicities. Equation (15) leads to the following conditions:

$$\nabla \times \mathbf{B} = 2(\beta_+ + \beta_-) \mathbf{B}^* + 2(\beta_+ \lambda_+^{-1} + \beta_- \lambda_-^{-1}) \nabla \times \mathbf{v}, \quad (16)$$

$$\rho \mathbf{v} = 2(\beta_+ \lambda_+^{-1} + \beta_- \lambda_-^{-1}) \mathbf{B}^* + 2(\beta_+ \lambda_+^{-2} + \beta_- \lambda_-^{-2}) \nabla \times \mathbf{v}, \quad (17)$$

$$\frac{v^2}{2} + h(\rho) - d_e^2 \frac{|\mathbf{J}|^2}{2\rho^2} + 2 \frac{d_e^2 (\beta_+ + \beta_-)}{\rho^2} \mathbf{B}^* \cdot \mathbf{J} + 2 \frac{d_e^2 (\beta_+ \lambda_+^{-1} + \beta_- \lambda_-^{-1})}{\rho^2} \mathbf{J} \cdot (\nabla \times \mathbf{v}) = \alpha, \quad (18)$$

where the enthalpy $h(\rho) = [\rho U(\rho)]_{\rho}$. The enthalpy is related to pressure $P(\rho)$ through the following relation:

$$h(\rho) = \int \frac{dP(\rho)}{\rho}. \quad (19)$$

That the enthalpy h of (19) depends only on a single thermodynamic variable, the specific volume ρ^{-1} , follows from the barotropic assumption embodied in the choice of internal energy per unit mass U . For simplicity, our Hamiltonian formulation was restricted in this way, but it can be generalized to include more thermodynamic variables such as entropy

per unit mass (see, e.g., Ref. 18 for MHD and Ref. 22 for XMHD). A common choice for this barotropic thermodynamic closure is the polytropic equation of state, where $P(\rho) = \kappa \rho^\gamma$ with a constant κ (independent of entropy). With this choice,

$$h(\rho) = \frac{\gamma}{\gamma-1} \frac{P(\rho)}{\rho}. \quad (20)$$

From (16) and (18), we obtain the XMHD analogue of the Bernoulli equation, which reveals the distribution of the pressure, for the velocity and the magnetic field described by the mutual solutions of the coupled equations (16) and (17)

$$\tilde{P}(\rho) = \alpha\rho - \rho \frac{v^2}{2} - \frac{d_c^2}{2\rho} |\nabla \times \mathbf{B}|^2, \quad (21)$$

where $\tilde{P} = \gamma P / (\gamma - 1)$.

III. SYMMETRIC FORMULATION VIA CHAIN RULE ON FUNCTIONAL DERIVATIVES

A. Translationally symmetric Poisson bracket

Assuming continuous translational symmetry and adopting a Cartesian system (x, y, z) , the fields \mathbf{B}^* and \mathbf{v} can be written as follows:

$$\mathbf{B}^* = B_z^*(x, y, t) \hat{z} + \nabla \psi^*(x, y, t) \times \hat{z}, \quad (22)$$

$$\mathbf{v} = v_z(x, y, t) \hat{z} + \nabla \chi(x, y, t) \times \hat{z} + \nabla \Upsilon(x, y, t), \quad (23)$$

where z is the ignorable coordinate with the corresponding unit vector \hat{z} along the direction of translational invariance, ψ^* is the poloidal flux function of \mathbf{B}^* and χ and Υ are Clebsch-like potentials of the poloidal velocity. The form of Eq. (22) ensures that the condition $\nabla \cdot \mathbf{B} = 0$ holds, while the existence of the term $\nabla \Upsilon$ in Eq. (23) allows for compressibility of the flow [provided the function $\Upsilon(x, y)$ is not harmonic]. Upon setting $\Upsilon = 0$ or $\Delta \Upsilon = 0$ ($\Delta \equiv \nabla^2$, the Laplacian), we can impose incompressibility of the flow. Note that in view of the translational symmetry, the representation adopted for the velocity field is consistent with the Helmholtz decomposition theorem and hence it is generic for the description of any kind of symmetric flow. Taking the divergence and the curl of Eqs. (22) and (23) give

$$\nabla \cdot \mathbf{v} = \Delta \Upsilon, \quad (24)$$

$$\nabla \times \mathbf{v} = -\Delta \chi \hat{z} + \nabla v_z \times \hat{z}, \quad (25)$$

$$\nabla \cdot \mathbf{B}^* = 0, \quad (26)$$

$$\nabla \times \mathbf{B}^* = -\Delta \psi^* \hat{z} + \nabla B_z^* \times \hat{z}. \quad (27)$$

For convenience, we define the following quantities: $w := \Delta \Upsilon$ or $\Upsilon = \Delta^{-1} w$ and $\Omega = -\Delta \chi$ or $\chi = -\Delta^{-1} \Omega$.

The transition from the general 3D Hamiltonian model to a translationally symmetric one is accomplished by expressing the Hamiltonian (6) and the Poisson bracket (7), which are expressed in terms of the state vector $\mathbf{u} = \{\rho, \mathbf{v}, \mathbf{B}^*\}$, to those in terms of the symmetric state vector $\mathbf{u}_{TS} = \{\rho, v_z, \chi, \Upsilon, B_z^*, \psi^*\}$. This reduction of phase space is achieved by

mapping the functional derivatives with respect to the original variables \mathbf{u} to functional derivatives with respect to the variables \mathbf{u}_{TS} . This mapping is computed using the chain rule for functional derivatives, obtained by equating first variations in terms of the two sets of variables. The variation of a functional $F[\rho, \mathbf{v}, \mathbf{B}^*]$ is

$$\delta F[\mathbf{u}] = \int_V d^3x (F_\rho \delta \rho + F_{\mathbf{v}} \delta \mathbf{v} + F_{\mathbf{B}^*} \delta \mathbf{B}^*), \quad (28)$$

while that in terms of \mathbf{u}_{TS} is

$$\delta F[\mathbf{u}_{TS}] = \int_D d^2x [F_\rho \delta \rho + F_{v_z} \delta v_z + F_\chi \delta \chi + F_\Upsilon \delta \Upsilon + F_{B_z^*} \delta B_z^* + F_{\psi^*} \delta \psi^*], \quad (29)$$

where $D \subseteq R^2$ is a restriction of V to R^2 . Using $\delta \chi = -\Delta^{-1} \delta \Omega = -\Delta^{-1} (\hat{z} \cdot \nabla \times \delta \mathbf{v})$, $\delta \Upsilon = \Delta^{-1} \delta w = \Delta^{-1} (\nabla \cdot \delta \mathbf{v})$, and $\delta \psi^* = -\Delta^{-1} (\hat{z} \cdot \nabla \times \delta \mathbf{B}^*)$, (29) can be rewritten as

$$\delta F = \int_D d^2x \{ F_\rho \delta \rho + F_{v_z} \hat{z} \cdot \delta \mathbf{v} - F_\chi \Delta^{-1} (\hat{z} \cdot \nabla \times \delta \mathbf{v}) + F_\Upsilon \Delta^{-1} \nabla \cdot \delta \mathbf{v} + F_{B_z^*} \hat{z} \cdot \delta \mathbf{B}^* - F_{\psi^*} \Delta^{-1} (\hat{z} \cdot \nabla \times \delta \mathbf{B}^*) \}. \quad (30)$$

Then, from the self-adjointness of the operator Δ^{-1} and for appropriate boundary conditions such that the boundary terms vanish, we obtain

$$\begin{aligned} \delta F &= \int_D d^2x (F_\rho \delta \rho + F_{\mathbf{v}} \cdot \delta \mathbf{v} + F_{\mathbf{B}^*} \cdot \delta \mathbf{B}^*), \quad (31) \\ &= \int_D d^2x [F_\rho \delta \rho + (F_{v_z} \hat{z} + \nabla F_\Omega \times \hat{z} - \nabla F_w) \cdot \delta \mathbf{v} \\ &\quad + (F_{B_z^*} \hat{z} - \nabla (\Delta^{-1} F_{\psi^*}) \times \hat{z}) \cdot \delta \mathbf{B}^*], \quad (32) \end{aligned}$$

where we have used the following relations:

$$F_\Upsilon = \Delta F_w, \quad F_\chi = -\Delta F_\Omega, \quad (33)$$

which come from

$$\int_D d^2x F_\Upsilon \delta \Upsilon = \int_D d^2x F_w \delta w = \int_D d^2x \Delta F_w \delta \Upsilon, \quad (34)$$

$$\int_D d^2x F_\chi \delta \chi = \int_D d^2x F_\Omega \delta \Omega = -\int_D d^2x \Delta F_\Omega \delta \chi, \quad (35)$$

since the variations $\delta \chi$ and $\delta \Upsilon$ are arbitrary. Upon comparing (31) with (32), the following relations are deduced:

$$F_{\mathbf{v}} = F_{v_z} \hat{z} + \nabla F_\Omega \times \hat{z} - \nabla F_w, \quad (36)$$

$$F_{\mathbf{B}^*} = F_{B_z^*} \hat{z} - \nabla (\Delta^{-1} F_{\psi^*}) \times \hat{z}, \quad (37)$$

$$\nabla \times F_{\mathbf{B}^*} = F_{\psi^*} \hat{z} + \nabla F_{B_z^*} \times \hat{z}. \quad (38)$$

Substituting Eqs. (22), (23), (36), (37), and (38) into the Poisson bracket of XMHD given by (7), we obtain the translationally symmetric Poisson bracket of barotropic XMHD (see Appendix A for details)

$$\begin{aligned}
\{F, G\}_{TS}^{XMHD} = & \int_D d^2x \{F_\rho \Delta G_w - G_\rho \Delta F_w + \rho^{-1} \Omega ([F_\Omega, G_\Omega] + [F_w, G_w] + \nabla F_w \cdot \nabla G_\Omega - \nabla F_\Omega \cdot \nabla G_w) + v_z ([F_\Omega, \rho^{-1} G_{v_z}] \\
& - [G_\Omega, \rho^{-1} F_{v_z}] + \nabla (\rho^{-1} G_{v_z}) \cdot \nabla F_w - \nabla (\rho^{-1} F_{v_z}) \cdot \nabla G_w + \rho^{-1} F_\Upsilon G_{v_z} - \rho^{-1} G_\Upsilon F_{v_z}) + \psi^* ([F_\Omega, \rho^{-1} G_{\psi^*}] \\
& - [G_\Omega, \rho^{-1} F_{\psi^*}] + [F_{B_z^\pm}, \rho^{-1} G_{v_z}] - [G_{B_z^\pm}, \rho^{-1} F_{v_z}] + \nabla F_w \cdot \nabla (\rho^{-1} G_{\psi^*}) - \nabla G_w \cdot \nabla (\rho^{-1} F_{\psi^*}) + \rho^{-1} F_\Upsilon G_{\psi^*} \\
& - \rho^{-1} G_\Upsilon F_{\psi^*}) + \rho^{-1} B_z^* ([F_\Omega, G_{B_z^\pm}] - [G_\Omega, F_{B_z^\pm}] + \nabla F_w \cdot \nabla G_{B_z^\pm} - \nabla G_w \cdot \nabla F_{B_z^\pm}) + d_i \psi^* ([G_{B_z^\pm}, \rho^{-1} F_{\psi^*}] \\
& - [F_{B_z^\pm}, \rho^{-1} G_{\psi^*}]) - d_i \rho^{-1} B_z^* [F_{B_z^\pm}, G_{B_z^\pm}] + d_e^2 \rho^{-1} \Omega [F_{B_z^\pm}, G_{B_z^\pm}] + d_e^2 v_z ([F_{B_z^\pm}, \rho^{-1} G_{\psi^*}] - [G_{B_z^\pm}, \rho^{-1} F_{\psi^*}]) \}, \quad (39)
\end{aligned}$$

where $[a, b] := (\nabla a \times \nabla b) \cdot \hat{z} = (\partial_x a)(\partial_y b) - (\partial_x b)(\partial_y a)$. Here, we exploited the identity

$$\int_D d^2x [a, b]c = \int_D d^2x [c, a]b = \int_D d^2x [b, c]a, \quad (40)$$

which holds for arbitrary functionals a, b, c under appropriate boundary conditions (e.g., periodic boundary conditions).

The antisymmetry of the bracket (39) follows naturally from the antisymmetry of the Poisson bracket $[a, b] = -[b, a]$ and the vanishing of the boundary terms arising from integration by parts and Gauss' theorem. The Jacobi identity of (39) follows because of the reduction procedure. Similarly, by substitution, the symmetric representation of the Hamiltonian is given by

$$\begin{aligned}
\mathcal{H}_{TS}^{XMHD} = & \frac{1}{2} \int_D d^2x \left\{ \rho (v_z^2 + |\nabla \chi|^2 + |\nabla \Upsilon|^2) + 2\rho ([\Upsilon, \chi] + U(\rho)) + B_z^2 + |\nabla \psi|^2 + \frac{d_e^2}{\rho} [(\Delta \psi)^2 + |\nabla B_z|^2] \right\} \\
= & \int_D d^2x \left\{ \frac{\rho}{2} (v_z^2 + |\nabla \chi|^2 + |\nabla \Upsilon|^2) + \rho ([\Upsilon, \chi] + U(\rho)) + \frac{B_z^* B_z}{2} + \frac{\nabla \psi^* \cdot \nabla \psi}{2} \right\}. \quad (41)
\end{aligned}$$

With (41), the translationally symmetric equations of motion take the form $\partial_t \mathbf{u}_{TS} = \{\mathbf{u}_{TS}, \mathcal{H}_{TS}\}_{TS}^{XMHD}$. The bracket (39) has a more complicated form than its MHD counterpart obtained in Ref. 14, due to the terms that originate from the ion and electron contributions, having coefficients d_i and d_e , respectively. However, a remarkable transformation introduced in Ref. 4 can simplify it. The new transformed bracket has the form of the translationally symmetric HMHD Poisson bracket, which can be obtained by setting $d_e = 0$ in (39), but with dependence on a generalized magnetic field variable

$$\mathbf{B}_\pm = \mathbf{B}^* + \lambda_\pm^{-1} \nabla \times \mathbf{v}. \quad (42)$$

The new magnetic field variable \mathbf{B}_\pm , in view of Eqs. (22) and (23) can be written as

$$\mathbf{B}_\pm = \mathbf{B}^* + \lambda_\pm^{-1} \nabla \times \mathbf{v} = (B_z^* + \lambda_\pm^{-1} \Omega) \hat{z} + \nabla (\psi^* + \lambda_\pm^{-1} v_z) \times \hat{z} = B_z^\pm \hat{z} + \nabla \psi_\pm \times \hat{z}, \quad (43)$$

i.e., we have

$$B_z^\pm = B_z^* + \lambda_\pm^{-1} \Omega, \quad \psi_\pm = \psi^* + \lambda_\pm^{-1} v_z. \quad (44)$$

We can prove that under the change

$$\{\rho, v_z, \Omega, \Upsilon, B_z^*, \psi^*\} \leftrightarrow \{\rho, v_z, \Omega, \Upsilon, B_z^\pm, \psi_\pm\}$$

the functional derivatives change as follows:

$$F_{v_z} \rightarrow F_{v_z} + \lambda_\pm^{-1} F_{\psi_\pm}, \quad F_\Omega \rightarrow F_\Omega + \lambda_\pm^{-1} F_{B_z^\pm}, \quad F_\Upsilon \rightarrow F_\Upsilon, \quad F_{\psi^*} \rightarrow F_{\psi_\pm}, \quad F_{B_z^*} \rightarrow F_{B_z^\pm}, \quad (45)$$

with the change in variables of (44). Upon inserting the transformation of the functional derivatives of (45) into (39), we obtain the following bracket:

$$\begin{aligned}
\{F, G\}_{TS}^{XMHD} = & \int_D d^2x \{F_\rho \Delta G_w - G_\rho \Delta F_w + \rho^{-1} \Omega ([F_\Omega, G_\Omega] + [F_w, G_w] + \nabla F_w \cdot \nabla G_\Omega - \nabla F_\Omega \cdot \nabla G_w) + v_z ([F_\Omega, \rho^{-1} G_{v_z}] - [G_\Omega, \rho^{-1} F_{v_z}] \\
& + \nabla (\rho^{-1} G_{v_z}) \cdot \nabla F_w - \nabla (\rho^{-1} F_{v_z}) \cdot \nabla G_w + \rho^{-1} F_\Upsilon G_{v_z} - \rho^{-1} G_\Upsilon F_{v_z}) + \psi_\pm ([F_\Omega, \rho^{-1} G_{\psi_\pm}] - [G_\Omega, \rho^{-1} F_{\psi_\pm}] \\
& + [F_{B_z^\pm}, \rho^{-1} G_{v_z}] - [G_{B_z^\pm}, \rho^{-1} F_{v_z}] + \nabla F_w \cdot \nabla (\rho^{-1} G_{\psi_\pm}) - \nabla G_w \cdot \nabla (\rho^{-1} F_{\psi_\pm}) + \rho^{-1} F_\Upsilon G_{\psi_\pm} - \rho^{-1} G_\Upsilon F_{\psi_\pm}) \\
& + \rho^{-1} B_z^\pm ([F_\Omega, G_{B_z^\pm}] - [G_\Omega, F_{B_z^\pm}] + \nabla F_w \cdot \nabla G_{B_z^\pm} - \nabla G_w \cdot \nabla F_{B_z^\pm}) - \nu_\pm \rho^{-1} B_z^\pm [F_{B_z^\pm}, G_{B_z^\pm}] \\
& + \nu_\pm \psi_\pm ([G_{B_z^\pm}, \rho^{-1} F_{\psi_\pm}] - [F_{B_z^\pm}, \rho^{-1} G_{\psi_\pm}]) \}, \quad (46)
\end{aligned}$$

where $\nu_\pm := d_i - 2\lambda_\pm^{-1}$.

As was the case for (39), the bracket (46) with the Hamiltonian (41) generate the translationally symmetric XMHD equations of motion according to $\partial_t \mathbf{u}_{TS} = \{\mathbf{u}_{TS}, \mathcal{H}_{TS}\}_{TS}^{XMHD}$.

B. Translationally symmetric Casimirs

As in the 3D case, there exist Casimir invariants conserved by the translationally symmetric dynamics. As already mentioned, the Casimirs satisfy $\{F, C\} = 0, \forall F$. For the bracket (46), this gives

$$\int_D d^2x (F_\rho Q_1 + F_{v_z} Q_2 + F_\Omega Q_3 + F_w Q_4 + F_{B_z^\pm} Q_5 + F_{\psi_\pm} Q_6) = 0, \quad (47)$$

where the quantities Q_i ($i = 1, 2, \dots, 6$) are given by the following expressions:

$$Q_1 = \Delta C_w = C_\Upsilon, \quad (48)$$

$$Q_2 = [C_\Omega, v_z] + \nabla \cdot (v_z \nabla C_w) - v_z C_\Upsilon - [\psi_\pm, C_{B_z^\pm}], \quad (49)$$

$$Q_3 = \nabla \cdot (\rho^{-1} \Omega \nabla C_w) - [\rho^{-1} \Omega, C_\Omega] - [v_z, \rho^{-1} C_{v_z}] - [\psi_\pm, \rho^{-1} C_{\psi_\pm}] - [\rho^{-1} B_z^\pm, C_{B_z^\pm}], \quad (50)$$

$$Q_4 = \Delta(\rho^{-1} v_z C_{v_z}) + \Delta(\rho^{-1} \psi_\pm C_{\psi_\pm}) - \Delta C_\rho - [\rho^{-1} \Omega, C_w] - \nabla \cdot (\rho^{-1} \Omega \nabla C_\Omega + v_z \nabla(\rho^{-1} C_{v_z}) + \psi_\pm \nabla(\rho^{-1} C_{\psi_\pm}) + \rho^{-1} B_z^\pm \nabla C_{B_z^\pm}), \quad (51)$$

$$Q_5 = [\rho^{-1} C_{v_z}, \psi_\pm] + [C_\Omega, \rho^{-1} B_z^\pm] + \nabla \cdot (\rho^{-1} B_z^\pm \nabla C_w) + \nu_\pm [\psi_\pm, \rho^{-1} C_{\psi_\pm}] + \nu_\pm [\rho^{-1} B_z^\pm, C_{B_z^\pm}], \quad (52)$$

$$Q_6 = [C_\Omega, \psi_\pm] + \nabla \cdot (\psi_\pm \nabla C_w) - \psi_\pm C_\Upsilon + \nu_\pm [\psi_\pm, C_{B_z^\pm}]. \quad (53)$$

For (47) to be satisfied for arbitrary variations, the coefficients Q_i must vanish separately, i.e.,

$$Q_i = 0, \quad i = 1, 2, \dots, 6. \quad (54)$$

Equation $Q_1 = 0$, i.e., $C_\Upsilon = 0$, implies that the Casimirs are independent of Υ . Equations $Q_4 = 0$ and $Q_3 = 0$ are, respectively, the divergence and the z component of the curl of the following equation:

$$\begin{aligned} & \nabla(\rho^{-1} v_z C_{v_z}) + \nabla(\rho^{-1} \psi_\pm C_{\psi_\pm}) - \nabla C_\rho \\ & - v_z \nabla(\rho^{-1} C_{v_z}) - \psi_\pm \nabla(\rho^{-1} C_{\psi_\pm}) - \rho^{-1} B_z^\pm \nabla C_{B_z^\pm} \\ & - \rho^{-1} \Omega \nabla C_w \times \hat{z} - \rho^{-1} \Omega \nabla C_\Omega = 0. \end{aligned} \quad (55)$$

We observe that (55) is satisfied automatically for $C_\rho = \text{constant}$, which gives the first Casimir

$$C_m = \int_D d^2x \rho. \quad (56)$$

Note that, in general, a solution to $Q_4 = 0$ could be satisfied by $C_m = \int_D d^2x \rho \Phi$ with Φ being a harmonic function, $\Delta \Phi = 0$. The equations $Q_2 = 0$ and $Q_6 = 0$ can be

combined by multiplying the first by ν_\pm and adding it to the second

$$\nu_\pm Q_2 + Q_6 = [C_\Omega, \psi_\pm + \nu_\pm v_z] + \nabla \cdot [(\psi_\pm + \nu_\pm v_z) \nabla C_w] = 0, \quad (57)$$

where we have used that $C_\Upsilon = 0$. With the new variable $\xi_\pm = \psi_\pm + \nu_\pm v_z$, (57) becomes

$$[\xi_\pm, C_\Omega] - \nabla \cdot (\xi_\pm \nabla C_w) = 0. \quad (58)$$

Equivalently, we can write

$$\nabla C_\Omega - \hat{z} \times \nabla C_w = \xi_\pm^{-1} \nabla A_\pm, \quad (59)$$

for $\xi_\pm \neq 0$ and A_\pm being arbitrary functions. The z -component of the curl of Eq. (59) is

$$\Delta C_w = C_\Upsilon = [A_\pm, \xi_\pm^{-1}] = 0. \quad (60)$$

Therefore, A_\pm are arbitrary functions of ξ_\pm , i.e.,

$$A_\pm = A_\pm(\psi_\pm + \nu_\pm v_z). \quad (61)$$

Now, divergence of (59) translates into

$$\Delta C_\Omega = \nabla \cdot (\xi_\pm^{-1} \nabla A_\pm) = \nabla \cdot (A'_\pm \xi_\pm^{-1} \nabla \xi_\pm) = \Delta A_\pm, \quad (62)$$

where the functions \mathcal{A}_\pm are related to A_\pm via $A'_\pm := \xi_\pm^{-1} A'_\pm$. According to (62), we have $C_\Omega = \mathcal{A}_\pm$, up to a harmonic function, therefore

$$C = \int_D d^2x \Omega \mathcal{A}_\pm(\psi_\pm + \nu_\pm v_z) + \mathcal{F}(B_z^\pm, \psi_\pm, v_z). \quad (63)$$

Inserting (63) into $Q_6 = 0$, we obtain

$$[A_\pm, \psi_\pm] + \nu_\pm [\psi_\pm, \mathcal{F}_{B_z^\pm}] = 0. \quad (64)$$

From (64), we derive $\mathcal{F}_{B_z^\pm} = \nu_\pm^{-1} A_\pm + \mathcal{G}_\pm(\psi_\pm)$, with \mathcal{G}_\pm being an arbitrary function of ψ_\pm , which when combined with (63) gives the following families of solutions

$$C_1^\pm = \int_D d^2x (B_z^\pm + \nu_\pm \Omega) \mathcal{A}_\pm(\psi_\pm + \nu_\pm v_z) + \tilde{\mathcal{F}}_\pm(v_z, \psi_\pm), \quad (65)$$

$$C_2^\pm = \int_D d^2x B_z^\pm \mathcal{G}_\pm(\psi_\pm) + \tilde{\mathcal{F}}_\pm(v_z, \psi_\pm), \quad (66)$$

for $\mathcal{G}_\pm = 0$ and $A_\pm = 0$ respectively. We remark here that if after Eq. (62) one takes $C_\Omega = \mathcal{A}_\pm + \Phi(x, y)$, with $\Phi(x, y)$ being a harmonic function, then it is not difficult to prove that the additional functional, coming from Φ will be a Casimir only if $\Phi = \Phi(\psi_\pm + \nu_\pm v_z)$ or $\Phi = \Phi(\psi_\pm)$. This would result in special cases of C_1^\pm and C_2^\pm , which may be valid if the motion of the variables ψ_\pm and v_z is restricted by a differential constraint. Having found the dependencies of the Casimir invariants on Ω and B_z^\pm , it remains to investigate any additional dependencies on v_z and ψ_\pm , represented by $\tilde{\mathcal{F}}$. Upon substituting (65) into $Q_5 = 0$, the latter reduces to

$$[\rho^{-1} (\tilde{\mathcal{F}}_{v_z} - \nu_\pm \tilde{\mathcal{F}}_{\psi_\pm}), \psi_\pm] = 0, \quad (67)$$

which additionally gives the following functionals:

$$C_3^\pm = \int_D d^2x \rho \mathcal{K}_\pm(\psi_\pm + \nu_\pm v_z), \quad (68)$$

$$C_4^\pm = \int_D d^2x \rho \mathcal{M}_\pm(\psi_\pm), \quad (69)$$

where \mathcal{M}_\pm and \mathcal{K}_\pm are arbitrary functions. The above functionals express the conservation of canonical-like momenta in the direction of symmetry. Also, they encapsulate conservation of mass, since C_m is the special case with $\mathcal{K}_\pm = 1$, and the conservation of the mechanical momentum along the axis of symmetry. To make this clear, note that C_4 , for example, if \mathcal{M}_\pm is differentiable, can be written as $C_4 = \int_D d^2x \rho \int_1^{\psi_\pm} ds \mathcal{N}_\pm(s)$ with $\mathcal{N}_\pm(\psi_\pm) = \mathcal{M}'_\pm(\psi_\pm)$. Under a change of the integration variable, this takes the form $C_4 = \int_D d^2x \rho \int_1^{v_z} ds \mathcal{N}_\pm(\psi_\pm + \nu_\pm v_z - \nu_\pm s)$. For $\mathcal{N}_\pm = 1$, we recover the conservation of mechanical momentum along the z -axis. Notice that in view of (68) and (69), the term $\tilde{\mathcal{F}}_\pm$ in (65) and (66) can be subtracted. It is not difficult to verify that C_1^\pm , C_2^\pm , C_3^\pm , C_4^\pm satisfy $\mathcal{Q}_{2,3,4} = 0$ as well and therefore all Casimir-determining equations (54) are satisfied. Also, since $\psi_\pm = \psi_\mp + \nu_\mp v_z$ and $B_z^\pm = B_z^\mp + \nu_\mp \Omega$ (because $\lambda_\pm^{-1} = d_i - \lambda_\mp^{-1}$), the functionals (65)–(69) represent just four independent families of invariants. Therefore, one may freely keep either the set denoted by (+) or the (−) representation. In terms of the original magnetic variables (B_z^*, ψ^*) , the XMHD Casimir invariants are written as

$$C_1 = \int_D d^2x (B_z^* + \mu \Omega) \mathcal{A}(\psi^* + \mu v_z), \quad (70)$$

$$C_2 = \int_D d^2x (B_z^* + \lambda^{-1} \Omega) \mathcal{G}(\psi^* + \lambda^{-1} v_z), \quad (71)$$

$$C_3 = \int_D d^2x \rho \mathcal{K}(\psi^* + \mu v_z), \quad (72)$$

$$C_4 = \int_D d^2x \rho \mathcal{M}(\psi^* + \lambda^{-1} v_z), \quad (73)$$

where the parameters λ and μ are either $(\lambda, \mu) = (\lambda_+, \mu_+)$ or $(\lambda, \mu) = (\lambda_-, \mu_-)$, with $\mu_\pm := \nu_\pm + \lambda_\pm^{-1} = d_i - \lambda_\pm^{-1} = \lambda_\mp^{-1}$.

As discussed above, the Casimirs $C_{3,4}$ express the conservation of mass and the conservation of (canonical) momenta in the direction of symmetry. In addition, the Casimirs $C_{1,2}$ are the symmetric counterparts of the generalized helicities (14). Unlike the 3D Casimirs, the symmetric invariants form infinite families, due to the existence of the arbitrary functions \mathcal{A} , \mathcal{G} , \mathcal{K} , \mathcal{M} . Later, we will see that these arbitrary functions are transferred, by the variational principle, into the equilibrium equations giving in principle the possibility of constructing infinitely many classes of equilibria, unlike the 3D case where all equilibria obtained from an energy-Casimir variational principle belong to the same class [see Eqs. (16)–(18)].

C. Hall MHD limit

Hall-MHD neglects electron inertia and therefore is recovered by the XMHD model for $d_e \rightarrow 0$. If we assume

$(\lambda, \mu) = (\lambda_+, \mu_+)$ and take the limit $d_e \rightarrow 0$, then $B_z^* \rightarrow B_z$, $\psi^* \rightarrow \psi$, $\lambda^{-1} \rightarrow d_i$ and $\mu \rightarrow 0$. In this case, the Hamiltonian becomes identical in form to the ordinary MHD symmetric Hamiltonian, that is

$$\mathcal{H}_{TS}^{HMHD} = \frac{1}{2} \int_D d^2x \left\{ \rho (v_z^2 + |\nabla \chi|^2 + |\nabla \Upsilon|^2) + 2\rho ([\Upsilon, \chi] + U(\rho)) + B_z^2 + |\nabla \psi|^2 \right\}.$$

Also, the HMHD Casimir invariants are

$$C_1^{HMHD} = \int_D d^2x B_z \mathcal{A}(\psi), \quad (74)$$

$$C_2^{HMHD} = \int_D d^2x (B_z + d_i \Omega) \mathcal{G}(\psi + d_i v_z), \quad (75)$$

$$C_3^{HMHD} = \int_D d^2x \rho \mathcal{K}(\psi), \quad (76)$$

$$C_4^{HMHD} = \int_D d^2x \rho \mathcal{M}(\psi + d_i v_z). \quad (77)$$

If we return to the symmetric XMHD Poisson bracket and set $d_e = 0$, we can verify that the HMHD bracket possesses the Casimirs (74)–(77). We remark that the “generalized variables” $B_z + d_i \Omega$, $\psi + d_i v_z$ appear in (74)–(77) since the ion canonical helicity $\int_V d^3x (\mathbf{A} + d_i \mathbf{v}) \cdot (\mathbf{B} + d_i \nabla \times \mathbf{v})$ is a Casimir invariant in 3D HMHD.⁴

D. MHD limit

For the MHD limit, we additionally require $d_i \rightarrow 0$ in (74)–(77), which yields only two of the translationally symmetric ideal MHD Casimir invariants of Refs. 14–16. However, it was observed in the first Hamiltonian structure that contained Hall physics,²³ that care must be taken with this limit (see also Ref. 9) which appears at the face value to not obviously yield the MHD versions of the Casimirs C_2^{HMHD} and C_3^{HMHD} .

To see how this transpires, we rewrite the invariants C_2^{HMHD} and C_4^{HMHD} as follows:

$$\begin{aligned} C_2^{HMHD} &= \int_D d^2x d_i^{-1} (B_z + d_i \Omega) \times [\mathcal{G}(\psi) + d_i v_z \mathcal{G}'(\psi) + \mathcal{O}(d_i^2)] \\ &= \int_D d^2x [d_i^{-1} B_z \mathcal{G}(\psi) + \Omega \mathcal{G}(\psi) + v_z B_z \mathcal{G}'(\psi) \\ &\quad + d_i \Omega v_z \mathcal{G}'(\psi) + \mathcal{O}(d_i)], \end{aligned} \quad (78)$$

$$C_4^{HMHD} = \int_D d^2x \rho d_i^{-1} [\mathcal{M}(\psi) + d_i v_z \mathcal{M}'(\psi) + \mathcal{O}(d_i^2)], \quad (79)$$

where we have scaled the arbitrary functions \mathcal{G} and \mathcal{M} by a factor of d_i . If we then take $d_i \rightarrow 0$, the first term of C_2^{HMHD} in (78) is seen to diverge. However, this term is itself a special case of C_1^{HMHD} , so it can be subtracted from (78), giving

$$\begin{aligned} C_2^{MHD} &= \int_D d^2x (\Omega \mathcal{G}(\psi) + v_z B_z \mathcal{G}'(\psi)) \\ &= \int_D d^2x (\nabla \chi \cdot \nabla \psi + v_z B_z) \mathcal{G}'(\psi). \end{aligned} \quad (80)$$

A similar argument applies for the limit of the Casimir \mathcal{C}_4 of (79). Therefore, in the MHD limit $d_i \rightarrow 0$, all Casimirs approach their translationally symmetric MHD counterparts of Refs. 14 and 15. To summarize (with a redefinition of the arbitrary functions \mathcal{G} and \mathcal{M}), the following translationally symmetric MHD Casimirs are obtained from the XMHD Casimirs in the limit $d_e \rightarrow 0$ followed by $d_i \rightarrow 0$:

$$\mathcal{C}_1^{MHD} = \int_D d^2x B_z \mathcal{A}(\psi), \quad (81)$$

$$\mathcal{C}_2^{MHD} = \int_D d^2x (B_z v_z + \nabla\psi \cdot \nabla\chi) \mathcal{G}(\psi), \quad (82)$$

$$\mathcal{C}_3^{MHD} = \int_D d^2x \rho \mathcal{K}(\psi), \quad (83)$$

$$\mathcal{C}_4^{MHD} = \int_D d^2x \rho v_z \mathcal{M}(\psi). \quad (84)$$

Note that the Casimirs \mathcal{C}_1^{MHD} , \mathcal{C}_3^{MHD} are identical to the HMHD Casimir functionals given by (74) and (76). This follows from the fact that the magnetic helicity is a common Casimir invariant for both models. The MHD limit of the HMHD model is also discussed in Refs. 24 and 25, although it is not shown how to limit the HMHD Casimirs to their MHD values.

E. Inertial MHD limit

Inertial MHD (IMHD) occurs upon setting $d_i = 0$, while $d_e \neq 0$, the reverse of the limit of Sec. III C. IMHD is valid when the characteristic time scale for changes in the current \mathbf{J} is significantly shorter than the electron gyro-period.² The Hamiltonian of translationally symmetric IMHD is \mathcal{H}_{TS}^{IMHD} , as given by (41). In the inertial MHD limit $d_i \rightarrow 0$, the parameters $\lambda_{\pm} = (-d_i \pm \sqrt{d_i^2 + 4d_e^2}) / (2d_e^2)$ go to $\pm d_e^{-1}$ and hence $\lim_{d_i \rightarrow 0} \mu_{\pm} = \mp d_e$, which leads to the following form for the Casimir invariants:

$$\mathcal{C}_1^{IMHD} = \int_D d^2x (B_z^* + d_e \Omega) \mathcal{A}(\psi^* + d_e v_z), \quad (85)$$

$$\mathcal{C}_2^{IMHD} = \int_D d^2x (B_z^* - d_e \Omega) \mathcal{G}(\psi^* - d_e v_z), \quad (86)$$

$$\mathcal{C}_3^{IMHD} = \int_D d^2x \rho \mathcal{K}(\psi^* + d_e v_z), \quad (87)$$

$$\mathcal{C}_4^{IMHD} = \int_D d^2x \rho \mathcal{M}(\psi^* - d_e v_z). \quad (88)$$

Upon taking $d_e \rightarrow 0$ in a manner similar to the d_i limits of Sec. III D, one can show that the Casimirs of (85)–(88) become the MHD Casimirs of (81)–(84). For example, upon setting $\mathcal{K} = \mathcal{M}$, $\lim_{d_e \rightarrow 0} (\mathcal{C}_3^{IMHD} - \mathcal{C}_4^{IMHD}) / d_e$ becomes \mathcal{C}_4^{MHD} . The Casimir \mathcal{C}_2^{MHD} follows similarly.

An interesting property of IMHD is that the well-known MHD cross helicity is also a Casimir for IMHD, if $\mathbf{B} \rightarrow \mathbf{B}^*$,⁶ that is

$$\mathcal{C}_c = \int_V d^3x \mathbf{v} \cdot \mathbf{B}^*, \quad (89)$$

is a Casimir invariant of the general 3D IMHD model. For a translationally symmetric system, inserting the representations of (22) and (23) into (89) and assuming appropriate boundary conditions, the symmetric version of the functional above is

$$\mathcal{C}_c = \int_D d^2x (v_z B_z^* + \Omega \psi^*), \quad (90)$$

which at a first glance is not included in (85)–(88). However, it is easy to see that upon choosing $\mathcal{A} = \psi^* + d_e v_z$ and $\mathcal{G} = \psi^* - d_e v_z$, the Casimir (90) is recovered from $(\mathcal{C}_1^{IMHD} - \mathcal{C}_2^{IMHD}) / (2d_e)$.

IV. ENERGY-CASIMIR VARIATIONAL PRINCIPLE WITH SYMMETRY

A. The variational principle

Having determined the invariants of the translationally symmetric XMHD, we can easily construct the Energy-Casimir variational principle of (12) for XMHD equilibria that have translation symmetry. Similar variational principles with symmetry can be found in Refs. 14–16 and 26. Gathering together relations (41) and (70)–(73), the Energy-Casimir principle $\delta \tilde{\mathcal{F}} = 0$ reads as follows:

$$\begin{aligned} \delta \int_D d^2x \left\{ \frac{\rho}{2} (v_z^2 + |\nabla\chi|^2 + |\nabla\Upsilon|^2) + \rho([\Upsilon, \chi] + U(\rho)) \right. \\ \left. + \frac{B_z^* B_z}{2} + \frac{\nabla\psi^* \cdot \nabla\psi}{2} - (B_z^* + \mu\Omega) \mathcal{A}(\psi^* + \mu v_z) \right. \\ \left. - (B_z^* + \lambda^{-1}\Omega) \mathcal{G}(\psi^* + \lambda^{-1} v_z) \right. \\ \left. - \rho \mathcal{M}(\psi^* + \lambda^{-1} v_z) - \rho \mathcal{K}(\psi^* + \mu v_z) \right\} = 0. \end{aligned} \quad (91)$$

Note, in (91), the Casimir \mathcal{C}_m with the harmonic function Φ has been omitted.

For the first variation of (91) to vanish, the coefficients of the arbitrary variations must separately vanish, yielding the following conditions:

$$\begin{aligned} \delta\rho : \frac{v^2}{2} + [\rho U(\rho)]_{\rho} - \mathcal{M}(\phi) - \mathcal{K}(\phi) - \frac{d_e^2}{\rho^2} \left\{ \frac{1}{2} (\Delta\psi)^2 \right. \\ \left. + \frac{1}{2} |\nabla B_z|^2 - \nabla B_z \cdot \nabla [\mathcal{A}(\phi) + \mathcal{G}(\phi)] \right. \\ \left. + \Delta\psi [(B_z^* + \mu\Omega) \mathcal{A}'(\phi) + (B_z^* + \lambda^{-1}\Omega) \mathcal{G}'(\phi)] \right. \\ \left. + \rho (\mathcal{M}'(\phi) + \mathcal{K}'(\phi)) \right\} = 0, \end{aligned} \quad (92)$$

$$\begin{aligned} \delta v_z : \rho v_z - \lambda^{-1} \rho \mathcal{M}'(\phi) - \mu \rho \mathcal{K}'(\phi) \\ - \mu (B_z^* + \mu\Omega) \mathcal{A}'(\phi) - \lambda^{-1} (B_z^* + \lambda^{-1}\Omega) \mathcal{G}'(\phi) = 0, \end{aligned} \quad (93)$$

$$\delta\chi : \nabla \cdot (\rho \nabla\chi) - [\rho, \Upsilon] = \mu \Delta \mathcal{A}(\phi) + \lambda^{-1} \Delta \mathcal{G}(\phi), \quad (94)$$

$$\delta\Upsilon : \nabla \cdot (\rho \nabla\Upsilon) = [\chi, \rho], \quad (95)$$

$$\delta B_z^* : B_z = \mathcal{A}(\phi) + \mathcal{G}(\phi), \quad (96)$$

$$\begin{aligned} \delta\psi^* : \Delta\psi + \rho \mathcal{M}'(\phi) + \rho \mathcal{K}'(\phi) + (B_z^* + \mu\Omega) \mathcal{A}'(\phi) \\ + (B_z^* + \lambda^{-1}\Omega) \mathcal{G}'(\phi) = 0, \end{aligned} \quad (97)$$

where $\phi := \psi^* + \lambda^{-1}v_z$, $\varphi := \psi^* + \mu w_z$, and $'$ denotes the derivative with respect to argument. For the derivation of the equilibrium equations above, we used the expressions for B_z^* , ψ^* in terms of the ordinary magnetic field variables ψ and B_z according to $\mathbf{B}^* := \mathbf{B} + d_e^2 \nabla \times (\rho^{-1} \nabla \times \mathbf{B}) = B_z^*(x, y) \hat{z} + \nabla \psi^*(x, y) \times \hat{z}$ with $\mathbf{B} = B_z(x, y) \hat{z} + \nabla \psi(x, y) \times \hat{z}$

$$B_z^* = B_z - d_e^2 \nabla \cdot (\rho^{-1} \nabla B_z), \quad (98)$$

$$\psi^* = \psi - d_e^2 \rho^{-1} \Delta \psi. \quad (99)$$

Equation (92) is a Bernoulli law, which describes the effects of macroscopic ion flows and electron inertia on the total pressure. Using (96) and (97), the Bernoulli equation takes the form

$$\tilde{P}(\rho) = \rho [\mathcal{M}(\phi) + \mathcal{K}(\varphi)] - \rho \frac{v^2}{2} - \frac{d_e^2}{2\rho} [(\Delta \psi)^2 + |\nabla B_z|^2], \quad (100)$$

with the components of the flow velocity being described by Eqs. (93)–(95) and $\tilde{P} := \rho [\rho U(\rho)]_\rho = \gamma P / (\gamma - 1)$, where P is the total pressure (see Sec. II). Also, note that Eqs. (96) and (98) can be used to express the quantity B_z^* in terms of the arbitrary functions \mathcal{A} and \mathcal{G}

$$B_z^* = \mathcal{A}(\varphi) + \mathcal{G}(\phi) - d_e^2 \nabla \cdot [\rho^{-1} \nabla \mathcal{A}(\varphi)] - d_e^2 \nabla \cdot [\rho^{-1} \nabla \mathcal{G}(\phi)]. \quad (101)$$

B. The Grad-Shafranov-Bernoulli system

We can show (see Appendix B) that (93) and (97), with the help of (94), (95), and the definition (99), can be written as a Grad-Shafranov-like system of the form

$$\begin{aligned} \alpha_1 \mathcal{A}'(\varphi) \nabla \cdot \left(\frac{\mathcal{A}'(\varphi)}{\rho} \nabla \varphi \right) + \alpha_2 \rho (\varphi - \phi) - \alpha_3 \frac{\rho}{d_e^2} \\ \times \left(\psi - \frac{\varphi - \lambda \mu \phi}{1 - \lambda \mu} \right) = [\mathcal{A}(\varphi) + \mathcal{G}(\phi)] \mathcal{A}'(\varphi) + \rho \mathcal{K}'(\varphi), \end{aligned} \quad (102)$$

$$\begin{aligned} \gamma_1 \mathcal{G}'(\phi) \nabla \cdot \left(\frac{\mathcal{G}'(\phi)}{\rho} \nabla \phi \right) + \gamma_2 \rho (\varphi - \phi) + \gamma_3 \frac{\rho}{d_e^2} \\ \times \left(\psi - \frac{\varphi - \lambda \mu \phi}{1 - \lambda \mu} \right) = [\mathcal{A}(\varphi) + \mathcal{G}(\phi)] \mathcal{G}'(\phi) + \rho \mathcal{M}'(\phi), \end{aligned} \quad (103)$$

$$\Delta \psi = \frac{\rho}{d_e^2} \left(\psi - \frac{\varphi - \lambda \mu \phi}{1 - \lambda \mu} \right), \quad (104)$$

where

$$\begin{aligned} \alpha_1 = \mu^2 + d_e^2, \quad \alpha_2 = \frac{\lambda^2}{(1 - \lambda \mu)^2}, \quad \alpha_3 = \frac{1}{1 - \lambda \mu}, \\ \gamma_1 = \lambda^{-2} + d_e^2, \quad \gamma_2 = -\alpha_2, \quad \gamma_3 = \lambda \mu \alpha_3. \end{aligned} \quad (105)$$

The above equilibrium equations are coupled to the Bernoulli law (100), comprising a Grad-Shafranov-Bernoulli (GSB) system. The existence of three coupled equations for three

different flux functions, namely ψ, ϕ, φ , is a direct verification that in the XMHD model, the ions and the electrons are allowed to move individually and separate from the magnetic surfaces, forming their own flow surfaces. Upon specifying the free functions $\mathcal{A} = \mathcal{A}(\varphi)$, $\mathcal{G} = \mathcal{G}(\phi)$, $\mathcal{K} = \mathcal{K}(\varphi)$ and $\mathcal{M} = \mathcal{M}(\phi)$, and adopting an equation of state $P = P(\rho)$, one in principle, can solve the GSB system, at least numerically, to determine the functions φ, ϕ, ψ and ρ . The level sets of the flux function ψ give the magnetic surfaces, on which the magnetic field lines lie. From φ and ϕ , we can compute ψ^*

$$\psi^* = \frac{\varphi - \lambda \mu \phi}{1 - \lambda \mu}, \quad (106)$$

while the poloidal ion flow velocity is given by

$$\mathbf{v}_p = \frac{1}{\rho} \left(\mu \mathcal{A}'(\varphi) \nabla \varphi + \lambda^{-1} \mathcal{G}'(\phi) \nabla \phi \right) \times \hat{z}, \quad (107)$$

and the longitudinal velocity component follows from:

$$v_z = \frac{\lambda}{1 - \lambda \mu} (\phi - \varphi). \quad (108)$$

Note that the longitudinal component of the magnetic field is directly related to φ and ϕ through (96), and the poloidal field is simply given by $\mathbf{B}_p = \nabla \psi \times \hat{z}$. Thus, all equilibrium quantities of interest can be specified upon solving the system (102)–(104).

C. Special cases of equilibria

1. Equilibria with longitudinal flow ($\mathbf{v}_p = 0$)

From (107), requiring $\mathbf{v}_p = 0$, we deduce that

$$\mathcal{G}(\phi) = -\lambda \mu \mathcal{A}(\varphi), \quad (109)$$

hence $\varphi = f(\phi)$. According to (106), $\psi^* = (f(\phi) - \lambda \mu \phi) / (1 - \lambda \mu)$ i.e., two sets of flux surfaces exist, the electron surfaces and the magnetic surfaces. The ions and the electrons can flow in the poloidal direction on the same surfaces, but their relative velocities are constrained so that the total poloidal velocity vanishes. Substituting (109) into the system (102) and (103) and using $\varphi = \varphi(\psi^*)$, $\phi = \phi(\psi^*)$ yields

$$\begin{aligned} \gamma_1 \mathcal{G}'(\psi^*) \nabla \cdot \left(\frac{\mathcal{G}'(\psi^*)}{\rho} \nabla \psi^* \right) + \gamma_2 \rho (\varphi - \phi) + \gamma_3 \frac{\rho}{d_e^2} (\psi - \psi^*) \\ = \frac{\lambda \mu - 1}{\lambda \mu} \mathcal{G}(\psi^*) \mathcal{G}'(\psi^*) + \rho \mathcal{M}'(\psi^*), \end{aligned} \quad (110)$$

$$\Delta \psi = \frac{\rho}{d_e^2} (\psi - \psi^*), \quad (111)$$

with $\varphi - \phi$ given by

$$\varphi - \phi = (\mu - \lambda^{-1}) [\mu \mathcal{K}'(\psi^*) + \lambda^{-1} \mathcal{M}'(\psi^*)]. \quad (112)$$

2. Static equilibria

For the case of static XMHD equilibria, where the macroscopic flow is neglected completely, we require additionally $v_z = 0$. Hence, the flux functions ϕ and φ are equal

to $\psi^* = \psi - d_e^2 \rho^{-1} \Delta \psi$ i.e., $f(\phi) = \phi = \psi^*$. Hence, Eqs. (110) and (111) reduce to

$$\begin{aligned} \gamma_1 \mathcal{G}'(\psi^*) \nabla \cdot \left(\frac{\mathcal{G}'(\psi^*)}{\rho} \nabla \psi^* \right) + \gamma_3 \frac{\rho}{d_e^2} (\psi - \psi^*) \\ = \frac{\lambda \mu - 1}{\lambda \mu} \mathcal{G}(\psi^*) \mathcal{G}'(\psi^*) + \rho \mathcal{M}'(\psi^*), \end{aligned} \quad (113)$$

$$\Delta \psi = \frac{\rho}{d_e^2} (\psi - \psi^*). \quad (114)$$

As above, two sets of flux surfaces exist, the electron-ion surfaces and the magnetic surfaces. Note that the electrons and the ions are allowed to move (in order to carry the electric current) but their velocities should satisfy the constraint $m_i \mathbf{v}_i + m_e \mathbf{v}_e = 0$. In this static case, the Bernoulli equation (100) becomes

$$\tilde{P} = \rho [\mathcal{M}(\psi^*) + \mathcal{K}(\psi^*)] - \frac{d_e^2}{2\rho} [(\Delta \psi)^2 + |\nabla B_z|^2], \quad (115)$$

closing the GSB system.

3. Hall MHD equilibria

The HMHD GSB equilibrium equations can be obtained from the system of Eqs. (102)–(103) and (100) upon setting $d_e = 0$. To take properly this limit, one should substitute the third term of the LHS of Eqs. (102) and (103) by (104). Adopting $(\lambda, \mu) = (\lambda_+, \mu_+)$, for $d_e \rightarrow 0$, we have $\mu \rightarrow 0$ and $\lambda^{-1} \rightarrow d_i$; therefore, the independent flux functions are the poloidal magnetic flux function ψ and the ion flow function $\phi := \psi + d_i v_z$. Using the definition of ϕ , v_z becomes

$$v_z = d_i^{-1} (\phi - \psi). \quad (116)$$

Also, from (107), we take

$$\mathbf{v}_p = \frac{d_i}{\rho} \mathcal{G}'(\phi) \mathbf{B}_{ip}, \quad \text{where} \quad \mathbf{B}_{ip} := \nabla \phi \times \hat{z}. \quad (117)$$

Next, with $d_e = 0$, Eqs. (102) and (103), in view of (104), reduce to

$$\begin{aligned} d_i^2 \mathcal{G}'(\phi) \nabla \cdot \left(\frac{\mathcal{G}'(\phi)}{\rho} \nabla \phi \right) + \frac{\rho}{d_i^2} (\phi - \psi) \\ - [\mathcal{G}(\phi) + \mathcal{A}(\psi)] \mathcal{G}'(\phi) - \rho \mathcal{M}'(\phi) = 0, \end{aligned} \quad (118)$$

$$\Delta \psi + \frac{\rho}{d_i^2} (\phi - \psi) + \rho \mathcal{K}'(\psi) + [\mathcal{G}(\phi) + \mathcal{A}(\psi)] \mathcal{A}'(\psi) = 0. \quad (119)$$

Finally, we close the system by writing the Bernoulli equation (100) with $d_e = 0$, in terms of ρ and the ion and magnetic flux functions. To do so, we express the kinetic term using (116) and (117) arriving at

$$\tilde{P}(\rho) = \rho \left[\mathcal{K}(\psi) + \mathcal{M}(\phi) - \frac{(\phi - \psi)^2}{2d_i^2} \right] - \frac{d_i^2}{2\rho} (\mathcal{G}'(\phi))^2 |\nabla \phi|^2. \quad (120)$$

To summarize, translationally symmetric barotropic Hall MHD equilibria are governed by the GSB system (118)–(120) with $\mathcal{K}(\psi)$, $\mathcal{M}(\phi)$, $\mathcal{A}(\psi)$, $\mathcal{G}(\phi)$ being arbitrary functions,

and the pressure $\tilde{P}(\rho)$ obeys a barotropic equation of state. These are the barotropic translationally symmetric counterparts of the baroclinic axisymmetric equilibrium equations derived by an Euler-Lagrange variational principle in Ref. 25 and of the barotropic axisymmetric equilibrium equations derived in Ref. 27 by a direct projection of the 3D equilibrium equations. Other derivations of the two-fluid equilibrium equations, which do not ignore electron inertia, have been made by various authors, e.g., Refs. 28–30. As expected, the sets of equilibrium equations derived there are mostly of the type of the system (102)–(104) because XMHD is closer to a full two-fluid description than HMHD.

Despite the simpler structure of HMHD, the system of (118)–(120) forms rather complex classes of equilibria. It requires the simultaneous solution of two coupled nonlinear PDEs, the Grad-Shafranov equations, which are additionally coupled to a Bernoulli equation and generally the existence of equilibrium solutions is not guaranteed. Due to this strong coupling, studies of two-fluid equilibria have been carried out numerically e.g., see Refs. 31 and 32. Here, we follow this approach for Hall MHD, giving an example of an equilibrium configuration (Fig. 1) computed by means of a simple finite difference iterative code, implemented on Matlab. More information and possible improvements of this computation will be given in a future work. For the sake of clarity, we mention that we used an MHD initial guess for ψ , and the ion flux function ϕ was initialized on the basis of this initial

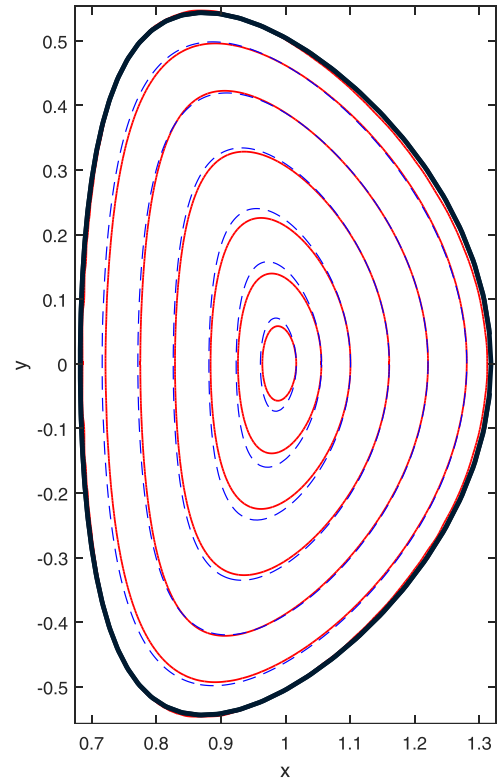


FIG. 1. Ion flow surfaces (solid-red) ($\phi = \text{constant}$) and magnetic surfaces (dashed-blue) ($\psi = \text{constant}$) with a dimensionless Hall parameter $d_i = 0.03$ (normalized ion skin depth) for a “straight” Tokamak HMHD equilibrium. The solid black line represents the boundary. Departure of the flow surfaces from the magnetic surfaces due to the Hall term in Ohm’s law is observed, with a separation distance of the order of $0.04 L_0$.

guess. The initial density ρ was set as a linear function of initialized flux function. These quantities are used for the calculation of their updated counterparts in the next iteration, and so on, until the resulting state converges. For this particular example, we adopted the following choices:

$$\begin{aligned}\mathcal{G}(\phi) &= g_0 + g_1\phi + g_2\phi^2, \\ \mathcal{A}(\psi) &= a_0 + a_1\psi + a_2\psi^2, \\ \mathcal{M}(\phi) &= m_0 + m_1\phi + m_2\phi^2, \\ \mathcal{K}(\psi) &= k_0 + k_1\psi + k_2\psi^2, \\ \tilde{P}(\rho) &= p_1\rho^\gamma,\end{aligned}\quad (121)$$

where $\gamma = 5/3$ is the specific heat ratio. Note that because we assumed that the plasma is barotropic, p_1 can be a constant, or at most a function of ρ . If we assume additionally that p_1 is a function of ψ and ϕ , then the mass density should also be a function of ψ and ϕ , and due to the Bernoulli equation, Equation (120), v^2 should be a function of the poloidal ion and magnetic fluxes, a property that demands certain restrictions on the permissible equilibrium configurations. The present study though, can be extended to the more generic case of baroclinic closure, i.e., when the internal energy is a function of the density and specific entropies,²⁵ which yield a dependence of the pressure on the flux functions without restricting the equilibria. This will be considered in our future work.

For the computation of the equilibrium, we imposed Dirichlet boundary conditions on the fluxes ψ and ϕ on a D-shaped boundary relevant to fusion experiments with elongation $\kappa = 1.7$ and triangularity $\delta = 0.4$. In Fig. 1, we observe the “departure” of the flow surfaces from magnetic surfaces, a result qualitatively consistent with the configurations presented in Ref. 32, where the baroclinic, axisymmetric, HMHD equilibrium equations were solved by means of the FLOW2 code. The observed departure is due to the Hall term $d_i\mathbf{J} \times \mathbf{B}/\rho$ in Ohm’s law, which “breaks” the frozen flux condition of ideal MHD. In Hall MHD, the flow surfaces are frozen into the “ion fluid,” while the magnetic surfaces are frozen into the “electron fluid.” An estimate of the poloidal separation distance Δr , a measure of the departure of the ion flow surfaces from the magnetic surfaces, was given in Ref. 25. For typical Tokamak experiments, this quantity is of the order of the ion poloidal Larmor radius, which is used as a typical step size in neoclassical transport studies. The separation distance can be approximated by $\Delta r \sim d_i v_z/B_p$. For our computed equilibrium depicted in Fig. 1, the normalized poloidal separation distance is $\overline{\Delta r} \sim 0.04$ (for $d_i = 0.03$, and using the average values of v_z and B_p).

Although our purpose of this numerical example was to demonstrate the qualitative way ion surfaces depart from the magnetic surfaces, which is predicted by HMHD theory, we briefly mention some equilibrium characteristics of our example. The maximum β in the plasma core is $\beta_{max} = 1.2\%$, the current density profile is peaked on the axis, i.e., it appears to have a maximum in the central region with maximum values of the order of $1 \times B_0/(\mu_0 L_0)$, while it reverses in the outer region. The plasma response to the external magnetic field is

purely diamagnetic, since the center drops to $0.8 \times B_0$ from $1 \times B_0$ at the boundary. Lastly, the flow in the z -direction is peaked on the axis with a maximum value at $0.25 \times v_A$, where $v_A = B_0/\sqrt{\mu_0 \rho_0}$, and the poloidal velocity component has a maximum value of $0.1 \times v_A$. The constants B_0 , ρ_0 , and L_0 are reference values for the magnetic field, the mass density, and the characteristic length scale, respectively. The values and the shapes of the profiles can be adjusted by regulating the free parameters in Ansatz (121) and adding some additional nonlinear terms. However, for the sake of simplicity, here, we consider this Ansatz with parametric values that favor fast convergence and results in configurations with distinct surface separation.

As a final note, a similar numerical procedure as that employed above for Hall MHD equilibria can be utilized for the numerical integration of the GS systems (110)–(111) and (113)–(114) for XMHD equilibria with longitudinal flows and static XMHD equilibria respectively.

V. CONCLUSION

In this paper, we presented the Hamiltonian formulation of translationally symmetric barotropic extended magnetohydrodynamics. We derived the symmetric Casimir integrals of motion and produced the Energy-Casimir variational principle for obtaining the generalized equilibrium equations, which govern XMHD stationary states. These states may be particularly interesting for the study of 2D collisionless reconnection configurations. Also, since two-fluid effects become significant for smaller length scales, increased values of plasma β , and flows approaching the ion diamagnetic drift speed, equilibrium studies based on XMHD equations could be useful for an adequate description of magnetically confined plasmas with such characteristics. The equilibrium system of equations was shown to be of Grad-Shafranov-Bernoulli type, and we studied special cases of XMHD equilibria and HMHD equilibria with arbitrary flow. In the case of HMHD equilibria with arbitrary flow, we computed a numerical equilibrium on a D-shaped domain, relevant to fusion experiments. The resulting configuration is representative of the predicted separation of the ion-flow and magnetic surfaces. Extension of the present study to cases of arbitrary symmetry, as done for MHD in Ref. 15, in particular, for helically symmetric configurations, is in progress and will be published in a future work.

ACKNOWLEDGMENTS

This work has been carried out within the framework of the EUROfusion Consortium and has received funding from (a) the National Programme for the Controlled Thermonuclear Fusion, Hellenic Republic and (b) Euratom research and training program 2014–2018 under Grant Agreement No. 633053. The views and opinions expressed herein do not necessarily reflect those of the European Commission. P.J.M. was supported by the U.S. Department of Energy Contract No. DE-FG05-80ET-53088 and a Forschungspreis from the Alexander von Humboldt Foundation. He warmly acknowledges the hospitality of the Numerical Plasma Physics

Division of Max Planck IPP, Garching. D.A.K. and G.N.T. would like to thank George Poulipoulis and Apostolos Kuiroukidis for useful discussions regarding the construction of the numerical equilibrium.

APPENDIX A: DERIVATION OF THE SYMMETRIC POISSON BRACKET (39)

1. Compressional part

$$\{F, G\}^c = - \int_V d^3x (F_\rho \nabla \cdot G_{\mathbf{v}} - G_\rho \nabla \cdot F_{\mathbf{v}}). \quad (\text{A1})$$

Using the relation $\nabla \cdot F_{\mathbf{v}} = -\Delta F_w$, we obtain

$$\{F, G\}_{TS}^c = \int_D d^2x (F_\rho \Delta G_w - G_\rho \Delta F_w). \quad (\text{A2})$$

2. Vortical part

$$\{F, G\}^v = \int_V d^3x \rho^{-1} (\nabla \times \mathbf{v}) \cdot (F_{\mathbf{v}} \times G_{\mathbf{v}}). \quad (\text{A3})$$

Using (23) and $\Omega := -\Delta\chi$, the vorticity is

$$\nabla \times \mathbf{v} = \Omega \hat{z} + \nabla v_z \times \hat{z}; \quad (\text{A4})$$

therefore, we have

$$\begin{aligned} & \rho^{-1} (\nabla \times \mathbf{v}) \cdot (F_{\mathbf{v}} \times G_{\mathbf{v}}) \\ &= \rho^{-1} \Omega \hat{z} \cdot (F_{\mathbf{v}} \times G_{\mathbf{v}})_z + \rho^{-1} (\nabla v_z \times \hat{z}) \cdot (F_{\mathbf{v}} \times G_{\mathbf{v}})_p. \end{aligned}$$

The subscripts z and p denote the z and poloidal components, respectively, which read as follows:

$$\begin{aligned} (F_{\mathbf{v}} \times G_{\mathbf{v}})_z &= \nabla F_w \times \nabla G_w - \nabla F_\Omega \cdot (\hat{z} \times \nabla G_\Omega) \hat{z} \\ &+ (\nabla F_w \cdot \nabla G_\Omega) \hat{z} - (\nabla F_\Omega \cdot \nabla G_w) \hat{z}, \quad (\text{A5}) \end{aligned}$$

$$(F_{\mathbf{v}} \times G_{\mathbf{v}})_p = F_{v_z} \nabla G_\Omega - G_{v_z} \nabla F_\Omega + F_{v_z} \nabla G_w \times \hat{z} - G_{v_z} \nabla F_w \times \hat{z}. \quad (\text{A6})$$

Using $[a, b] = (\nabla a \times \nabla b) \cdot \hat{z}$,

$$\begin{aligned} \rho^{-1} \Omega \hat{z} \cdot (F_{\mathbf{v}} \times G_{\mathbf{v}})_z &= \rho^{-1} \Omega ([F_\Omega, G_\Omega] + \nabla F_w \cdot \nabla G_\Omega \\ &- \nabla F_\Omega \cdot \nabla F_w + [F_w, G_w]), \quad (\text{A7}) \end{aligned}$$

$$\begin{aligned} \rho^{-1} (\nabla v_z \times \hat{z}) \cdot (F_{\mathbf{v}} \times G_{\mathbf{v}})_p &= \rho^{-1} (F_{v_z} [G_\Omega, v_z] - G_{v_z} [F_\Omega, v_z] \\ &+ F_{v_z} \nabla v_z \cdot \nabla G_w - G_{v_z} \nabla v_z \cdot \nabla F_w), \quad (\text{A8}) \end{aligned}$$

integrating over the domain D and exploiting (40), it gives

$$\begin{aligned} & \int_D d^2x \rho^{-1} (\nabla v_z \times \hat{z}) \cdot (F_{\mathbf{v}} \times G_{\mathbf{v}})_p \\ &= \int_D d^2x v_z \{ [F_\Omega, \rho^{-1} G_{v_z}] - [G_\Omega, \rho^{-1} F_{v_z}] \\ &+ \nabla F_w \cdot \nabla (\rho^{-1} G_{v_z}) - \nabla G_w \cdot \nabla (\rho^{-1} F_{v_z}) \\ &+ \rho^{-1} F_{\Gamma} G_{v_z} - \rho^{-1} G_{\Gamma} F_{v_z} \}. \quad (\text{A9}) \end{aligned}$$

Integrating (A7) over D and adding it to (A9) gives the vortical part of the translationally symmetric bracket.

3. Magnetic field-flow part

The magnetic field-flow (MHD) contribution is

$$\{F, G\}^{mf} = \int_V d^3x \rho^{-1} \mathbf{B}^* \cdot [F_{\mathbf{v}} \times (\nabla \times G_{\mathbf{B}^*}) - G_{\mathbf{v}} \times (\nabla \times F_{\mathbf{B}^*})]; \quad (\text{A10})$$

hence, one needs to compute $\rho^{-1} \mathbf{B}^* \cdot [F_{\mathbf{v}} \times (\nabla \times G_{\mathbf{B}^*})]$, since the second term of (A10) follows by interchanging F and G . From (36) and (38), we get

$$\begin{aligned} F_{\mathbf{v}} \times (\nabla \times G_{\mathbf{B}^*}) &= (F_{v_z} \hat{z} + \nabla F_\Omega \times \hat{z} - \nabla F_w) \\ &\times (G_{\psi^*} \hat{z} + \nabla G_{B_z^*} \times \hat{z}) \\ &= F_{v_z} \nabla G_{B_z^*} - G_{\psi^*} \nabla F_\Omega + [F_\Omega, G_{B_z^*}] \hat{z} \\ &- G_{\psi^*} \nabla F_w \times \hat{z} + \nabla F_w \cdot \nabla G_{B_z^*} \hat{z}, \quad (\text{A11}) \end{aligned}$$

and since $\mathbf{B}^* = B_z^* \hat{z} + \nabla \psi^* \times \hat{z}$, one can derive

$$\begin{aligned} \rho^{-1} \mathbf{B}^* \cdot F_{\mathbf{v}} \times (\nabla \times G_{\mathbf{B}^*}) &= \rho^{-1} \{ B_z^* ([F_\Omega, G_{B_z^*}] + \nabla F_w \cdot \nabla G_{B_z^*}) + F_{v_z} [G_{B_z^*}, \psi^*] \\ &+ G_{\psi^*} [\psi^*, F_\Omega] - G_{\psi^*} \nabla F_w \cdot \nabla \psi^* \}. \quad (\text{A12}) \end{aligned}$$

Integrating over D and using (40) gives

$$\begin{aligned} & \int_D d^2x \left\{ \rho^{-1} B_z^* ([F_\Omega, G_{B_z^*}] + \nabla F_w \cdot \nabla G_{B_z^*}) \right. \\ &+ \psi^* \left([F_\Omega, \rho^{-1} G_{\psi^*}] + [\rho^{-1} F_{v_z}, G_{B_z^*}] \right. \\ &\left. \left. + \nabla F_w \cdot \nabla (\rho^{-1} G_{\psi^*}) + \rho^{-1} F_{\Gamma} G_{\psi^*} \right) \right\}. \quad (\text{A13}) \end{aligned}$$

The second term of (A10) can be computed by (A13) upon interchanging F and G .

4. Hall part

The Hall part of the bracket (7) is

$$\{F, G\}^{hall} = -d_i \int_V d^3x \rho^{-1} \mathbf{B}^* \cdot [(\nabla \times F_{\mathbf{B}^*}) \times (\nabla \times G_{\mathbf{B}^*})].$$

Using Eq. (38), we obtain

$$\begin{aligned} & (\nabla \times F_{\mathbf{B}^*}) \times (\nabla \times G_{\mathbf{B}^*}) \\ &= [F_{B_z^*}, G_{B_z^*}] \hat{z} + F_{\psi^*} \nabla F_{B_z^*} - G_{\psi^*} \nabla F_{B_z^*}. \quad (\text{A14}) \end{aligned}$$

Taking the inner product with $\rho^{-1} \mathbf{B}^*$, the expression above gives

$$\rho^{-1} \left(B_z^* [F_{B_z^*}, G_{B_z^*}] + F_{\psi^*} [G_{B_z^*}, \psi^*] - G_{\psi^*} [F_{B_z^*}, \psi^*] \right).$$

which upon integrating over D and using (40) gives

$$\begin{aligned} \{F, G\}_{TS}^{hall} &= -d_i \int_D d^2x \left\{ \rho^{-1} B_z^* [F_{B_z^*}, G_{B_z^*}] \right. \\ &\left. + \psi^* \left([F_{B_z^*}, \rho^{-1} G_{\psi^*}] - [G_{B_z^*}, \rho^{-1} F_{\psi^*}] \right) \right\}. \end{aligned}$$

5. Electron inertial part

$$\{F, G\}^{inertial} = d_e^2 \int_V d^3x \rho^{-1} (\nabla \times \mathbf{v}) \cdot [(\nabla \times F_{\mathbf{B}^*}) \times (\nabla \times G_{\mathbf{B}^*})].$$

For this part, we take the inner product of (A14) with $\rho^{-1} \nabla \times \mathbf{v}$, where the curl of \mathbf{v} is given by (A4). Following the same steps, as before, gives

$$\begin{aligned} \{F, G\}_{TS}^{inertial} &= d_e^2 \int_D d^2x \left\{ \rho^{-1} \Omega [F_{B_z^*}, G_{B_z^*}] \right. \\ &\quad \left. + v_z \left([F_{B_z^*}, \rho^{-1} G_{\psi^*}] - [G_{B_z^*}, \rho^{-1} F_{\psi^*}] \right) \right\}. \end{aligned}$$

APPENDIX B: DERIVATION OF (102)–(104)

From (94) and (95), we deduce the following (except for the gradients of two harmonic functions that can be neglected):

$$\mu \nabla \mathcal{A} + \lambda^{-1} \nabla \mathcal{G} = \rho \nabla \chi - \rho \nabla \Upsilon \times \hat{z}, \quad (\text{B1})$$

$$\rho \nabla \Upsilon = \chi \nabla \rho \times \hat{z}. \quad (\text{B2})$$

Taking the cross product of (B1) with \hat{z} gives

$$\mathbf{v}_p = \rho^{-1} \left[\mu \mathcal{A}'(\varphi) \nabla \varphi + \lambda^{-1} \mathcal{G}'(\phi) \nabla \phi \right] \times \hat{z}. \quad (\text{B3})$$

Now, from the curl of (B3), we obtain

$$\Delta \chi = \mu \nabla \cdot \left(\frac{\mathcal{A}'}{\rho} \nabla \varphi \right) + \lambda^{-1} \nabla \cdot \left(\frac{\mathcal{G}'}{\rho} \nabla \phi \right) = -\Omega, \quad (\text{B4})$$

and substituting the expression above into (93) gives

$$\begin{aligned} &(\mu^2 \mathcal{A}' + \lambda^{-2} \mathcal{G}') \nabla \cdot \left(\mu \frac{\mathcal{A}'}{\rho} \nabla \varphi + \lambda^{-1} \frac{\mathcal{G}'}{\rho} \nabla \phi \right) \\ &= \frac{\lambda \rho}{1 - \lambda \mu} (\varphi - \phi) + \mu \rho \mathcal{K}' + \lambda^{-1} \rho \mathcal{M}' + B_z^* (\mu \mathcal{A}' + \lambda^{-1} \mathcal{G}'), \end{aligned} \quad (\text{B5})$$

where we used $v_z = \lambda(\varphi - \phi)/(\lambda\mu - 1)$, which follows from the definitions of φ and ϕ . Using Eq. (101), we can write Eq. (B5) as

$$\begin{aligned} &(\mu^3 + \mu d_e^2) \mathcal{A}' \nabla \cdot \left(\frac{\mathcal{A}'}{\rho} \nabla \varphi \right) + (\lambda^{-3} + \lambda^{-1} d_e^2) \mathcal{G}' \nabla \cdot \left(\frac{\mathcal{G}'}{\rho} \nabla \phi \right) \\ &= \frac{\lambda \rho}{1 - \lambda \mu} (\varphi - \phi) + \rho (\mu \mathcal{K}' + \lambda^{-1} \mathcal{M}') \\ &\quad + (\mu \mathcal{A}' + \lambda^{-1} \mathcal{G}') (\mathcal{A} + \mathcal{G}). \end{aligned} \quad (\text{B6})$$

Inserting Eqs. (98) and (B4) into (97) and following a similar procedure as above, we derive a second GS-like equation,

$$\begin{aligned} &(\mu^2 + d_e^2) \mathcal{A}' \nabla \cdot \left(\frac{\mathcal{A}'}{\rho} \nabla \varphi \right) + (\lambda^{-2} + d_e^2) \mathcal{G}' \nabla \cdot \left(\frac{\mathcal{G}'}{\rho} \nabla \phi \right) \\ &= \frac{\rho}{d_e^2} \left(\psi - \frac{\varphi - \lambda \mu \phi}{1 - \lambda \mu} \right) + \rho (\mathcal{K}' + \mathcal{M}') + (\mathcal{A}' + \mathcal{G}') (\mathcal{A} + \mathcal{G}), \end{aligned} \quad (\text{B7})$$

and ψ is connected to φ and ϕ by

$$\Delta \psi = \frac{\rho}{d_e^2} \left(\psi - \frac{\varphi - \lambda \mu \phi}{1 - \lambda \mu} \right). \quad (\text{B8})$$

The last equation can be derived from (99), using the definitions of φ and ϕ . Finally, we may refine the GS system a bit more by combining (B6) and (B7). After careful manipulation, this system leads to (102) and (103).

¹R. Lüst, *Fortschr. Phys.* **7**, 503 (1959).

²K. Kimura and P. J. Morrison, *Phys. Plasmas* **21**, 082101 (2014).

³H. M. Abdelhamid, Y. Kawazura, and Z. Yoshida, *J. Phys. A: Math. Theor.* **48**, 235502 (2015).

⁴M. Lingam, P. J. Morrison, and G. Miloshevich, *Phys. Plasmas* **22**, 072111 (2015).

⁵M. J. Lighthill, *Philos. Trans. R. Soc. London* **25**, 397 (1960).

⁶M. Lingam, P. J. Morrison, and E. Tassi, *Phys. Lett. A* **379**, 570 (2015).

⁷M. Lingam, G. Miloshevich, and P. J. Morrison, *Phys. Lett. A* **380**, 2400 (2016).

⁸E. C. D'Avignon, P. J. Morrison, and M. Lingam, *Phys. Plasmas* **23**, 062101 (2016).

⁹D. Grasso, E. Tassi, H. M. Abdelhamid, and P. J. Morrison, *Phys. Plasmas* **24**, 012110 (2017).

¹⁰E. Tassi, P. J. Morrison, F. L. Waelbroeck, and D. Grasso, *Plasma Phys. Controlled Fusion* **50**, 085014 (2008).

¹¹E. Tassi, "Hamiltonian closures in fluid models for plasmas," *Eur. Phys. J. D* (unpublished).

¹²P. J. Morrison and J. M. Greene, *Phys. Rev. Lett.* **45**, 790 (1980).

¹³P. J. Morrison, *Rev. Mod. Phys.* **70**, 467 (1998).

¹⁴T. Andreussi, P. J. Morrison, and F. Pegoraro, *Plasma Phys. Controlled Fusion* **52**, 055001 (2010).

¹⁵T. Andreussi, P. J. Morrison, and F. Pegoraro, *Phys. Plasmas* **19**, 052102 (2012).

¹⁶S. M. Moawad, *J. Plasma Phys.* **79**, 873 (2013).

¹⁷T. Andreussi, P. J. Morrison, and F. Pegoraro, *Phys. Plasmas* **20**, 092104 (2013); Erratum *ibid.* **22**, 039903 (2015).

¹⁸T. Andreussi, P. J. Morrison, and F. Pegoraro, *Phys. Plasmas* **23**, 102112 (2016).

¹⁹P. J. Morrison, *Phys. Plasmas* **24**, 055502 (2017).

²⁰Y. Kawazura, *Phys. Rev. E* **96**, 013207 (2017).

²¹Z. Yoshida, P. J. Morrison, and F. Dobarro, *J. Math. Fluid Mech.* **16**, 41 (2014).

²²I. K. Charidakos, M. Lingam, P. J. Morrison, R. L. White, and A. Wurm, *Phys. Plasmas* **21**, 092118 (2014).

²³R. D. Hazeltine, C. T. Hsu, and P. J. Morrison, *Phys. Fluids* **30**, 3204 (1987).

²⁴Z. Yoshida and E. Hameiri, *J. Phys. A: Math. Theor.* **46**, 335502 (2013).

²⁵E. Hameiri, *Phys. Plasmas* **20**, 092503 (2013).

²⁶T. Andreussi and F. Pegoraro, *Phys. Plasmas* **15**, 092108 (2008).

²⁷G. N. Throumoulopoulos and H. Tasso, *Phys. Plasmas* **13**, 102504 (2006).

²⁸L. C. Steinhauer, *Phys. Plasmas* **6**, 2734 (1999).

²⁹K. G. McClements and A. Thyagaraja, *Mon. Not. R. Astron. Soc.* **323**, 733 (2001).

³⁰J. P. Goedbloed, *Phys. Plasmas* **11**, L81 (2004).

³¹A. Ishida, L. C. Steinhauer, and Y. K. M. Peng, *Phys. Plasmas* **17**, 122507 (2010).

³²L. Guazzotto and R. Betti, *Phys. Plasmas* **22**, 092503 (2015).