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## ABSTRACT

The formal stability analysis of Eulerian extended magnetohydrodynamics (XMHD) equilibria is considered within the noncanonical Hamiltonian framework by means of the energy-Casimir variational principle and the dynamically accessible stability method. Specifically, we find explicit sufficient stability conditions for axisymmetric XMHD and Hall MHD (HMHD) equilibria with toroidal flow and for equilibria with arbitrary flow under constrained perturbations. The dynamically accessible, second-order variation of the Hamiltonian, which can potentially provide explicit stability criteria for generic equilibria, is also obtained. Moreover, we examine the Lagrangian stability of the general quasineutral two-fluid model written in terms of MHD-like variables, by finding the action and the Hamiltonian functionals of the linearized dynamics, working within a mixed Lagrangian-Eulerian framework. Upon neglecting electron mass, we derive a HMHD energy principle, and in addition, the perturbed induction equation arises from Hamilton's equations of motion in view of a consistency condition for the relation between the perturbed magnetic potential and the canonical variables.

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## I. INTRODUCTION

The stability of plasma equilibria is crucial for the attainment of long lived states of magnetically confined plasmas, with sufficient confinement of thermal energy for the self-sustained operation of thermonuclear reactors. In general, the most drastic way to lose the confinement of plasma energy is the development of either macroinstabilities, e.g., the current driven kink and the pressure driven ballooning instabilities, associated with plasma disruption (which effectively put upper limits on the attainable pressure and current), or microinstabilities that result in enhanced turbulence and anomalous transport. Stability analyses are usually performed using the standard MHD energy principle<sup>1</sup> that was generalized for flowing equilibria in Ref. 2. The stability of stationary plasma states with macroscopic sheared flows, albeit a tough problem from the mathematical point of view, is important since it is believed that plasma rotation, either being self-generated or driven externally, may have beneficial effects in terms of confinement. Indeed, plasma flows are associated with the suppression of turbulence<sup>3</sup> and the L-H transitions<sup>4</sup> observed in Tokamaks. Also,

there are many studies proposing that plasma sheared rotation variously affects the stability properties of Tokamak equilibria in several cases, either inducing stabilization or destabilization (e.g., Refs. 5–9) with the main destabilizing mechanism being the Kelvin-Helmholtz instability.<sup>10</sup>

Furthermore, many astrophysical phenomena, such as the development of turbulence in various stages of the solar wind and in magnetized accretion disks, are consequences of flow-driven instabilities such as the Kelvin-Helmholtz (e.g., see Ref. 11) and the magnetorotational instability (MRI).<sup>12</sup> It is evident that plasma instability is the reason for the emergence of new structures, but, most importantly for fusion physics, instabilities are the main mechanisms behind the undesirable interchange of energy, which should be sufficiently reduced in fusion experiments. This pursuit is the main reason for performing stability studies for over 60 years, trying to refine the resulting stability or instability criteria and incorporate as much physics as possible.

It is widely agreed that ordinary MHD, despite being a successful model for describing macroscopic phenomena, provides a rather

rough description of plasmas because it neglects the presence of multi-fluid effects. This is especially true when there exist characteristic length scales comparable to the ion and electron skin depths, e.g., due to the presence of current sheets or thin boundary layers. In such cases, multifluid models are needed to describe phenomena arising due to the coexistence of different particle species and the decoupling of their respective motions, even at the macroscopic level. Regarding stability, when mode frequencies comparable to the particle gyrofrequencies are present, then MHD becomes clearly an insufficient framework. This intuitive reasoning about the insufficiency of the MHD model is corroborated when MHD theory fails to adequately predict the experimental observations: the observed stability of elongated Field Reversed Configurations (FRCs)<sup>13,14</sup> and the high magnetic reconnection rates (see, e.g., Refs. 15 and 16) are examples where two-fluid models work significantly better than MHD. Moreover, there exist recent views on Tokamak physics, suggesting that the Hall drift term cannot be neglected in both equilibrium and dynamics computations; also, it has been suggested that Hall effects may be associated with the pressure pedestals, formed in the L-H transitions.<sup>17,18</sup>

For the reasons described above, very often, we need to invoke multifluid descriptions since they capture finer dynamical effects, taking place in shorter length and temporal scales. If rotation is neglected, the two-fluid effects are incorporated more easily through the multi-fluid pressure (e.g., see Ref. 19) because no decoupling of electron and ion motion occurs. However, as was stressed earlier, plasma flows are consequential, and therefore, it is important to take them into account. A characteristic consequence of including flows in stability methods based on energy functionals is the nonseparability of the kinetic and potential energy contributions, rendering the resulting stability criteria sufficient but not necessary. A typical example is the MHD energy principle, which for static equilibria provides a necessary and sufficient condition,<sup>1</sup> while for stationary states,<sup>2</sup> it provides only sufficient conditions. These are the Lagrange and Dirichlet conditions of Hamiltonian dynamics, as pointed out in Ref. 20, respectively. As we shall see later, the nonseparability is even stiffer in the two-fluid case. Hence, we understand that forming sufficient and necessary stability criteria for flowing equilibria would require the introduction of several restrictions on the equilibrium states or/and the perturbations under consideration.

Given the historical precedent, it would appear desirable to apply formal stability analysis methods, similar to those originating from the MHD energy principle to flowing multifluid plasma equilibria, because this framework is already well known from MHD theory and also because this would facilitate comparisons with the MHD results. By formal stability, we mean an analysis based on a quantity, a kind of energy, which is conserved by the full nonlinear dynamics of the system. The first variation of the quantity must vanish, and the second variation must be positive (or negative) definite at equilibrium. When this is the case, the second variation serves as a Lyapunov functional for the linear dynamics. At present, only a limited number of studies have led to appropriate Lyapunov functionals and ultimately to stability conclusions within the two-fluid context, primarily in the Hall MHD (HMHD) limit,<sup>19,21–24</sup> and a few of them employing the complete two-fluid model.<sup>25,26</sup>

A very useful apparatus for conducting stability analysis is the Hamiltonian description of ideal fluid and plasma models. The Hamiltonian framework, when adopting either a canonical description

within the Lagrangian viewpoint or a noncanonical description within the Eulerian one, is a convenient framework for studying linearized dynamics and constructing functionals that can be exploited to establish stability criteria. Fluid and plasma criteria, such as the MHD energy principle and the Rayleigh criterion for shear flow, ultimately exist because of the Hamiltonian form that can serve as a guide.

In this paper, we conduct formal stability analyses within the framework of a quasineutral two-fluid model with electron inertia, the so-called extended MHD (XMHD) model (e.g., see Refs. 27 and 28). Attention has been drawn to XMHD because of the recent discovery of its Hamiltonian structure<sup>29</sup> and its remarkable similarities with the Hamiltonian structure of Hall MHD.<sup>30–32</sup> We exploit this noncanonical Hamiltonian description of the model to employ the energy-Casimir (EC) and dynamically accessible (DA) methods<sup>30,33,34</sup> for deriving sufficient stability criteria upon constructing appropriate Lyapunov functionals. Moreover, using the action formalism developed in Refs. 31 and 35, we examine the Lagrangian stability of the quasineutral two-fluid model by deriving the Hamiltonian of the corresponding linearized system in terms of Lagrangian displacements. Neglecting electron inertia, we derive a Hall MHD Lagrangian stability criterion that also takes into account the electron pressure contribution. Each one of the above stability methods has certain advantages and disadvantages, which are discussed in detail in their respective sections. We can briefly say though that when applied under the same conditions, an ordering between them emerges from the dynamical point of view.<sup>36</sup> The EC variations, being dynamically unconstrained, are more generic than the Lagrangian ones, which are generated through certain relations from arbitrary displacement vectors. In turn, the latter are more generic than the DA set of variations that are restricted by Hamiltonian dynamics.

The aim of this study is to provide a framework for formal stability analyses within a two-fluid description, which is more accurate and generic than that for MHD, staying though conceptually and formalistically as close as possible to MHD. In addition, this work emphasizes that the Hamiltonian approach provides a unifying framework for studying equilibrium and stability employing the same principles.

The main ingredients of the Hamiltonian formulation of XMHD are the Hamiltonian functional,<sup>28,29</sup>

$$\begin{aligned} \mathcal{H} &= \frac{1}{2} \int_V d^3x \left[ \rho v^2 + 2\rho U(\rho) + B^2 + d_e^2 \frac{|\nabla \times \mathbf{B}|^2}{\rho} \right], \\ &= \frac{1}{2} \int_V d^3x \left[ \rho v^2 + 2\rho U(\rho) + \mathbf{B} \cdot \mathbf{B}^* \right], \end{aligned} \quad (1)$$

where  $V \subseteq \mathbb{R}^3$ , and the noncanonical Poisson bracket,<sup>29</sup>

$$\begin{aligned} \{F, G\} &= \int_V d^3x \{ G_\rho \nabla \cdot F_{\mathbf{v}} - F_\rho \nabla \cdot G_{\mathbf{v}} + \rho^{-1} (\nabla \times \mathbf{v}) \cdot (F_{\mathbf{v}} \times G_{\mathbf{v}}) \\ &\quad + \rho^{-1} \mathbf{B}^* \cdot [F_{\mathbf{v}} \times (\nabla \times G_{\mathbf{B}^*}) - G_{\mathbf{v}} \times (\nabla \times F_{\mathbf{B}^*})] \\ &\quad - d_i \rho^{-1} \mathbf{B}^* \cdot [(\nabla \times F_{\mathbf{B}^*}) \times (\nabla \times G_{\mathbf{B}^*})] \\ &\quad + d_e^2 \rho^{-1} (\nabla \times \mathbf{v}) \cdot [(\nabla \times F_{\mathbf{B}^*}) \times (\nabla \times G_{\mathbf{B}^*})] \}, \end{aligned} \quad (2)$$

where  $F_u := \delta F / \delta u$  denotes the functional derivative of  $F$  with respect to the dynamical variable  $u$ . The Poisson bracket of (2) is a generalization of that first given for MHD in Ref. 37. Here, the set of dynamical variables, say  $\mathbf{u}$ , are the mass density  $\rho$ , the fluid velocity  $\mathbf{v}$ , and the generalized magnetic field  $\mathbf{B}^*$  suggested in Ref. 38, given by

$$\mathbf{B}^* = \mathbf{B} + d_e^2 \nabla \times \left( \frac{\nabla \times \mathbf{B}}{\rho} \right). \quad (3)$$

The parameters  $d_i$  and  $d_e$  are the normalized ion and electron skin depths, respectively. The equations of motion for XMHD arising from  $\partial_t \mathbf{u} = \{\mathbf{u}, \mathcal{H}\}$  are the following:

$$\partial_t \rho = -\nabla \cdot (\rho \mathbf{v}), \quad (4)$$

$$\partial_t \mathbf{v} = \mathbf{v} \times \boldsymbol{\omega} - \nabla \left( h + \frac{v^2}{2} + d_e^2 \frac{|\mathbf{J}|^2}{2\rho^2} \right) + \frac{\mathbf{J} \times \mathbf{B}^*}{\rho}, \quad (5)$$

$$\partial_t \mathbf{B}^* = \nabla \times \left( \mathbf{v} \times \mathbf{B}^* - d_i \frac{\mathbf{J} \times \mathbf{B}^*}{\rho} + d_e^2 \frac{\mathbf{J} \times \boldsymbol{\omega}}{\rho} \right), \quad (6)$$

where  $\boldsymbol{\omega} := \nabla \times \mathbf{v}$  and  $\mathbf{J} = \nabla \times \mathbf{B}$ .

The degeneracy and explicit dependence of the noncanonical Poisson bracket on the dynamical variables  $\mathbf{u} = (\rho, \mathbf{v}, \mathbf{B}^*)$  result in the emergence of topological constants of motion, called Casimirs, satisfying  $\{F, C\} = 0, \forall F$ . The presence of these invariants and their topological consequences give rise to the EC and DA methods. Exploiting these methodologies, we construct Lyapunov functionals suitable for establishing sufficient stability criteria without any reference to the dynamical equations: the perturbative procedure is implemented exclusively at the Hamiltonian level.

This paper is organized as follows: in Sec. II, we employ the EC method for studying the stability of axisymmetric XMHD equilibria. In this framework, several sufficient stability criteria are derived, concerning either special equilibria or special perturbations. In Sec. III, we find the dynamically accessible variations for the XMHD model, i.e., variations that keep the phase space trajectory on Casimir leaves. In addition, the second order, dynamically accessible variation of the Hamiltonian is utilized in order to establish a stability criterion for generic equilibria. Finally, in Sec. IV, we compute the second order variation of the Lagrangian in a mixed Eulerian-Lagrangian framework and furthermore employ a Lagrange-Euler map to express the Lagrangian completely in terms of Eulerian coordinates. These results are used to construct the Hamiltonians for the linearized dynamics of the quasineutral two-fluid model and Hall MHD.

## II. ENERGY-CASIMIR STABILITY OF AXISYMMETRIC EQUILIBRIA

In Ref. 39, we derived the equilibrium equations for helically symmetric and axisymmetric barotropic plasmas described by XMHD, using the EC principle. That principle can be extended to the computation of the second order variation which when evaluated on the EC equilibrium, denoted here as  $\mathbf{u}_e$ , is conserved by the linearized dynamics (e.g., Refs. 20 and 40), and therefore a sufficient linear stability condition can be established by requiring that  $\delta^2(\mathcal{H} - \sum_i C_i)[\mathbf{u}_e, \delta \mathbf{u}]$  has a definite sign. In general, however, the applicability of the EC method is not guaranteed since it requires a sufficient number of Casimir invariants in order to be established. This is the reason why in three-dimensional systems, EC stability is usually not possible, other than the special cases when there exist some kind of Ertel's invariants, emerging usually due to entropy advection and providing additional Casimirs.<sup>40</sup> This would also be the case for XMHD if a baroclinic thermodynamic closure had been used. Ultimately, the lack of Casimirs was shown to be caused by the kind of degeneracy of the Poisson bracket in Ref. 20. However, if a continuous spatial

symmetry is present, the usual helicities are converted to infinite families of invariants in view of the symmetric decomposition of the fields, thus rendering the EC method applicable, as, for example, in Refs. 36 and 41–44 for the MHD model. One has to keep in mind though that this symmetric decomposition of the fields restricts the variations so as to respect the geometrical symmetry of the system as well.

### A. Axisymmetric XMHD energy-Casimir functional

The axisymmetric velocity and magnetic fields can be Helmholtz-decomposed as follows:

$$\mathbf{v} = rv_\phi \nabla \phi + \nabla \chi \times \nabla \phi + \nabla \Upsilon, \quad (7)$$

$$\mathbf{B} = rB_\phi \nabla \phi + \nabla \psi \times \nabla \phi, \quad (8)$$

inducing a similar form for the generalized magnetic field  $\mathbf{B}^*$ . From Eqs. (4.10)–(4.13) in Ref. 39, we can easily obtain the following axisymmetric Casimirs:

$$C_1 = \int_D d^2x \left( r^{-1} B_\phi^* + \gamma \Omega \right) \mathcal{F}(\psi^* + \gamma rv_\phi), \quad (9)$$

$$C_2 = \int_D d^2x \left( r^{-1} B_\phi^* + \mu \Omega \right) \mathcal{G}(\psi^* + \mu rv_\phi), \quad (10)$$

$$C_3 = \int_D d^2x \rho \mathcal{M}(\psi^* + \gamma rv_\phi), \quad (11)$$

$$C_4 = \int_D d^2x \rho \mathcal{N}(\psi^* + \mu rv_\phi), \quad (12)$$

where  $\Omega := (\nabla \times \mathbf{v}_\perp) \cdot \nabla \phi$  with  $\mathbf{v}_\perp := \nabla \chi \times \nabla \phi + \nabla \Upsilon$  and  $\psi^* = \psi - d_e^2 \rho^{-1} \Delta^* \psi$ ,  $B_\phi^* = B_\phi - d_e^2 r \nabla \cdot [r^{-2} \rho^{-1} \nabla(rB_\phi)]$ , with  $\Delta^* := r^2 \nabla \cdot (r^{-2} \nabla)$  being the so-called Shafranov operator. The parameters  $\gamma$  and  $\mu$  are  $(\gamma, \mu) = (\gamma_+, \gamma_-)$ , where  $\gamma_\pm = (d_i \pm \sqrt{d_i^2 + 4d_e^2})/2$ . The axisymmetric Hamiltonian is given by

$$\mathcal{H} = \int_D d^2x \left( \rho \frac{v_\phi^2}{2} + \rho \frac{|\nabla \chi|^2}{2r^2} + \rho \frac{|\nabla \Upsilon|^2}{2} + \rho [\Upsilon, \chi] + \rho U(\rho) + \frac{B_\phi^* B_\phi}{2} + \frac{\nabla \psi^* \cdot \nabla \psi}{2r^2} \right). \quad (13)$$

The vanishing of the first order variation of the EC functional, i.e.,  $\delta \mathcal{H}_C = \delta(\mathcal{H} - \sum_i C_i) = 0$ , yields the EC equilibrium equations, given by Eqs. (4.25)–(4.31) of Ref. 39 with  $\ell = 0, n = -1$  therein, which can be written in a Grad-Shafranov-Bernoulli form [see Eqs. (5.1)–(5.4) in the same reference]. In this case,  $\delta \mathcal{H}_C$  assumes the form

$$\begin{aligned} \delta \mathcal{H}_C = \int_D d^2x \left\{ \left[ h(\rho) - \mathcal{M} - \mathcal{N} + \frac{v_\phi^2}{2} + \frac{|\mathbf{v}_\perp|^2}{2} + \frac{d_e^2}{2r^2 \rho^2} \right. \right. \\ \times \left. \left. \left( (\Delta^* \psi)^2 + |\nabla(rB_\phi)|^2 \right) \right] \delta \rho + [B_\phi - r^{-1}(\mathcal{F} + \mathcal{G})] \delta B_\phi^* \right. \\ + [\rho \mathbf{v}_\perp - \gamma \nabla \mathcal{F} \times \nabla \phi - \mu \nabla \mathcal{G} \times \nabla \phi] \cdot \delta \mathbf{v}_\perp \\ + \left[ \rho v_\phi - \gamma r(r^{-1} B_\phi^* + \gamma \Omega) \mathcal{F}' - \mu r(r^{-1} B_\phi^* + \mu \Omega) \mathcal{G}' \right. \\ \left. - \gamma r \rho \mathcal{M}' - \mu r \rho \mathcal{N}' \right] \delta v_\phi - \left[ r^{-2} \Delta^* \psi + (r^{-1} B_\phi^* + \gamma \Omega) \mathcal{F}' \right. \\ \left. + (r^{-1} B_\phi^* + \mu \Omega) \mathcal{G}' + \rho \mathcal{M}' + \rho \mathcal{N}' \right] \delta \psi^* \left. \right\}. \quad (14) \end{aligned}$$

**B. Second order variation**

The expressions into the square brackets in (14) vanish on the EC equilibrium solution; therefore, the second order variation would involve only the first order variations of the fields. After some manipulations,  $\delta^2 \mathcal{H}_C[\mathbf{u}_e; \delta \mathbf{u}]$  can be written in the following form:

$$\begin{aligned} \delta^2 \mathcal{H}_C[\mathbf{u}_e; \delta \mathbf{u}] = & \int_D d^2x \left\{ \frac{d_e^2}{\rho r^2} |\nabla(r\delta B_\phi)|^2 + \frac{|\nabla\delta\psi|^2}{r^2} \right. \\ & + \frac{d_e^2 r^2}{\rho} [\nabla \cdot (r^{-2} \nabla \delta\psi)]^2 + \rho (\delta v_\phi + \rho^{-1} v_\phi \delta\rho)^2 \\ & + \rho |\delta \mathbf{v}_\perp + \rho^{-1} \mathbf{v}_\perp \delta\rho|^2 - 2 \frac{d_e^2}{r^2 \rho} \nabla(\delta\mathcal{F} + \delta\mathcal{G}) \cdot \nabla(r\delta B_\phi) \\ & + 2 \frac{d_e^2}{r^2 \rho^2} \nabla(\delta\mathcal{F} + \delta\mathcal{G}) \cdot \nabla(rB_\phi) \delta\rho \\ & \left. - 2[(\gamma \nabla \delta\mathcal{F} + \mu \nabla \delta\mathcal{G}) \times \nabla\phi] \cdot \delta \mathbf{v}_\perp \right\} + \mathcal{Q}, \end{aligned} \quad (15)$$

where

$$\mathcal{Q} = \int_D d^2x (\delta B_\phi \delta\varphi \delta\xi \delta\rho) \mathcal{A} (\delta B_\phi \delta\varphi \delta\xi \delta\rho)^T, \quad (16)$$

with

$$\mathcal{A} = \begin{pmatrix} 1 & A_{\varphi B_\phi} & A_{\xi B_\phi} & 0 \\ A_{\varphi B_\phi} & A_{\varphi\varphi} & 0 & A_{\varphi\rho} \\ A_{\xi B_\phi} & 0 & A_{\xi\xi} & A_{\xi\rho} \\ 0 & A_{\varphi\rho} & A_{\xi\rho} & A_{\rho\rho} \end{pmatrix}, \quad (17)$$

and the elements of  $\mathcal{A}$  are given explicitly by

$$A_{\varphi\varphi} = -\left(r^{-1} B_\phi^* + \gamma\Omega\right) \mathcal{F}'' - \rho \mathcal{M}'', \quad (18)$$

$$A_{\xi\xi} = -\left(r^{-1} B_\phi^* + \mu\Omega\right) \mathcal{G}'' - \rho \mathcal{N}'', \quad (19)$$

$$A_{\varphi B_\phi} = -r^{-1} \mathcal{F}', \quad A_{\xi B_\phi} = -r^{-1} \mathcal{G}', \quad (20)$$

$$A_{\varphi\rho} = -\mathcal{M}', \quad A_{\xi\rho} = -\mathcal{N}', \quad (21)$$

$$\begin{aligned} A_{\rho\rho} = & \rho^{-1} \left[ c_s^2 - v_\phi^2 - |\mathbf{v}_\perp|^2 - \frac{d_e^2}{\rho^2} \right. \\ & \left. \times \left( r^2 [\nabla \cdot (r^{-2} \nabla \psi)]^2 + r^{-2} |\nabla(rB_\phi)|^2 \right) \right], \end{aligned} \quad (22)$$

where  $c_s^2 := \rho h'(\rho)$ . In deriving (15), we integrated by parts, omitted the surface integrals, and completed squares in terms involving the mass density and velocity field variations.

For  $\mathcal{Q}$  alone to be positive definite, the matrix  $\mathcal{A}$  has to be positive definite, which is equivalent to the requirement that the principal minors of  $\mathcal{A}$  satisfy

$$A_{\varphi\varphi} - A_{\varphi B_\phi}^2 > 0, \quad (23)$$

$$A_{\xi\xi} (A_{\varphi\varphi} - A_{\varphi B_\phi}^2) - A_{\varphi\varphi} A_{\xi B_\phi}^2 > 0, \quad (24)$$

$$\begin{aligned} A_{\rho\rho} \left[ A_{\xi\xi} (A_{\varphi\varphi} - A_{\varphi B_\phi}^2) - A_{\varphi\varphi} A_{\xi B_\phi}^2 \right] \\ + (A_{\varphi B_\phi} A_{\xi\rho} - A_{\xi B_\phi} A_{\varphi\rho})^2 - A_{\xi\xi} A_{\varphi\rho}^2 - A_{\varphi\varphi} A_{\xi\rho}^2 > 0. \end{aligned} \quad (25)$$

However,  $\mathcal{Q} > 0$  does not imply stability because there are several indefinite terms in  $\delta^2 \mathcal{H}_C$ . More precisely, the first five terms in  $\delta^2 \mathcal{H}_C$

are always non-negative, with the magnetic terms expressing the magnetic field line bending, while the other two terms contain kinetic energy and compressional contributions of the perturbation. These kinetic-compressional terms constitute an example of the nonseparability of energies mentioned in the introduction, rendering the resulting stability conditions sufficient but not necessary. The nonseparability is even more severe, since kinetic and potential energy contributions are intertwined also via other terms in  $\delta^2 \mathcal{H}_C$  reflecting the fact that in the two-fluid framework, the coupling between flows and magnetic fields is more complicated. In particular, what really makes life difficult are the last three terms into the curly bracket in (15) because they are clearly sources of indefiniteness, a characteristic that has been identified in previous EC stability analyses of similar models<sup>45,46</sup> and can potentially be related to linear instability or the presence of Negative Energy Modes (NEMs). Both can lead to disastrous destabilization and loss of confinement. In order to remove the indefiniteness, we can eliminate or conflate these “problematic” terms into other terms in view of certain constraints imposed on the variations  $\delta B_\phi^*$  and  $\delta\Omega$  or by considering special equilibria.

**C. Special equilibria**

**1. Extended MHD**

For purely toroidal flow and current, i.e.,  $\mathcal{F}' = \mathcal{G}' = 0$ , it is clear that  $\mathcal{Q} > 0$  implies  $\delta^2 \mathcal{H}_C > 0$ . For our special class of equilibria, we have  $A_{\varphi B_\phi} = A_{\xi B_\phi} = 0$ , and consequently conditions (23)–(25) yield

$$\mathcal{M}'' < 0, \quad \mathcal{N}'' < 0, \quad (26)$$

$$\begin{aligned} \mathcal{M}'' \mathcal{N}'' \left[ c_s^2 - v_\phi^2 - \frac{d_e^2}{\rho^2} \left( r^2 [\nabla \cdot (r^{-2} \nabla \psi)]^2 \right) \right] \\ + \mathcal{M}'' (\mathcal{N}')^2 + \mathcal{N}'' (\mathcal{M}')^2 > 0. \end{aligned} \quad (27)$$

The first two conditions imply that  $\mathcal{M}$  and  $\mathcal{N}$  must be concave functions. For the condition (27) to be satisfied, the quantity inside the square bracket must necessarily be positive, that is, the toroidal velocity modified by an electron inertial correction has to be lower than the speed of sound, thus preventing shock formation.

**2. Hall MHD**

In the limit  $d_e \rightarrow 0$ ,  $\mu \rightarrow 0$  as well, so there is only one indefinite term in (15), which can be removed upon selecting  $\mathcal{F}' = 0$ . In this case, the flow is purely toroidal, but there is poloidal current created by the electrons. From (23)–(25), we obtain the following sufficient stability conditions:

$$\mathcal{M}'' < 0, \quad (28)$$

$$r^{-2} \mathcal{G}\mathcal{G}'' + \rho \mathcal{N}'' + r^{-2} (\mathcal{G}')^2 < 0, \quad (29)$$

$$\begin{aligned} \left[ \mathcal{M}'' (c_s^2 - v_\phi^2) + (\mathcal{M}')^2 \right] \left[ r^{-2} \mathcal{G}\mathcal{G}'' + \rho \mathcal{N}'' + r^{-2} (\mathcal{G}')^2 \right] \\ + \rho \mathcal{M}'' (\mathcal{N}')^2 > 0. \end{aligned} \quad (30)$$

The conditions above necessarily entail  $c_s^2 - v_\phi^2 > 0$ . This special case is interesting because the stability condition is expressed explicitly in terms of equilibrium quantities, and furthermore, it allows us to study the stability of nontrivial equilibria. For this reason, we proceed by constructing a Hall MHD equilibrium with purely toroidal rotation

and applying the criterion (28)–(30). From  $\delta\mathcal{H}_c = 0$  [see Eq. (14)], setting  $d_e = 0$  and imposing  $\mathbf{v}_\perp = \delta\mathbf{v}_\perp = 0$ , we can easily extract the equilibrium equations of interest. These are

$$\Delta^*\psi + \mathcal{G}\mathcal{G}'(\psi) + \rho \frac{\varphi - \psi}{d_i^2} + r^2 \rho \mathcal{N}'(\psi) = 0, \quad (31)$$

$$h(\rho) = \mathcal{M}(\varphi) + \mathcal{N}(\psi) - \frac{v_\phi^2}{2}, \quad (32)$$

$$B_\phi = r^{-1}\mathcal{G}(\psi), \quad v_\phi = d_i r \mathcal{M}'(\varphi), \quad (33)$$

$$\varphi - d_i^2 r^2 \mathcal{M}'(\varphi) = \psi, \quad (34)$$

where we have used the definition of  $\varphi$  to write  $v_\phi = \frac{\varphi - \psi}{d_i r}$ . Additionally, we consider the following nonlinear ansatz for the free functions  $\mathcal{G}$ ,  $\mathcal{M}$ , and  $\mathcal{N}$ :

$$\begin{aligned} \mathcal{G} &= g_0 + g_1\psi + \frac{1}{2}g_2\psi^2 + \frac{1}{3}g_3\psi^3, \\ \mathcal{M} &= m_0 + m_1\varphi + \frac{1}{2}m_2\varphi^2 + \frac{1}{3}m_3\varphi^3, \\ \mathcal{N} &= n_0 + n_1\psi + \frac{1}{2}n_2\psi^2 + \frac{1}{3}n_3\psi^3. \end{aligned} \quad (35)$$

We set  $m_1 = 0$ , which implies that there exists a solution to (34) for which  $\varphi = 0$  wherever  $\psi = 0$ ; therefore, the two flux functions satisfy the same boundary condition. We consider an adiabatic equation of state, i.e.,  $h(\rho) = \Gamma/(\Gamma - 1)p_1\rho^{\Gamma-1}$ , where  $\Gamma = 5/3$  is the adiabatic index and  $p_1$  is a constant. Then, Eq. (31) was solved numerically using finite differences and a simple SOR iterative solver on an up-down poloidally asymmetric domain with a prescribed diverted boundary having a lower x-point and tokamak pertinent values for the free parameters.

It is not difficult to adjust the free parameters in (35) to make conditions (28)–(29) to be satisfied everywhere in the plasma. However, when it comes to (30), we observe that for  $\beta > 1\%$ , the condition is satisfied only within a narrow annular region, wider on the high field side and narrower on the low field side. For  $\beta > 10\%$ , this region is even narrower forming a thin layer spreading across the high field side only (Fig. 1). For  $\beta < 1\%$ , we were able to find equilibria that satisfy all three conditions (28)–(30) all over the computational domain. This indicates that condition (30) is potentially related to the stabilization of pressure driven modes. To capture the influence of the Hall parameter  $d_i$  on stability, we considered an equilibrium with  $d_i = 0.04$  where all three stability conditions are satisfied everywhere outside a small region near the core. Then, we increased gradually  $d_i$ , observing that this region was continuously shrinking until it disappeared for  $d_i = 0.24$ . Thereby, we conclude that upon increasing  $d_i$ , the stability properties may be improved (see Fig. 2). We also corroborated that including the linear term in  $\mathcal{M}$ , which is related to rigid rotation and therefore being intrinsically destabilizing, shrinks the “stable” region toward the high field side. In closing, we underline that an equilibrium that fails to satisfy the stability conditions is not necessarily unstable, because the criteria we derived are only sufficient.

#### D. Conditional stability (constrained variations)

As mentioned earlier, the indefiniteness in  $\delta^2\mathcal{H}_C$  comes from the terms in (15) containing  $\delta\mathcal{F}$  and  $\delta\mathcal{G}$  and multiplied by  $\nabla \times \delta\mathbf{v}_\perp$ ,  $\delta B_\phi$  and  $\delta\rho$ . Hence, a simple way to get rid of the indefiniteness is to

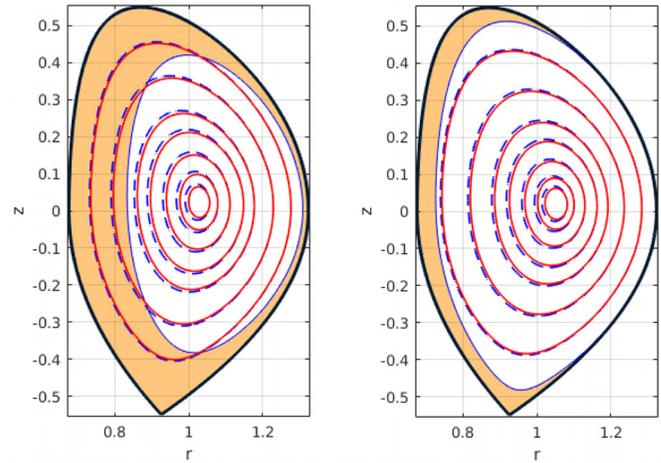


FIG. 1. The stability diagrams for two ITER-like equilibria with maximum  $\beta \sim 2\%$  (left) and  $\sim 20\%$  (right). In the colored regions, all three conditions (28)–(30) are satisfied. The Hall parameter is  $d_i = 0.04$  in both cases. The solid red lines represent the magnetic surfaces, while the dashed blue ones are surfaces of constant angular velocity.

assume  $\delta\rho = \delta B_\phi = \nabla \times \delta\mathbf{v}_\perp = 0$ . However, such a severe restriction of the permitted perturbations should be justified on physical grounds. Another possibility is to assume only  $\delta\rho = 0$ , which can be justified by the fact that incompressible variations are considered to be the most dangerous ones and then try to eliminate the explicit appearance of  $\delta B_\phi$  and  $\delta\mathbf{v}_\perp$  into  $\delta^2\mathcal{H}_C$  by other means. A way to do so is to partially minimize the functional (15) with respect to  $\delta\mathbf{v}_\perp$  and  $\delta B_\phi$ . This is a standard procedure to obtain simplified forms of the Lyapunov functional and improved stability criteria (e.g., see Refs. 36, 47, 54, and 55). The minimization can be realized upon considering  $\delta^2\mathcal{H}_C$  as a functional of the variations  $\delta\mathbf{u}$  and require its variation

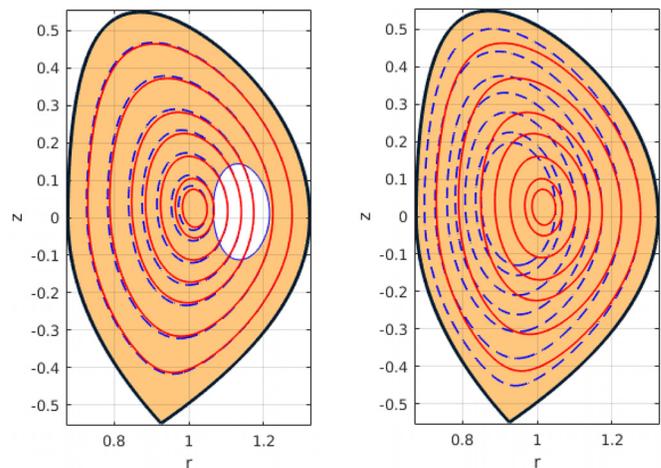


FIG. 2. The stability diagrams for equilibria with maximum  $\beta \sim 0.8\%$  with  $d_i = 0.04$  (left) and  $d_i = 0.24$  (right). While for  $d_i = 0.04$ , there is a hole within which (30) is not satisfied, increasing  $d_i$  results in a completely stable configuration, under EC variations.

with respect to  $\delta B_\phi$  and  $\delta v_\perp$  to vanish. The resulting Euler-Lagrange equations,

$$\delta B_\phi = r^{-1}(\delta\mathcal{F} + \delta\mathcal{G}), \tag{36}$$

$$\delta v_\perp = -\mathbf{v}_\perp \delta\rho + (\gamma\nabla\delta\mathcal{F} + \mu\nabla\delta\mathcal{G}) \times \nabla\phi, \tag{37}$$

are indeed minimizers of the functional, since the second variation with respect to  $\delta B_\phi$  and  $\delta v_\perp$  is positive definite. Henceforth, we set  $\delta\rho = 0$ . Upon substituting Eqs. (36)–(37) into (15), we find

$$\begin{aligned} \delta^2\tilde{\mathcal{H}}_C = \int_D d^2x \left\{ \frac{|\nabla\delta\psi|^2}{r^2} + \frac{d_e^2 r^2}{\rho} [\nabla \cdot (r^{-2}\nabla\delta\psi)]^2 \right. \\ \left. + \rho(\delta v_\phi)^2 + r^{-2}(\delta\mathcal{F} - \delta\mathcal{G})^2 \right\} + \tilde{\mathcal{Q}}, \end{aligned} \tag{38}$$

and therefore,  $\tilde{\mathcal{Q}} > 0$  implies stability. We have

$$\begin{aligned} \tilde{\mathcal{Q}} = \int_D d^2x \left[ A_{\varphi\varphi}(\delta\varphi)^2 + A_{\xi\xi}(\delta\xi)^2 \right] \\ - \int_D d^2x \left\{ \frac{\gamma^2 + d_e^2}{r^2\rho} [(\mathcal{F}')^2 |\nabla\delta\varphi|^2 + 2\mathcal{F}'\delta\varphi\nabla\mathcal{F}' \cdot \nabla\delta\varphi] \right. \\ \left. + \frac{\mu^2 + d_e^2}{r^2\rho} [(\mathcal{G}')^2 |\nabla\delta\xi|^2 + 2\mathcal{G}'\delta\xi\nabla\mathcal{G}' \cdot \nabla\delta\xi] \right. \\ \left. + \left[ 2r^{-2}(\mathcal{F}')^2 + \frac{\gamma^2 + d_e^2}{\rho r^2} |\nabla\mathcal{F}'|^2 \right] (\delta\varphi)^2 \right. \\ \left. + \left[ 2r^{-2}(\mathcal{G}')^2 + \frac{\mu^2 + d_e^2}{\rho r^2} |\nabla\mathcal{G}'|^2 \right] (\delta\xi)^2 \right\}. \end{aligned} \tag{39}$$

Following Ref. 40, let us define the vectors  $\mathbf{k}_\varphi := \nabla\delta\varphi/\delta\varphi$ ,  $\mathbf{k}_\xi := \nabla\delta\xi/\delta\xi$ . In view of this definition, we can write (39) in diagonal form with

$$\begin{aligned} \tilde{A}_{\varphi\varphi} = -\left( r^{-1}B_\phi^* + \gamma\Omega \right) \mathcal{F}'' - \rho\mathcal{M}'' - 2r^{-2}(\mathcal{F}')^2 \\ - \frac{\gamma^2 + d_e^2}{\rho r^2} \left[ |\nabla\mathcal{F}'|^2 + (\mathcal{F}')^2 |\mathbf{k}_\varphi|^2 + \mathbf{k}_\varphi \cdot \nabla(\mathcal{F}')^2 \right], \end{aligned} \tag{40}$$

$$\begin{aligned} \tilde{A}_{\xi\xi} = -\left( r^{-1}B_\phi^* + \mu\Omega \right) \mathcal{G}'' - \rho\mathcal{N}'' - 2r^{-2}(\mathcal{G}')^2 \\ - \frac{\mu^2 + d_e^2}{\rho r^2} \left[ |\nabla\mathcal{G}'|^2 + (\mathcal{G}')^2 |\mathbf{k}_\xi|^2 + \mathbf{k}_\xi \cdot \nabla(\mathcal{G}')^2 \right]. \end{aligned} \tag{41}$$

Invoking the Cauchy-Schwartz inequality, it is clear that the conditions

$$\begin{aligned} -\left( r^{-1}B_\phi^* + \gamma\Omega \right) \mathcal{F}'' - \rho\mathcal{M}'' - 2r^{-2}(\mathcal{F}')^2 \\ - \frac{\gamma^2 + d_e^2}{\rho r^2} \left[ |\nabla\mathcal{F}'|^2 + (\mathcal{F}')^2 |\mathbf{k}_\varphi|^2 + |\mathbf{k}_\varphi| |\nabla(\mathcal{F}')^2| \right] \\ \equiv a_\varphi |\mathbf{k}_\varphi|^2 + b_\varphi |\mathbf{k}_\varphi| + c_\varphi > 0, \end{aligned} \tag{42}$$

$$\begin{aligned} -\left( r^{-1}B_\phi^* + \mu\Omega \right) \mathcal{G}'' - \rho\mathcal{N}'' - 2r^{-2}(\mathcal{G}')^2 \\ - \frac{\mu^2 + d_e^2}{\rho r^2} \left[ |\nabla\mathcal{G}'|^2 + (\mathcal{G}')^2 |\mathbf{k}_\xi|^2 + |\mathbf{k}_\xi| |\nabla(\mathcal{G}')^2| \right] \\ \equiv a_\xi |\mathbf{k}_\xi|^2 + b_\xi |\mathbf{k}_\xi| + c_\xi > 0, \end{aligned} \tag{43}$$

are sufficient for  $\tilde{A}_{\varphi\varphi} > 0$  and  $\tilde{A}_{\xi\xi} > 0$  and therefore for  $\tilde{\mathcal{Q}} > 0$ . The two polynomials in  $|\mathbf{k}_\varphi|$  and  $|\mathbf{k}_\xi|$  must have at least one real positive root. Given that  $a_\varphi < 0$ ,  $b_\varphi < 0$  and  $a_\xi < 0$ ,  $b_\xi < 0$ , we understand that one root will always be negative; thus, in order for the second one to be positive, the products of the roots, given by  $c_\varphi/a_\varphi$ ,  $c_\xi/a_\xi$ , must be negative. Therefore, we conclude that the conditions under which there exists exactly one real positive root for each polynomial are

$$\begin{aligned} c_\varphi := -\left( r^{-1}B_\phi^* + \gamma\Omega \right) \mathcal{F}'' - \rho\mathcal{M}'' \\ - 2r^{-2}(\mathcal{F}')^2 - \frac{\gamma^2 + d_e^2}{\rho r^2} |\nabla\mathcal{F}'|^2 > 0, \end{aligned} \tag{44}$$

$$\begin{aligned} c_\xi := -\left( r^{-1}B_\phi^* + \mu\Omega \right) \mathcal{G}'' - \rho\mathcal{N}'' \\ - 2r^{-2}(\mathcal{G}')^2 - \frac{\mu^2 + d_e^2}{\rho r^2} |\nabla\mathcal{G}'|^2 > 0. \end{aligned} \tag{45}$$

Now, in view of (44)–(45), the two polynomials are also positive in the domain  $0 \leq |\mathbf{k}_\varphi| < k_\varphi^+$ ,  $0 \leq |\mathbf{k}_\xi| < k_\xi^+$ , where  $k_\varphi^+$  and  $k_\xi^+$  are the real positive roots of the polynomials in (42) and (43), respectively. This is true, since they do not change sign within this domain and furthermore they are positive for  $|\mathbf{k}_\varphi| = 0$ ,  $|\mathbf{k}_\xi| = 0$ . We thereby conclude that conditions (44) and (45) are sufficient for  $\tilde{A}_{\varphi\varphi} > 0$  and  $\tilde{A}_{\xi\xi} > 0$ , if  $|\mathbf{k}_\varphi| < k_\varphi^+$  and  $|\mathbf{k}_\xi| < k_\xi^+$ . On the other hand, there is a topological lower bound on the admissible values of  $k_\varphi$ ,  $k_\xi$  due to the Poincaré inequality,

$$\int_D d^2x |\mathbf{k}_\varphi|^2 (\delta\varphi)^2 = \int_D d^2x |\nabla\delta\varphi|^2 \geq C^{-1} \int_D d^2x (\delta\varphi)^2, \tag{46}$$

i.e.,  $\langle (|\mathbf{k}_\varphi|^2 - C^{-1})(\delta x)^2 \rangle \geq 0$ , where  $x = \varphi, \xi$ . Here,  $C$  is the Poincaré constant depending on the geometry of the domain  $D$ . Note that for smooth and bounded domains, the smallest eigenvalue of the Laplacian is an optimal value for  $C^{-1}$  since it minimizes the Rayleigh quotient. Finally, note that if we do not assume  $\delta\rho = 0$ , then an additional inequality of the form  $\tilde{A}_{\varphi\varphi}\tilde{A}_{\xi\xi}\tilde{A}_{\rho\rho} - \tilde{A}_{\varphi\varphi}\tilde{A}_{\xi\xi}^2 - \tilde{A}_{\xi\xi}\tilde{A}_{\varphi\varphi}^2 > 0$  will emerge. In this case, it turns out that  $|\mathbf{v}|^2 + d_e^2|\mathbf{j}|^2/\rho^2 < c_s^2$  is again necessary but not sufficient for stability. Possibly, similar manipulations to those employed above to arrive at sufficient conditions could be used; such a treatment though would introduce additional constraints on the admissible equilibria and the values of  $|\mathbf{k}_x|$ , restricting the range of applicability of the resulting stability criterion, which will diverge even more from necessity. Summarizing, the following sufficient conditional stability criterion holds for incompressible perturbations:

$$\begin{aligned} c_\varphi > 0, \quad c_\xi > 0, \\ \text{for } |\mathbf{k}_\varphi| < k_\varphi^+, \quad |\mathbf{k}_\xi| < k_\xi^+, \\ \langle (|\mathbf{k}_x|^2 - C^{-1})(\delta x)^2 \rangle \geq 0, \end{aligned} \tag{47}$$

where

$$k_x^+ = \frac{1}{2a_x} \left( -b_x - \sqrt{b_x^2 - 4a_x c_x} \right), \quad x = \varphi, \xi. \tag{48}$$

Note that the last inequality in (47) is satisfied for sure if  $\min(|\mathbf{k}_x|^2) \geq C^{-1}$  and hence,  $c_x > 0$ ,  $x = \varphi, \xi$  are sufficient stability conditions

if  $C^{-1} \leq |\mathbf{k}_x|^2 < k_x^+$ . As a final point, we stress that this stability criterion is general enough to capture a large variety of modes as long as  $k^+$ 's are large enough. Hence, this criterion is practically useful to assess the stability properties of equilibria, when the equilibrium states under consideration render  $k^+$ 's as large as possible.

### III. DYNAMICALLY ACCESSIBLE VARIATIONS

Within the noncanonical Hamiltonian framework, one can also consider the so-called dynamically accessible variations (DAVs) introduced in Refs. 20, 33, and 34 and used in the MHD context in Refs. 36, 44, and 47. The EC method is valid for general perturbations, but applicable only for EC equilibria, and as mentioned in Sec. II, many times, the perturbations need be restricted to be spatially symmetric. On the other hand, this defect is removed for DA stability analyses, which allow one to treat generic equilibria by restricting perturbations to adhere to phase space constraints; i.e., perturbations are restricted to lie on the symplectic leaves, which are essentially the level sets of the Casimirs. Because DAVs lie on the symplectic leaves, they conserve the Casimirs, that is,  $\delta C_{da} = 0$ , regardless of the equilibrium conditions.

In Refs. 20, 33, and 34, it was argued that the stability under DAVs is important because perturbations away from the symplectic leaf of the equilibrium under consideration, although well posed as an initial value problem, must come from physics outside the dynamical model being considered, since that dynamics preserves the Casimirs. If such physics is operative, then one might need to incorporate it into the dynamical model under consideration. If this were done, then EC or any other kind of stability analysis would likely change. Viewed this way, DA stability is quite natural to consider.

In addition to satisfying  $\delta C_{da} = 0$ , the first order DAVs nullify the Hamiltonian on generic equilibrium points, including the energy-Casimir ones; thus,

$$\delta \mathcal{H}[\mathbf{u}_e; \delta \mathbf{u}_{da}] = 0 \tag{49}$$

is a variational principle for generic equilibria. The sufficient stability criterion is provided by the positive definiteness of perturbation energy

$$\delta^2 \mathcal{H}_{da}[\mathbf{u}_e] = \int d^3x \left( \frac{\delta^2 \mathcal{H}}{\delta u^i \delta u^j} \Big|_{\mathbf{u}_e} \delta u_{da}^i \delta u_{da}^j + \frac{\delta \mathcal{H}}{\delta u^i} \Big|_{\mathbf{u}_e} \delta^2 u_{da}^i \right), \tag{50}$$

where  $\delta u_{da}$  and  $\delta^2 u_{da}$  are the first order and second order projections of arbitrary variations onto the symplectic leaves, respectively. Such DAVs are obtained from the generating functional given by  $\mathcal{W} = \int d^3x u_i g^i$ , where  $\mathbf{g}$  is a state vector embodying the arbitrariness of the perturbations of the various dynamical variables. The DAVs to the first order are given by  $\delta \mathbf{u}_{da} = \{\mathbf{u}, \mathcal{W}\}$ . In our case, one has

$$\mathcal{W} = \int_V d^3x (g_0 \rho + \mathbf{g}_1 \cdot \mathbf{v} + \mathbf{g}_2 \cdot \mathbf{B}^*), \tag{51}$$

generating the following variations:

$$\delta \rho_{da} = \{\rho, \mathcal{W}\} = -\nabla \cdot \mathbf{g}_1, \tag{52}$$

$$\delta \mathbf{v}_{da} = \{\mathbf{v}, \mathcal{W}\} = -\nabla g_0 + \rho^{-1} \mathbf{g}_1 \times \boldsymbol{\omega} + \rho^{-1} (\nabla \times \mathbf{g}_2) \times \mathbf{B}^*, \tag{53}$$

$$\delta \mathbf{B}_{da}^* = \{\mathbf{B}^*, \mathcal{W}\} = \nabla \times [\rho^{-1} (\mathbf{g}_1 - d_i \nabla \times \mathbf{g}_2) \times \mathbf{B}^* + d_e^2 \rho^{-1} (\nabla \times \mathbf{g}_2) \times \boldsymbol{\omega}]. \tag{54}$$

To show that the dynamically accessible variation of the Hamiltonian vanishes at general equilibria, we consider

$$\delta \mathcal{H}_{da} = \int_V d^3x \left[ \rho \mathbf{v} \cdot \delta \mathbf{v}_{da} + \left( h + \frac{v^2}{2} + d_e^2 \frac{|\mathbf{J}|^2}{2\rho} \right) \delta \rho_{da} + \mathbf{B} \cdot \delta \mathbf{B}_{da}^* \right], \tag{55}$$

with expressions (52)–(54). Upon performing integrations by parts and omitting the surface integrals, we find

$$\begin{aligned} \delta \mathcal{H}_{da} = & - \int_V d^3x \left\{ -g_0 \nabla \cdot (\rho \mathbf{v}) \right. \\ & + \mathbf{g}_1 \cdot \left[ \mathbf{v} \times \boldsymbol{\omega} - \nabla \left( h + \frac{v^2}{2} + d_e^2 \frac{|\mathbf{J}|^2}{2\rho} \right) + \frac{\mathbf{J} \times \mathbf{B}^*}{\rho} \right] \\ & \left. + \mathbf{g}_2 \cdot \nabla \times \left[ \mathbf{v} \times \mathbf{B}^* - d_i \frac{\mathbf{J} \times \mathbf{B}^*}{\rho} + d_e^2 \frac{\mathbf{J} \times \boldsymbol{\omega}}{\rho} \right] \right\}. \end{aligned} \tag{56}$$

It is apparent that the coefficients of  $g_0, \mathbf{g}_1, \mathbf{g}_2$  vanish in view of generic XMHD equilibrium conditions and consequently  $\delta \mathcal{H}_{da}[\mathbf{u}_e] = 0$ .

To proceed with the derivation of stability criteria, we need to calculate the second order variation of the Hamiltonian, which, in view of Eq. (50), is

$$\begin{aligned} \delta^2 \mathcal{H}_{da} = & \int_V d^3x \left\{ \rho |\delta \mathbf{v}_{da}|^2 + \left( h + \frac{v^2}{2} + d_e^2 \frac{|\mathbf{J}|^2}{2\rho^2} \right) \delta^2 \rho_{da} \right. \\ & + \left[ h'(\rho) - d_e^2 \frac{|\mathbf{J}|^2}{\rho^3} \right] (\delta \rho_{da})^2 + 2\mathbf{v} \cdot \delta \mathbf{v}_{da} \delta \rho_{da} + \rho \mathbf{v} \cdot \delta^2 \mathbf{v}_{da} \\ & \left. + \delta \mathbf{B}_{da} \cdot \delta \mathbf{B}_{da}^* + \mathbf{B} \cdot \delta^2 \mathbf{B}_{da}^* + \frac{d_e^2}{\rho^2} \mathbf{J} \cdot \delta \mathbf{J}_{da} \delta \rho_{da} \right\}. \end{aligned} \tag{57}$$

From the definition of  $\mathbf{B}^*$ , one has

$$\delta \mathbf{B}_{da}^* = \delta \mathbf{B}_{da} - d_e^2 \nabla \times \left( \frac{\mathbf{J}}{\rho^2} \delta \rho_{da} \right) + d_e^2 \nabla \times \left( \frac{\delta \mathbf{J}_{da}}{\rho} \right). \tag{58}$$

Upon inserting (58) into (57), the second term of (58) cancels out the last term in (57), leading to

$$\begin{aligned} \delta^2 \mathcal{H}_{da} = & \int_V d^3x \left\{ \rho |\delta \mathbf{v}_{da} + \rho^{-1} \mathbf{v} \delta \rho_{da}|^2 + |\delta \mathbf{B}_{da}|^2 \right. \\ & + d_e^2 \frac{|\delta \mathbf{J}_{da}|^2}{\rho} + \rho^{-1} \left( c_s^2 - |\mathbf{v}|^2 - d_e^2 \frac{|\mathbf{J}|^2}{\rho^2} \right) (\delta \rho_{da})^2 \\ & + \rho \mathbf{v} \cdot \delta^2 \mathbf{v}_{da} + \mathbf{B} \cdot \delta^2 \mathbf{B}_{da}^* \\ & \left. + \left( h + \frac{|\mathbf{v}|^2}{2} + d_e^2 \frac{|\mathbf{J}|^2}{2\rho^2} \right) \delta^2 \rho_{da} \right\}. \end{aligned} \tag{59}$$

The second order variations of the field variables are given by

$$\delta^2 \rho_{da} = 0, \tag{60}$$

$$\begin{aligned} \delta^2 \mathbf{v}_{da} &= \rho^{-1} \mathbf{g}_1 \times \nabla \times \delta \mathbf{v}_{da} + \rho^{-1} (\nabla \times \mathbf{g}_2) \times \delta \mathbf{B}_{da}^* - \rho^{-2} [\mathbf{g}_1 \times \boldsymbol{\omega} + (\nabla \times \mathbf{g}_2) \times \mathbf{B}^*] \delta \rho_{da} \\ &= \rho^{-1} (\boldsymbol{\zeta} \times \boldsymbol{\omega} + \boldsymbol{\eta} \times \mathbf{B}^*) \nabla \cdot (\rho \boldsymbol{\zeta}) + \boldsymbol{\zeta} \times \nabla \times (\boldsymbol{\zeta} \times \boldsymbol{\omega} + \boldsymbol{\eta} \times \mathbf{B}^*) + \boldsymbol{\eta} \times \nabla \times [(\boldsymbol{\zeta} - d_i \boldsymbol{\eta}) \times \mathbf{B}^* + d_e^2 \boldsymbol{\eta} \times \boldsymbol{\omega}], \end{aligned} \quad (61)$$

$$\begin{aligned} \delta^2 \mathbf{B}_{da}^* &= \nabla \times \{ \rho^{-1} (\mathbf{g}_1 - d_i \nabla \times \mathbf{g}_2) \times \delta \mathbf{B}_{da}^* + d_e^2 \rho^{-1} (\nabla \times \mathbf{g}_2) \times \nabla \times \delta \mathbf{v}_{da} - \rho^{-2} [(\mathbf{g}_1 - d_i \nabla \times \mathbf{g}_2) \times \mathbf{B}^* + d_e^2 (\nabla \times \mathbf{g}_2) \times \boldsymbol{\omega}] \delta \rho_{da} \} \\ &= \nabla \times \{ (\boldsymbol{\zeta} - d_i \boldsymbol{\eta}) \times \nabla \times [(\boldsymbol{\zeta} - d_i \boldsymbol{\eta}) \times \mathbf{B}^* + d_e^2 \boldsymbol{\eta} \times \boldsymbol{\omega}] + d_e^2 \boldsymbol{\eta} \times \nabla \times (\boldsymbol{\zeta} \times \boldsymbol{\omega} + \boldsymbol{\eta} \times \mathbf{B}^*) + \rho^{-1} [(\boldsymbol{\zeta} - d_i \boldsymbol{\eta}) \times \mathbf{B}^* + d_e^2 \boldsymbol{\eta} \times \boldsymbol{\omega}] \nabla \cdot (\rho \boldsymbol{\zeta}) \}, \end{aligned} \quad (62)$$

where  $\boldsymbol{\zeta} := \rho^{-1} \mathbf{g}_1$  and  $\boldsymbol{\eta} := \rho^{-1} \nabla \times \mathbf{g}_2$  have been introduced to facilitate the comparison with the previous MHD and HMHD results.<sup>22,36,47</sup> Evidently,  $\nabla \cdot (\rho \boldsymbol{\eta}) = 0$  holds by definition of  $\boldsymbol{\eta}$ . Substituting (60)–(62) into (57), we find via some straightforward calculations an

expression for  $\delta^2 \mathcal{H}_{da}$  (see Appendix A) that is difficult to be compared with the corresponding HMHD and MHD expressions derived in Refs. 22 and 47, respectively. However, after some tedious but also straightforward manipulations, (A1) can be brought in the following form:

$$\begin{aligned} \delta^2 \mathcal{H}_{da} &= \int_V d^3x \left\{ \rho |\nabla \mathbf{g}_0 + \boldsymbol{\zeta} \times \boldsymbol{\omega} + \boldsymbol{\eta} \times \mathbf{B}^* + \boldsymbol{\zeta} \cdot \nabla \mathbf{v} - \mathbf{v} \cdot \nabla \boldsymbol{\zeta}|^2 + |\delta \mathbf{B}_{da}|^2 - \rho \boldsymbol{\zeta} \cdot \nabla [h'(\rho) \nabla \cdot (\rho \boldsymbol{\zeta})] \right. \\ &\quad - (\boldsymbol{\zeta} \cdot \nabla h) \nabla \cdot (\rho \boldsymbol{\zeta}) - \boldsymbol{\zeta} \cdot (\mathbf{v} \cdot \nabla \mathbf{v}) \nabla \cdot (\rho \boldsymbol{\zeta}) - (\boldsymbol{\zeta} \times \mathbf{J}) \cdot \nabla \times (\boldsymbol{\zeta} \times \mathbf{B}^*) - \rho \boldsymbol{\zeta} \cdot [(\boldsymbol{\zeta} \cdot \nabla \mathbf{v} - \mathbf{v} \cdot \nabla \boldsymbol{\zeta}) \cdot \nabla \mathbf{v}] \\ &\quad - \rho \boldsymbol{\zeta} \cdot (\mathbf{v} \cdot \nabla) (\boldsymbol{\zeta} \cdot \nabla \mathbf{v} - \mathbf{v} \cdot \nabla \boldsymbol{\zeta}) + 2d_i (\boldsymbol{\zeta} \times \mathbf{J}) \cdot \nabla \times (\boldsymbol{\eta} \times \mathbf{B}^*) - d_i \rho (\boldsymbol{\eta} \times \mathbf{B}^*) \cdot [\boldsymbol{\eta} \cdot \nabla (\mathbf{v} - d_i \mathbf{J} / \rho) - (\mathbf{v} - d_i \mathbf{J} / \rho) \cdot \nabla \boldsymbol{\eta}] \\ &\quad + d_e^2 \rho^{-1} |\nabla \times \delta \mathbf{B}_{da}|^2 - d_e^2 \boldsymbol{\zeta} \cdot \nabla \left( \frac{|\mathbf{J}|^2}{2\rho^2} \right) \nabla \cdot (\rho \boldsymbol{\zeta}) + d_e^2 \rho \boldsymbol{\zeta} \cdot \nabla \left[ \frac{|\mathbf{J}|^2}{\rho^3} \nabla \cdot (\rho \boldsymbol{\zeta}) \right] - d_e^2 (\boldsymbol{\eta} \times \mathbf{J}) \cdot \nabla \times (\boldsymbol{\eta} \times \mathbf{B}^*) \\ &\quad \left. - d_e^2 [(2\boldsymbol{\zeta} - d_i \boldsymbol{\eta}) \times \mathbf{J}] \cdot \nabla \times (\boldsymbol{\eta} \times \boldsymbol{\omega}) - d_e^2 \rho (\boldsymbol{\eta} \times \mathbf{v}) \cdot \nabla \times (\boldsymbol{\eta} \times \boldsymbol{\omega}) \right\}. \end{aligned} \quad (63)$$

Now, it becomes clear that the case  $d_e = 0$  corresponds to the barotropic counterpart of the HMHD  $\delta^2 \mathcal{H}_{da}$  given in Ref. 22, while if we further impose  $d_i = 0$ , we find  $\delta^2 \mathcal{H}_{da} = \int_V d^3x \rho |\delta \mathbf{v}_{da} + \boldsymbol{\zeta} \cdot \nabla \mathbf{v} - \mathbf{v} \cdot \nabla \boldsymbol{\zeta}|^2 + \delta W$ , where  $\delta W$  is the Frieman-Rosenbluth expression for the potential energy,<sup>2</sup> consistent with the results found in Refs. 36 and 47.

The correct MHD limit of (63) reveals an important advantage of the DA method compared to the EC one. As it has been highlighted in Refs. 49–51, the MHD limit of the Casimirs and variational functionals (e.g., the Lagrangian) of XMHD and HMHD presents certain peculiarities because the Hall term gives rise to singular perturbations, making the derivation of their MHD counterparts rather not straightforward, a difficulty that, as regards to the Casimirs, was treated in Refs. 49 and 51. Hence, it is natural that this complication is inherited by the variational principles involving the Casimirs, e.g., the EC method. However, in the derivation of  $\delta^2 \mathcal{H}_{da}$ , we did not make use of the Casimirs, and therefore their problematic MHD limit does not affect the MHD limit of the DA stability criterion.

The Dirichlet stability theorem, the condition  $\delta^2 \mathcal{H}_{da} > 0 \forall \boldsymbol{\zeta}, \boldsymbol{\eta}, \mathbf{g}_0$ , ensures the stability of generic XMHD equilibria under dynamically accessible perturbations. However, as long as the variation of the magnetic field is treated as arbitrary, i.e., independent of  $\boldsymbol{\zeta}$  and  $\boldsymbol{\eta}$ , even though it is not, the criterion is based on the positiveness of the terms that do not contain  $\delta \mathbf{B}_{da}$ . Thus, we understand that an improvement of the stability criterion can be obtained upon relating  $\delta \mathbf{B}_{da}$  with  $\boldsymbol{\zeta}$  and  $\boldsymbol{\eta}$  by solving the differential equation that connects  $\delta \mathbf{B}_{da}$  with  $\delta \mathbf{B}_{da}^*(\boldsymbol{\zeta}, \boldsymbol{\eta})$  and  $\delta \rho_{da}(\boldsymbol{\zeta})$  and follows from the definition of  $\mathbf{B}^*$ . The

solution can be effected by introducing a tensorial Green’s function as follows:

$$\begin{aligned} \delta \mathbf{B}_{da} &= \int_{V'} d^3x' \mathbf{G}(\mathbf{x}', \mathbf{x}) \cdot \nabla \\ &\quad \times \left[ (\boldsymbol{\zeta} - d_i \boldsymbol{\eta}) \times \mathbf{B}^* + d_e^2 \boldsymbol{\eta} \times \boldsymbol{\omega} - d_e^2 \frac{\mathbf{J}}{\rho^2} \nabla \cdot (\rho \boldsymbol{\zeta}) \right], \end{aligned} \quad (64)$$

with  $\mathbf{G}(\mathbf{x}', \mathbf{x})$  being the solution of

$$\left[ 1 + d_e^2 \nabla \times \left( \frac{\nabla \times}{\rho} \right) \right] \mathbf{G}_i(\mathbf{x}', \mathbf{x}) = \mathbf{e}_i \delta(\mathbf{x}' - \mathbf{x}), \quad (65)$$

with  $i = 1, 2, 3$ . For  $\rho = const.$ , things are simpler since the operator on the lhs of (65) becomes the Helmholtz operator (because  $\nabla \cdot \delta \mathbf{B}_{da} = 0$ ) and if Cartesian coordinates are employed, then the equation splits into a set of three independent differential equations, one for each spatial component, in which case Green’s tensor can be replaced by a scalar Green’s function that can be written as an infinite sum of Helmholtz basis functions. The problem, though, remains highly dependent on the particular boundary conditions.

#### IV. PERTURBATIONS IN MIXED EULERIAN-LAGRANGIAN FRAMEWORK

In the Lagrangian framework, the fluids are not described in terms of fields measured at a fixed position  $\mathbf{x} \in \mathbb{R}^3$  as in the Eulerian framework adopted above, but in terms of Lagrangian or material

variables suitable for tracking the motion of the individual fluid elements. The material variables are the positions of the fluid elements at a given instant:  $\mathbf{q}_s(\mathbf{a}_s, t)$  ( $s = i, e$  standing for the ion and electron species), where  $\mathbf{a}_s \in \mathbb{R}^3$  are the fluid element labels, usually taken as the element's position at  $t = 0$ . The two viewpoints are connected through the so-called Lagrange-Euler map, which has to be consistent in the sense that an action written in the Lagrangian framework is mapped to an action written exclusively in terms of Eulerian variables, a requirement called the Eulerian Closure Principle (ECP).<sup>52,53</sup> For a two-fluid theory, which is the starting point of the XMHD model, the Lagrange-Euler map is described by the following relations:

$$\mathbf{v}_s(\mathbf{x}, t) = \dot{\mathbf{q}}_s(\mathbf{a}_s, t)|_{\mathbf{a}_s=\mathbf{q}_s^{-1}(\mathbf{x},t)}, \tag{66}$$

$$n_s(\mathbf{x}, t) = \frac{n_{s0}(\mathbf{a}_s)}{\mathcal{J}_s(\mathbf{a}_s, t)} \Big|_{\mathbf{a}_s=\mathbf{q}_s^{-1}(\mathbf{x},t)}, \tag{67}$$

$$s_s(\mathbf{x}, t) = s_{s0}(\mathbf{a}_s)|_{\mathbf{a}_s=\mathbf{q}_s^{-1}(\mathbf{x},t)}, \tag{68}$$

where  $s_s$  are the specific entropies of the fluids and  $\mathcal{J}_s$  ( $s = i, e$ ) are the Jacobians of  $\mathbf{q}_s$  with respect to  $\mathbf{a}_s$ , i.e.,  $\mathcal{J}_s := \det(\partial q_i^j / \partial a_s^j)$ . For barotropic fluids,  $s_s$  are just constants. Equations (66)–(68) are nothing more than the well known single fluid Lagrange-Euler map, described in detail in Ref. 20, written for each one of the constituent fluids. The difference between the single-fluid MHD and the two-fluid case is that in the former model, the magnetic field can be expressed in terms of Lagrangian variables, due to the frozen-in property of the magnetic field lines. In the case of HMHD and XMHD, one can find similar frozen-in properties<sup>31,32</sup> as well. However, in XMHD, this property concerns generalized magnetic-vorticity fields, and as a result, only the field  $\mathbf{B}^*$  can be explicitly expressed in terms of the Lagrangian variables. This means that similar expressions for  $\mathbf{B}$  can be found only implicitly through a relation similar to (64). This makes a fully Lagrangian description of the XMHD model more involved and less universal than the corresponding description for MHD, since it requires the solution of a differential equation for  $\mathbf{B}$ , which depends on the specific boundary conditions. Another peculiarity is that in a fully Lagrangian description, the usual Legendre transform cannot be performed, and therefore one needs to start with a phase-space Lagrangian.<sup>31</sup> One way to get rid of those peculiarities is to sacrifice some information about the relationship of the magnetic field with the fluid motion, describing the former as an independent Eulerian variable. Despite this compromise, the resulting mixed Eulerian-Lagrangian description<sup>35</sup> is still sufficient in order to perform stability analyses and make comparisons with other stability methods.

Lagrangian stability, being applicable for all possible equilibria and also considering perturbations that are not dynamically restricted or constrained by spatial symmetry, most times appears to be the most generic method available. To perform a stability analysis in terms of Lagrangian displacements, within a fully Lagrangian framework, as in the work of Newcomb<sup>56</sup> for MHD or a mixed Eulerian-Lagrangian framework as was done by Vuillemin<sup>57</sup> for the complete two-fluid model (without quasineutrality), we need to start with the Lagrangian of the model and compute its second order variation induced by small perturbations. The two-fluid Lagrangian with Maxwell's term being neglected in view of the assumption  $v_A \ll c$  ( $v_A$  and  $c$  are the Alfvén speed and the speed of light, respectively)<sup>35</sup> is

$$\begin{aligned} \mathcal{L} = \sum_{s=i,e} \int d^3x \Big\{ & \frac{1}{2} m_s n_{s0}(\mathbf{a}_s) |\dot{\mathbf{q}}_s(\mathbf{a}_s, t)|^2 - m_s n_{s0}(\mathbf{a}_s) U_s \left( s_s, \frac{m_s n_{s0}(\mathbf{a}_s)}{\mathcal{J}_s(\mathbf{a}_s, t)} \right) \\ & + \int d^3x \delta(\mathbf{x} - \mathbf{q}_s(\mathbf{a}_s, t)) e_s n_{s0}(\mathbf{a}_s) [\dot{\mathbf{q}}_s(\mathbf{a}_s, t) \cdot \mathbf{A}(\mathbf{x}, t) - \Phi(\mathbf{x}, t)] \Big\} \\ & - \frac{1}{2\mu_0} \int d^3x |\nabla \times \mathbf{A}(\mathbf{x}, t)|^2, \end{aligned} \tag{69}$$

where  $\mathbf{A}$  and  $\Phi$  are the vector and electrostatic potentials, respectively. Now, since the trajectories  $\mathbf{q}_i, \mathbf{q}_e$  of the ion and electron fluid elements are in general, different, then at time  $t > 0$ , they will be located at different positions  $\mathbf{x}$  and  $\mathbf{x}'$  unless the fluid elements  $\mathbf{a}_i$  and  $\mathbf{a}_e$  are chosen appropriately so to make  $\mathbf{x}' = \mathbf{x}$ . Imposing locality at the Eulerian level is equivalent to matching up the ion and electron fluid elements on the basis of the map  $\mathbf{a}_e = \mathbf{q}_e^{-1}(\mathbf{q}_i(\mathbf{a}_i, t), t)$  (see Fig. 3 and also the corresponding explanation in Ref. 31). In view of these considerations, we understand that local quasineutrality at the Lagrangian level is ensured if

$$n_{i0}(\mathbf{a}_i) = n_{e0}(\mathbf{a}_e)|_{\mathbf{a}_e=\mathbf{q}_e^{-1}(\mathbf{q}_i(\mathbf{a}_i,t),t)}. \tag{70}$$

In view of (67), Eq. (70) along with the imposition of Eulerian quasineutrality  $n_i(\mathbf{x}, t) = n_e(\mathbf{x}, t)$  leads to

$$\mathcal{J}_i|_{\mathbf{a}_i=\mathbf{q}_i^{-1}(\mathbf{x},t)} = \mathcal{J}_e|_{\mathbf{a}_e=\mathbf{q}_e^{-1}(\mathbf{x},t)}.$$

The final step for obtaining an XMHD action is to replace the ion and electron Lagrangian variables with XMHD-like variables, which would play the roles of Lagrangian analogs for  $\mathbf{v}$  and  $\mathbf{J}/(en)$ . In this regard, we define two new Lagrangian quantities  $\mathbf{Q}$  and  $\mathbf{D}$  through the following relations:

$$\mathbf{Q}(\mathbf{a}_i, \mathbf{a}_e, t) := \frac{m_i}{m} \mathbf{q}_i(\mathbf{a}_i, t) + \frac{m_e}{m} \mathbf{q}_e(\mathbf{a}_e, t), \tag{71}$$

$$\mathbf{D}(\mathbf{a}_i, \mathbf{a}_e, t) := \mathbf{q}_i(\mathbf{a}_i, t) - \mathbf{q}_e(\mathbf{a}_e, t). \tag{72}$$

The inverse transformation reads as follows:

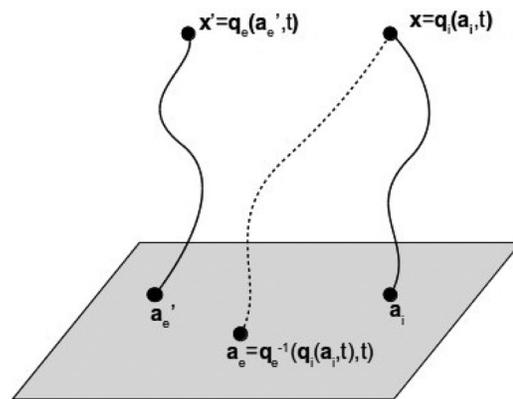


FIG. 3. The trajectories of a random pair of electron and ion fluid elements labeled by  $\mathbf{a}_e'$  and  $\mathbf{a}_i$ , respectively, end up at different locations at time  $t > 0$ . However, if the electron label is chosen so that  $\mathbf{a}_e = \mathbf{q}_e^{-1}(\mathbf{q}_i(\mathbf{a}_i, t), t)$ , then the trajectories intersect at time  $t > 0$ .

$$\begin{aligned} \mathbf{q}_i(\mathbf{a}_i, t) &= \mathbf{Q}(\mathbf{a}_i, \mathbf{a}_e, t) + \alpha_i \mathbf{D}(\mathbf{a}_i, \mathbf{a}_e, t), \\ \mathbf{q}_e(\mathbf{a}_e, t) &= \mathbf{Q}(\mathbf{a}_i, \mathbf{a}_e, t) + \alpha_e \mathbf{D}(\mathbf{a}_i, \mathbf{a}_e, t), \end{aligned} \quad (73)$$

where  $\alpha_i = m_e/m$  and  $\alpha_e = -m_i/m$ . We are now in a position to write down an XMHD Lagrangian in  $(\mathbf{Q}, \mathbf{D})$  variables as follows:

$$\begin{aligned} \mathcal{L} = & \int \int d^3 a_i d^3 a_e \delta(\mathbf{a}_e - \mathbf{q}_e^{-1}(\mathbf{q}_i(\mathbf{a}_i, t), t)) \sum_{s=i,e} n_{s0}(\mathbf{a}_s) \left[ \frac{m_s}{2} |\dot{\mathbf{Q}}|^2 \right. \\ & + \frac{m_s}{2} \alpha_s^2 |\dot{\mathbf{D}}|^2 + m_s \alpha_s \dot{\mathbf{Q}} \cdot \dot{\mathbf{D}} + e_s \dot{\mathbf{Q}} \cdot \mathbf{A}(\mathbf{q}_s(\mathbf{a}_s, t), t) - e_s \Phi(\mathbf{q}_s(\mathbf{a}_s, t), t) \\ & \left. + e_s \alpha_s \dot{\mathbf{D}} \cdot \mathbf{A}(\mathbf{q}_s(\mathbf{a}_s, t), t) - m_s U_s \left( s_s, \frac{m_s n_{s0}(\mathbf{a}_s)}{\mathcal{J}_s(\mathbf{a}_s, t)} \right) \right] \\ & - \frac{1}{2\mu_0} \int d^3 x |\nabla \times \mathbf{A}(\mathbf{x}, t)|^2. \end{aligned} \quad (74)$$

Note that  $\mathbf{Q}$  and  $\mathbf{D}$  depend on  $\mathbf{a}_i$  and  $\mathbf{a}_e$ . Also, the role of the delta function is to ensure the locality of the Eulerian version of (74), i.e., the trajectories  $\mathbf{q}_i$  and  $\mathbf{q}_e$  meet each other at  $t > 0$ . In general, we are interested in examining the stability of stationary equilibria in the Eulerian picture. It is well known<sup>20,36,56</sup> that not all Eulerian equilibria correspond to the Lagrangian ones, e.g., for a stationary Eulerian equilibrium, an infinite number of fluid elements have to be in motion for the realization of the flow. However, in the Lagrangian framework, moving fluid elements correspond to time dependent material variables. Therefore, we conclude that stationary Eulerian states correspond to time dependent Lagrangian trajectories  $\mathbf{q}_{s0} = \mathbf{q}_{s0}(\mathbf{a}_s, t)$ . Hence, we expand the material variables around time dependent reference trajectories considering a small perturbation, that is, the fields should be decomposed as follows:

$$\mathbf{Q}(\mathbf{a}_i, \mathbf{a}_e, t) = \mathbf{Q}_0(\mathbf{a}_i, \mathbf{a}_e, t) + \zeta(\mathbf{a}_i, \mathbf{a}_e, t), \quad (75)$$

$$\mathbf{D}(\mathbf{a}_i, \mathbf{a}_e, t) = \mathbf{D}_0(\mathbf{a}_i, \mathbf{a}_e, t) + \boldsymbol{\eta}(\mathbf{a}_i, \mathbf{a}_e, t), \quad (76)$$

$$\mathbf{A}(\mathbf{x}, t) = \mathbf{A}_0(\mathbf{x}) + \mathbf{A}_1(\mathbf{x}, t), \quad (77)$$

$$\Phi(\mathbf{x}, t) = \Phi_0(\mathbf{x}) + \Phi_1(\mathbf{x}, t), \quad (78)$$

where the quantities with subscript 0 define the equilibrium state, those with subscript 1 define the perturbed electromagnetic field, and  $\zeta, \boldsymbol{\eta}$  are Lagrangian displacements accounting for the perturbation of the fluid element trajectories. Hence, in view of (75)–(78), we find using (74) the perturbed  $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 + \dots$ . For stability, we are interested in  $\mathcal{L}_2$  because it describes the linearized dynamics, while  $\mathcal{L}_0$  is merely a constant and  $\mathcal{L}_1$  vanishes at equilibrium. To write down the second order perturbation of the Lagrangian, we need to expand the electromagnetic potentials and the internal energies. The magnetic and electric potentials are computed on the fluid trajectories; thus, up to second order, they are

$$\begin{aligned} \mathbf{A}(\mathbf{q}_{s0} + \zeta + \alpha_s \boldsymbol{\eta}, t) &= \mathbf{A}_0(\mathbf{q}_{s0}) + \mathbf{A}_1(\mathbf{q}_{s0}, t) + (\zeta + \alpha_s \boldsymbol{\eta}) \cdot \nabla_{\mathbf{q}_{s0}} \mathbf{A}_0(\mathbf{q}_{s0}) \\ &+ (\zeta + \alpha_s \boldsymbol{\eta}) \cdot \nabla_{\mathbf{q}_{s0}} \mathbf{A}_1(\mathbf{q}_{s0}, t) \\ &+ \frac{1}{2} (\zeta + \alpha_s \boldsymbol{\eta})(\zeta + \alpha_s \boldsymbol{\eta}) : \nabla_{\mathbf{q}_{s0}} \nabla_{\mathbf{q}_{s0}} \mathbf{A}_0(\mathbf{q}_{s0}), \end{aligned} \quad (79)$$

$$\begin{aligned} \Phi(\mathbf{q}_{s0} + \zeta + \alpha_s \boldsymbol{\eta}, t) &= \Phi_0(\mathbf{q}_{s0}) + \Phi_1(\mathbf{q}_{s0}, t) + (\zeta + \alpha_s \boldsymbol{\eta}) \cdot \nabla_{\mathbf{q}_{s0}} \Phi_0(\mathbf{q}_{s0}) \\ &+ (\zeta + \alpha_s \boldsymbol{\eta}) \cdot \nabla_{\mathbf{q}_{s0}} \Phi_1(\mathbf{q}_{s0}, t) \\ &+ \frac{1}{2} (\zeta + \alpha_s \boldsymbol{\eta})(\zeta + \alpha_s \boldsymbol{\eta}) : \nabla_{\mathbf{q}_{s0}} \nabla_{\mathbf{q}_{s0}} \Phi_0(\mathbf{q}_{s0}), \end{aligned} \quad (80)$$

where  $\mathbf{ab} : \mathbf{cd} := a_i b_j c^j d^i$ . The second order perturbative expansion of the internal energy terms is performed along lines similar to those of the single fluid case (see Ref. 20) in Appendix B. Henceforth, the subscript 0 will be dropped on the understanding that from now on  $\mathbf{A}, \Phi, \mathbf{Q}, \mathbf{D}, \mathbf{q}_s$ , and  $\mathcal{J}_s$  correspond to equilibrium. Using the results (79), (80) and (B3), we are able to construct  $\mathcal{L}_2$ ,

$$\begin{aligned} \mathcal{L}_2 = & \int \int d^3 a_i d^3 a_e \delta(\mathbf{a}_e - \mathbf{q}_e^{-1}(\mathbf{q}_i(\mathbf{a}_i, t), t)) \sum_s n_{s0}(\mathbf{a}_s) \left\{ \frac{m_s}{2} |\dot{\zeta}|^2 + \alpha_s^2 \frac{m_s}{2} |\dot{\boldsymbol{\eta}}|^2 + m_s \alpha_s \dot{\zeta} \cdot \dot{\boldsymbol{\eta}} + e_s (\dot{\mathbf{Q}} + \alpha_s \dot{\mathbf{D}}) \right. \\ & \cdot \left[ (\zeta + \alpha_s \boldsymbol{\eta}) \cdot \nabla_{\mathbf{q}_s} \mathbf{A}_1(\mathbf{q}_s, t) + \frac{1}{2} (\zeta + \alpha_s \boldsymbol{\eta})(\zeta + \alpha_s \boldsymbol{\eta}) : \nabla_{\mathbf{q}_s} \nabla_{\mathbf{q}_s} \mathbf{A}(\mathbf{q}_s) \right] \\ & + e_s (\dot{\zeta} + \alpha_s \dot{\boldsymbol{\eta}}) \cdot [\mathbf{A}_1(\mathbf{q}_s, t) + (\zeta + \alpha_s \boldsymbol{\eta}) \cdot \nabla_{\mathbf{q}_s} \mathbf{A}(\mathbf{q}_s)] - e_s (\zeta + \alpha_s \boldsymbol{\eta}) \cdot \nabla_{\mathbf{q}_s} \Phi_1(\mathbf{q}_s, t) \\ & - \frac{e_s}{2} (\zeta + \alpha_s \boldsymbol{\eta})(\zeta + \alpha_s \boldsymbol{\eta}) : \nabla_{\mathbf{q}_s} \nabla_{\mathbf{q}_s} \Phi(\mathbf{q}_s) - \frac{n_{s0}^2}{2\mathcal{J}_s^2} \frac{\partial^2 \mathcal{U}_s}{\partial n_s^2} (\nabla_{\mathbf{q}_s} \cdot \zeta + \alpha_s \nabla_{\mathbf{q}_s} \cdot \boldsymbol{\eta})^2 \\ & \left. - \frac{n_{s0}}{2\mathcal{J}_s} \frac{\partial \mathcal{U}_s}{\partial n_s} \left[ (\nabla_{\mathbf{q}_s} \cdot \zeta + \alpha_s \nabla_{\mathbf{q}_s} \cdot \boldsymbol{\eta})^2 + \nabla_{\mathbf{q}_s} (\zeta + \alpha_s \boldsymbol{\eta}) : \nabla_{\mathbf{q}_s} (\zeta + \alpha_s \boldsymbol{\eta}) \right] \right\} - \frac{1}{2\mu_0} \int d^3 x |\nabla \times \mathbf{A}_1(\mathbf{x}, t)|^2, \end{aligned} \quad (81)$$

where  $\mathcal{U}_s = m_s U_s$ . The locality of the perturbed Lagrangian density is imposed through the delta function in (81) by means of the equilibrium trajectories, i.e.,  $\delta(\mathbf{a}_e - \mathbf{q}_{e0}^{-1}(\mathbf{q}_{i0}(\mathbf{a}_i, t), t))$ . This is equivalent to imposing  $\mathbf{x}' = \mathbf{x}$ , after performing a single fluid Lagrange-Euler map (which involves the unperturbed trajectories, e.g., Ref. 20) for each constituent fluid. The Lagrangian (81) is not very different from the two-fluid result of Vuillemin;<sup>57</sup> actually, it is the quasineutral counterpart of his second order perturbed Lagrangian, written however, in terms of the XMHD Lagrangian displacements  $\zeta, \boldsymbol{\eta}$  instead of the

two-fluid ones  $\xi_i, \xi_e$ . Moreover, (81) is applicable for generic thermodynamic closures with scalar pressure, not only for fluids obeying the adiabatic ideal-gas law as in Ref. 57. The most important advantage of our formulation can be seen though, after employing the Lagrange-Euler map: first, because (81) explicitly dictates how the labels of the fluid elements are related so that the Lagrange-Euler map will result in a local Lagrangian and second because its Eulerian counterpart will be expressed in terms of the MHD-like variables  $\mathbf{v}$  and  $\mathbf{J}$ .

To employ the Lagrange-Euler map, we need to “Eulerianize” the displacement vectors. This procedure, along with the calculation of the Eulerian-field variations in terms of the Lagrangian displacements, which enables us to compare them with DAVs, is presented in [Appendix C](#). The Eulerian variations of the fields are

$$\delta \mathbf{v} = \partial_t \tilde{\boldsymbol{\zeta}} + \mathbf{v} \cdot \nabla \tilde{\boldsymbol{\zeta}} - \tilde{\boldsymbol{\zeta}} \cdot \nabla \mathbf{v} + \frac{m_i m_e}{m^2} (\mathbf{w} \cdot \nabla \tilde{\boldsymbol{\eta}} - \tilde{\boldsymbol{\eta}} \cdot \nabla \mathbf{w}), \tag{82}$$

$$\delta \mathbf{w} = \partial_t \tilde{\boldsymbol{\eta}} + \mathbf{v} \cdot \nabla \tilde{\boldsymbol{\eta}} - \tilde{\boldsymbol{\eta}} \cdot \nabla \mathbf{v} + \mathbf{w} \cdot \nabla \tilde{\boldsymbol{\zeta}} - \tilde{\boldsymbol{\zeta}} \cdot \nabla \mathbf{w} + \frac{m_e^2 - m_i^2}{m^2} (\mathbf{w} \cdot \nabla \tilde{\boldsymbol{\eta}} - \tilde{\boldsymbol{\eta}} \cdot \nabla \mathbf{w}), \tag{83}$$

where  $\mathbf{w} := \mathbf{J}/(en)$  and  $\tilde{\boldsymbol{\zeta}}, \tilde{\boldsymbol{\eta}}$  are the Eulerianized displacement vectors. Using the maps (C1) and (C3) from [Appendix C](#), and also relations (67) and (70) together with  $d^3x = \mathcal{J}_s d^3a_s$ , we compute the Eulerian expression for  $\mathcal{L}_s$  from the Lagrangian (81)

$$\begin{aligned} \mathcal{L}_2 = \int d^3x \left\{ \frac{mn}{2} |\partial_t \boldsymbol{\zeta}|^2 + \frac{m_i m_e}{2m} n |\partial_t \boldsymbol{\eta}|^2 + \partial_t \boldsymbol{\zeta} \cdot \left[ mn \left( \mathbf{v} \cdot \nabla \boldsymbol{\zeta} + \frac{m_i m_e}{m^2} \mathbf{w} \cdot \nabla \boldsymbol{\eta} \right) + en \boldsymbol{\eta} \cdot \nabla \mathbf{A} \right] \right. \\ \left. + \partial_t \boldsymbol{\eta} \cdot \left[ \frac{m_i m_e}{m} n \left( \mathbf{v} \cdot \nabla \boldsymbol{\eta} + \mathbf{w} \cdot \nabla \boldsymbol{\zeta} + \frac{m_e^2 - m_i^2}{m^2} \mathbf{w} \cdot \nabla \boldsymbol{\eta} \right) + en \left( \mathbf{A}_1 + \boldsymbol{\zeta} \cdot \nabla \mathbf{A} + \frac{m_e^2 - m_i^2}{m^2} \boldsymbol{\eta} \cdot \nabla \mathbf{A} \right) \right] + \mathfrak{B}(\boldsymbol{\zeta}, \boldsymbol{\eta}, \mathbf{A}_1, \Phi_1) \right\}, \tag{84} \end{aligned}$$

where

$$\begin{aligned} \mathfrak{B}(\boldsymbol{\zeta}, \boldsymbol{\eta}, \mathbf{A}_1, \Phi_1) = & -\frac{1}{2\mu_0} |\nabla \times \mathbf{A}_1|^2 + \frac{mn}{2} \left| \mathbf{v} \cdot \nabla \boldsymbol{\zeta} + \frac{m_i m_e}{m^2} \mathbf{w} \cdot \nabla \boldsymbol{\eta} \right|^2 \\ & + \frac{m_i m_e n}{2m} \left| \mathbf{v} \cdot \nabla \boldsymbol{\eta} + \mathbf{w} \cdot \nabla \boldsymbol{\zeta} + \frac{m_e^2 - m_i^2}{m^2} \mathbf{w} \cdot \nabla \boldsymbol{\eta} \right|^2 + en \left( \mathbf{v} \cdot \nabla \boldsymbol{\zeta} + \frac{m_i m_e}{m^2} \mathbf{w} \cdot \nabla \boldsymbol{\eta} \right) \cdot (\boldsymbol{\eta} \cdot \nabla \mathbf{A}) \\ & + en \left( \mathbf{v} \cdot \nabla \boldsymbol{\eta} + \mathbf{w} \cdot \nabla \boldsymbol{\zeta} + \frac{m_e^2 - m_i^2}{m^2} \mathbf{w} \cdot \nabla \boldsymbol{\eta} \right) \cdot \left( \mathbf{A}_1 + \boldsymbol{\zeta} \cdot \nabla \mathbf{A} + \frac{m_e^2 - m_i^2}{m^2} \boldsymbol{\eta} \cdot \nabla \mathbf{A} \right) \\ & + en \left[ \mathbf{v} \cdot (\boldsymbol{\eta} \cdot \nabla \mathbf{A}_1) + \mathbf{w} \cdot (\boldsymbol{\zeta} \cdot \nabla \mathbf{A}_1) + \mathbf{v} \cdot (\boldsymbol{\zeta} \boldsymbol{\eta} : \nabla \nabla \mathbf{A}) + \frac{1}{2} \mathbf{w} \cdot (\boldsymbol{\zeta} \boldsymbol{\zeta} : \nabla \nabla \mathbf{A}) + \frac{m_e^2 - m_i^2}{m^2} \mathbf{w} \cdot (\boldsymbol{\eta} \cdot \nabla \mathbf{A}_1) \right. \\ & + \frac{m_e^2 - m_i^2}{m^2} \mathbf{w} \cdot (\boldsymbol{\zeta} \boldsymbol{\eta} : \nabla \nabla \mathbf{A}) + \frac{m_e^2 - m_i^2}{2m^2} \mathbf{v} \cdot (\boldsymbol{\eta} \boldsymbol{\eta} : \nabla \nabla \mathbf{A}) + \frac{m_e^3 + m_i^3}{2m^3} \mathbf{w} \cdot (\boldsymbol{\eta} \boldsymbol{\eta} : \nabla \nabla \mathbf{A}) \\ & \left. - \boldsymbol{\zeta} \boldsymbol{\eta} : \nabla \nabla \Phi - \frac{m_e^2 - m_i^2}{2m^2} \boldsymbol{\eta} \boldsymbol{\eta} : \nabla \nabla \Phi - \boldsymbol{\eta} \cdot \nabla \Phi_1 \right] - \frac{p}{2} [\nabla \boldsymbol{\zeta} : \nabla \boldsymbol{\zeta} - (\nabla \cdot \boldsymbol{\zeta})^2] - \frac{1}{2} n \frac{\partial p}{\partial n} (\nabla \cdot \boldsymbol{\zeta})^2 \\ & - [\nabla \boldsymbol{\zeta} : \nabla \boldsymbol{\eta} - (\nabla \cdot \boldsymbol{\zeta})(\nabla \cdot \boldsymbol{\eta})] \left( \frac{m_e}{m} p_i - \frac{m_i}{m} p_e \right) - n \left( \frac{m_e}{m} \frac{\partial p_i}{\partial n} - \frac{m_i}{m} \frac{\partial p_e}{\partial n} \right) (\nabla \cdot \boldsymbol{\zeta})(\nabla \cdot \boldsymbol{\eta}) \\ & - \frac{1}{2} [\nabla \boldsymbol{\eta} : \nabla \boldsymbol{\eta} - (\nabla \cdot \boldsymbol{\eta})^2] \left[ \left( \frac{m_e}{m} \right)^2 p_i + \left( \frac{m_i}{m} \right)^2 p_e \right] - \frac{1}{2} n \left[ \left( \frac{m_e}{m} \right)^2 \frac{\partial p_i}{\partial n} + \left( \frac{m_i}{m} \right)^2 \frac{\partial p_e}{\partial n} \right] (\nabla \cdot \boldsymbol{\eta})^2. \tag{85} \end{aligned}$$

Here, we have used  $p_s = n^2 \partial \mathcal{U}_s / \partial n$ , Dalton’s law  $p = p_i + p_e$ , and in addition  $n^3 \partial^2 \mathcal{U}_s / \partial n^2 = n \partial p_s / \partial n - 2p_s$ . Also, the tildes have been dropped since we are working now in a completely Eulerian framework, and there is no need to distinguish from the Lagrangian variables. We should stress here that the version of the XMHD model we use in the previous sections was derived upon expanding the quasineutral two-fluid equations and keeping terms up to zeroth order in  $\mu := m_e/m_i$  in the Alfvén normalized equations of motion. In the derivations of this section, we have not performed such an expansion, and therefore until now, our results are fully two-fluid with quasineutrality. Hence, they can be used either to describe an ion-electron plasma or a positron-electron plasma, just by replacing the ion mass by the positron mass.

The Euler-Lagrange equations that correspond to (84) are obtained upon minimizing the action

$$\mathcal{S}_2 = \int_{t_1}^{t_2} dt \mathcal{L}_2, \tag{86}$$

with boundary conditions  $\boldsymbol{\zeta} \cdot \hat{\mathbf{n}} = \boldsymbol{\eta} \cdot \hat{\mathbf{n}} = 0$ , where  $\hat{\mathbf{n}}$  is the unit vector normal to the boundary and

$$\boldsymbol{\zeta}(\mathbf{x}, t = t_1) = \boldsymbol{\zeta}(\mathbf{x}, t = t_2) = \boldsymbol{\eta}(\mathbf{x}, t = t_1) = \boldsymbol{\eta}(\mathbf{x}, t = t_2) = 0.$$

These equations describe the linearized dynamics; more specifically, from the  $\boldsymbol{\zeta}$ -variation, one obtains the linearized momentum equation, while from  $\boldsymbol{\eta}$ -variations, a generalized Ohm’s law occurs. However, there are two redundant variables, namely,  $\mathbf{A}_1$  and  $\Phi_1$ , which do not appear in pairs of generalized coordinates and velocities. In some way, we need to express them in terms of the generalized coordinates so as to eliminate this redundancy. As regards  $\Phi_1$ , one can express it by selecting a particular gauge. Alternatively, we can compute the respective “Euler-Lagrange equations” that can be used either to eliminate  $\Phi_1$  and  $\mathbf{A}_1$  or as side conditions. Accordingly, extremizing the action with respect to the electromagnetic field variables, we find

$$\delta\Phi_1 : \quad e\nabla \cdot (n\boldsymbol{\eta}) = 0, \quad (87)$$

$$\begin{aligned} \delta\mathbf{A}_1 : \quad & en[\partial_t\boldsymbol{\eta} + \mathbf{v} \cdot \nabla\boldsymbol{\eta} - \boldsymbol{\eta} \cdot \nabla\mathbf{v} + \mathbf{w} \cdot \nabla\boldsymbol{\zeta} - \boldsymbol{\zeta} \cdot \nabla\mathbf{w} \\ & + \frac{m_e^2 - m_i^2}{m^2}(\mathbf{w} \cdot \nabla\boldsymbol{\eta} - \boldsymbol{\eta} \cdot \nabla\mathbf{w})] \\ & - \frac{\mathbf{J}}{n} \cdot \nabla \cdot (n\boldsymbol{\zeta}) - \mathbf{J}_1 = 0, \end{aligned} \quad (88)$$

where for the derivation of (88), we assumed  $(\mathbf{A}_1 \times \delta\mathbf{A}_1)|_{\partial D} \cdot \hat{\mathbf{n}} = 0$ . Equation (87) expresses the charge neutrality for the perturbed state. In view of this condition, the term that contains  $\Phi_1$  in  $\mathfrak{B}$  can be eliminated upon integrating by parts. Also, in principle, (88) can be used to express  $\mathbf{A}_1$  in terms of  $\boldsymbol{\zeta}$  and  $\boldsymbol{\eta}$ . Combining Eq. (88) with (83), we find the expression for the Eulerian variation of the particle density to be

$$n_1 = -\nabla \cdot (n\boldsymbol{\zeta}), \quad (89)$$

which is of the form of  $\delta\rho_{da}$  [see Eq. (52)].

To arrive at a sufficient stability condition, we need to calculate the Hamiltonian of the linearized dynamics. To this end, the standard procedure of Legendre transforming the Lagrangian (84) can be applied. The departing point for performing this transformation is to define the generalized momenta  $\boldsymbol{\pi}_\zeta$  and  $\boldsymbol{\pi}_\eta$  as follows:

$$\boldsymbol{\pi}_\zeta := \frac{\delta\mathcal{L}_2}{\delta\dot{\boldsymbol{\zeta}}} = mn \left( \partial_t\boldsymbol{\zeta} + \mathbf{v} \cdot \nabla\boldsymbol{\zeta} + \frac{m_i m_e}{m^2} \mathbf{w} \cdot \nabla\boldsymbol{\eta} \right) + en\boldsymbol{\eta} \cdot \nabla\mathbf{A}, \quad (90)$$

$$\begin{aligned} \boldsymbol{\pi}_\eta := \frac{\delta\mathcal{L}_2}{\delta\dot{\boldsymbol{\eta}}} = \frac{m_i m_e}{m} n \left( \partial_t\boldsymbol{\eta} + \mathbf{v} \cdot \nabla\boldsymbol{\eta} + \mathbf{w} \cdot \nabla\boldsymbol{\zeta} + \frac{m_e^2 - m_i^2}{m^2} \mathbf{w} \cdot \nabla\boldsymbol{\eta} \right) \\ + en \left( \mathbf{A}_1 + \boldsymbol{\zeta} \cdot \nabla\mathbf{A} + \frac{m_e^2 - m_i^2}{m^2} \boldsymbol{\eta} \cdot \nabla\mathbf{A} \right). \end{aligned} \quad (91)$$

With (90) and (91) at hand, we employ the usual Legendre transform,  $\mathcal{H}_2 = \int_D d^3x (\boldsymbol{\pi}_\zeta \cdot \partial_t\boldsymbol{\zeta} + \boldsymbol{\pi}_\eta \cdot \partial_t\boldsymbol{\eta}) - \mathcal{L}_2$ , to find

$$\begin{aligned} \mathcal{H}_2 = \int_D d^3x \left[ \frac{1}{2mn} \left| \boldsymbol{\pi}_\zeta - mn \left( \mathbf{v} \cdot \nabla\boldsymbol{\zeta} + \frac{m_i m_e}{m^2} \mathbf{w} \cdot \nabla\boldsymbol{\eta} \right) - en\boldsymbol{\eta} \cdot \nabla\mathbf{A} \right|^2 \right. \\ \left. + \frac{m}{2m_i m_e n} \left| \boldsymbol{\pi}_\eta - \frac{m_i m_e}{m} n \left( \mathbf{v} \cdot \nabla\boldsymbol{\eta} + \mathbf{w} \cdot \nabla\boldsymbol{\zeta} + \frac{m_e^2 - m_i^2}{m^2} \mathbf{w} \cdot \nabla\boldsymbol{\eta} \right) \right. \right. \\ \left. \left. - en \left( \mathbf{A}_1 + \boldsymbol{\zeta} \cdot \nabla\mathbf{A} + \frac{m_e^2 - m_i^2}{m^2} \boldsymbol{\eta} \cdot \nabla\mathbf{A} \right) \right|^2 - \mathfrak{B}(\boldsymbol{\zeta}, \boldsymbol{\eta}) \right]. \end{aligned} \quad (92)$$

From (92), we deduce that

$$- \int d^3x \mathfrak{B}(\boldsymbol{\zeta}, \boldsymbol{\eta}) \geq 0 \quad (93)$$

with  $\mathfrak{B}(\boldsymbol{\zeta}, \boldsymbol{\eta})$  given by (85) implies stability.

### V. HALL MHD

The HMHD case has an interesting peculiarity: to derive the HMHD perturbed Lagrangian, we assume massless electrons, i.e.,  $m_e = 0$ ; as a result,  $\partial_t\boldsymbol{\eta}$  appears linearly in  $\mathcal{L}_2$ , and therefore the definition of the canonical momentum  $\boldsymbol{\pi}_\eta$  results in a constraint instead of an equation that can be used to express  $\partial_t\boldsymbol{\eta}$  in terms of  $\boldsymbol{\pi}_\eta$ . But before addressing this peculiarity, we Alfvén normalize the HMHD Lagrangian term by term so as to facilitate comparisons with the already known results in this framework. The Alfvén normalization is effected by

$$\begin{aligned} \bar{n} = n/n_0, \quad \bar{t} = t/\tau_A, \quad \bar{\nabla} = \ell\nabla, \\ \bar{\mathbf{B}} = \mathbf{B}/B_0, \quad \bar{\mathbf{J}} = \mathbf{J}/(B_0/\ell\mu_0), \quad \bar{\mathbf{A}} = \mathbf{A}/(\ell B_0), \\ \bar{\mathbf{E}} = \mathbf{E}/(v_A B_0), \quad \bar{\Phi} = \Phi/(\ell v_A B_0), \quad \bar{p}_s = p_s/(B_0^2/\mu_0), \end{aligned} \quad (94)$$

where  $\ell$ ,  $n_0$ , and  $B_0$  are the reference length, the particle density, and the magnetic field, respectively;  $v_A = B_0/\sqrt{\mu_0 m_i n_0}$  is the Alfvén speed, and  $\tau_A = \ell/v_A$  is the Alfvén time. In order to write the Lagrangian in dimensionless form, we also need to introduce normalized displacements  $\boldsymbol{\zeta}$  and  $\boldsymbol{\eta}$ . Equations (82) and (83) suggest that an appropriate normalization is

$$\bar{\boldsymbol{\zeta}} = \boldsymbol{\zeta}/\ell, \quad \bar{\boldsymbol{\eta}} = \boldsymbol{\eta}/\sqrt{m_i/\mu_0 n_0 e^2} = \boldsymbol{\eta}/\lambda_i, \quad (95)$$

where  $\lambda_i$  is the ion skin depth ( $\lambda_i = d_i\ell$ ). In view of (94) and (95) and setting  $m_e = 0$ , the Lagrangian (84) can be brought into the following dimensionless form:

$$\begin{aligned} \mathcal{L}_2 = \int d^3x \left\{ \frac{\rho}{2} |\partial_t\boldsymbol{\zeta}|^2 + \rho(\partial_t\boldsymbol{\zeta}) \cdot (\boldsymbol{\eta} \cdot \nabla\mathbf{A} + \mathbf{v} \cdot \nabla\boldsymbol{\zeta}) \right. \\ \left. + \rho(\partial_t\boldsymbol{\eta}) \cdot (\mathbf{A}_1 + \boldsymbol{\zeta} \cdot \nabla\mathbf{A} - d_i\boldsymbol{\eta} \cdot \nabla\mathbf{A}) + \mathfrak{B}_{hmhd}(\boldsymbol{\zeta}, \boldsymbol{\eta}, \mathbf{A}_1) \right\}, \end{aligned} \quad (96)$$

where

$$\begin{aligned} \mathfrak{B}_{hmhd} = \frac{\rho}{2} |\mathbf{v} \cdot \nabla\boldsymbol{\zeta}|^2 + \rho(\mathbf{v} \cdot \nabla\boldsymbol{\zeta}) \cdot (\boldsymbol{\eta} \cdot \nabla\mathbf{A}) + \rho(\mathbf{v} \cdot \nabla\boldsymbol{\eta} + \rho^{-1}\mathbf{J} \cdot \nabla\boldsymbol{\zeta} - d_i\rho^{-1}\mathbf{J} \cdot \nabla\boldsymbol{\eta}) \cdot (\mathbf{A}_1 + \boldsymbol{\zeta} \cdot \nabla\mathbf{A} - d_i\boldsymbol{\eta} \cdot \nabla\mathbf{A}) + \rho\mathbf{v} \cdot (\boldsymbol{\eta} \cdot \nabla\mathbf{A}_1) \\ + \rho\mathbf{v} \cdot (\boldsymbol{\zeta}\boldsymbol{\eta} : \nabla\nabla\mathbf{A}) - \frac{d_i}{2}\rho\mathbf{v} \cdot (\boldsymbol{\eta}\boldsymbol{\eta} : \nabla\nabla\mathbf{A}) + \mathbf{J} \cdot (\boldsymbol{\zeta} \cdot \nabla\mathbf{A}_1) - d_i\mathbf{J} \cdot (\boldsymbol{\eta} \cdot \nabla\mathbf{A}_1) + \frac{1}{2}\mathbf{J} \cdot (\boldsymbol{\zeta}\boldsymbol{\zeta} : \nabla\nabla\mathbf{A}) \\ - d_i\mathbf{J} \cdot (\boldsymbol{\zeta}\boldsymbol{\eta} : \nabla\nabla\mathbf{A}) + \frac{d_i^2}{2}\mathbf{J} \cdot (\boldsymbol{\eta}\boldsymbol{\eta} : \nabla\nabla\mathbf{A}) - \rho\boldsymbol{\eta} \cdot \nabla\Phi_1 - \rho(\boldsymbol{\zeta}\boldsymbol{\eta} : \nabla\nabla\Phi) + \frac{d_i}{2}\rho\boldsymbol{\eta}\boldsymbol{\eta} : \nabla\nabla\Phi \\ - \frac{\rho}{2} [\nabla\boldsymbol{\zeta} : \nabla\boldsymbol{\zeta} - (\nabla \cdot \boldsymbol{\zeta})^2] - \frac{\rho}{2} c_s^2 (\nabla \cdot \boldsymbol{\zeta})^2 + d_i p_e [\nabla\boldsymbol{\zeta} : \nabla\boldsymbol{\eta} - (\nabla \cdot \boldsymbol{\zeta})(\nabla \cdot \boldsymbol{\eta})] + d_i \rho c_{se}^2 (\nabla \cdot \boldsymbol{\zeta})(\nabla \cdot \boldsymbol{\eta}) \\ - \frac{d_i^2}{2} p_e [\nabla\boldsymbol{\eta} : \nabla\boldsymbol{\eta} - (\nabla \cdot \boldsymbol{\eta})^2] - \frac{d_i^2}{2} \rho c_{se}^2 (\nabla \cdot \boldsymbol{\eta})^2 - \frac{1}{2} |\mathbf{B}_1|^2, \end{aligned} \quad (97)$$

and the bars have been dropped. Note that the term,  $\int d^3x \rho\boldsymbol{\eta} \cdot \nabla\Phi_1$ , vanishes in view of (87) and the boundary conditions. In addition, the perturbation of the velocity field and the field  $\mathbf{J}/\rho$  is given by

$$\begin{aligned} \delta \mathbf{v} &= \partial_t \boldsymbol{\zeta} + \mathbf{v} \cdot \nabla \boldsymbol{\zeta} - \boldsymbol{\zeta} \cdot \nabla \mathbf{v}, \quad (98) \\ \delta \left( \frac{\mathbf{J}}{\rho} \right) &= \partial_t \boldsymbol{\eta} + \mathbf{v} \cdot \nabla \boldsymbol{\eta} - \boldsymbol{\eta} \cdot \nabla \mathbf{v} \\ &+ \frac{\mathbf{J}}{\rho} \cdot \nabla \boldsymbol{\zeta} - \boldsymbol{\zeta} \cdot \nabla \frac{\mathbf{J}}{\rho} - d_i \left( \frac{\mathbf{J}}{\rho} \cdot \nabla \boldsymbol{\eta} - \boldsymbol{\eta} \cdot \nabla \frac{\mathbf{J}}{\rho} \right), \quad (99) \end{aligned}$$

while the generalized momenta  $\boldsymbol{\pi}_\zeta$  and  $\boldsymbol{\pi}_\eta$  are now computed as follows:

$$\boldsymbol{\pi}_\zeta = \frac{\delta \mathcal{L}_2}{\delta \boldsymbol{\zeta}} = \rho (\partial_t \boldsymbol{\zeta} + \mathbf{v} \cdot \nabla \boldsymbol{\zeta}) + \rho \boldsymbol{\eta} \cdot \nabla \mathbf{A}, \quad (100)$$

$$\boldsymbol{\pi}_\eta = \frac{\delta \mathcal{L}_2}{\delta \boldsymbol{\eta}} = \rho (\mathbf{A}_1 + \boldsymbol{\zeta} \cdot \nabla \mathbf{A} - d_i \boldsymbol{\eta} \cdot \nabla \mathbf{A}). \quad (101)$$

Note that Eq. (101) cannot be used in order to express  $\partial_t \boldsymbol{\eta}$  in terms of  $\boldsymbol{\pi}_\eta$ ; therefore, it can be interpreted as a constraint between the dynamical variables, which helps us though to express explicitly  $\mathbf{A}_1$  in terms of canonical variables via  $\mathbf{A}_1 = \rho^{-1} \boldsymbol{\pi}_\eta - (\boldsymbol{\zeta} - d_i \boldsymbol{\eta}) \cdot \nabla \mathbf{A}$ . A consistency condition is that this equation holds for all time, i.e., that it is preserved by the dynamics,

$$[\boldsymbol{\pi}_\eta - \rho (\mathbf{A}_1 + \boldsymbol{\zeta} \cdot \nabla \mathbf{A} - d_i \boldsymbol{\eta} \cdot \nabla \mathbf{A}), \mathcal{H}_2] = 0, \quad (102)$$

where

$$[f, g] = \int d^3x \left( \frac{\delta f}{\delta \boldsymbol{\zeta}} \cdot \frac{\delta g}{\delta \boldsymbol{\pi}_\zeta} - \frac{\delta g}{\delta \boldsymbol{\zeta}} \cdot \frac{\delta f}{\delta \boldsymbol{\pi}_\zeta} + \frac{\delta f}{\delta \boldsymbol{\eta}} \cdot \frac{\delta g}{\delta \boldsymbol{\pi}_\eta} - \frac{\delta g}{\delta \boldsymbol{\eta}} \cdot \frac{\delta f}{\delta \boldsymbol{\pi}_\eta} \right) \quad (103)$$

$$\begin{aligned} \partial_t \boldsymbol{\pi}_\zeta &= - \frac{\delta \mathcal{H}_2}{\delta \boldsymbol{\zeta}} = - \left\{ \rho \mathbf{v} \cdot \nabla [\rho^{-1} \boldsymbol{\pi}_\zeta - \mathbf{v} \cdot \nabla \boldsymbol{\zeta} - \boldsymbol{\eta} \cdot \nabla \mathbf{A}] + \rho \mathbf{v} \cdot \nabla (\boldsymbol{\eta} \cdot \nabla \mathbf{A}) + \rho \mathbf{v} \cdot \nabla (\mathbf{v} \cdot \nabla \boldsymbol{\zeta}) + \mathbf{J} \cdot \nabla \frac{\boldsymbol{\pi}_\eta}{\rho} \right. \\ &- \rho \nabla \mathbf{A} \cdot \left[ (\boldsymbol{\eta} \cdot \nabla \mathbf{v}) + (\boldsymbol{\zeta} - d_i \boldsymbol{\eta}) \cdot \nabla \frac{\mathbf{J}}{\rho} + \frac{\mathbf{J}}{\rho} \cdot \nabla (\rho \boldsymbol{\zeta}) + \nabla \times \nabla \times (\rho^{-1} \boldsymbol{\pi}_\eta - \boldsymbol{\zeta} \cdot \nabla \mathbf{A} + d_i \boldsymbol{\eta} \cdot \nabla \mathbf{A}) \right] \\ &- \rho (\boldsymbol{\eta} \cdot \nabla \nabla \mathbf{A}) \cdot \mathbf{v} - \nabla [\rho^{-1} \boldsymbol{\pi}_\eta - (\boldsymbol{\zeta} - d_i \boldsymbol{\eta}) \cdot \nabla \mathbf{A}] \cdot \mathbf{J} - (\boldsymbol{\zeta} \cdot \nabla \nabla \mathbf{A}) \cdot \mathbf{J} + d_i (\boldsymbol{\eta} \cdot \nabla \nabla \mathbf{A}) \cdot \mathbf{J} + \rho \boldsymbol{\eta} \cdot \nabla \nabla \Phi \\ &\left. + \nabla p \nabla \cdot \boldsymbol{\zeta} - \nabla \boldsymbol{\zeta} \cdot \nabla p - \nabla \left( \rho \frac{\partial p}{\partial \rho} \nabla \cdot \boldsymbol{\zeta} \right) - d_i \nabla p_e \nabla \cdot \boldsymbol{\eta} + d_i \nabla \boldsymbol{\eta} \cdot \nabla p_e + d_i \nabla \left( \rho \frac{\partial p_e}{\partial \rho} \nabla \cdot \boldsymbol{\eta} \right) \right\}. \quad (109) \end{aligned}$$

Combining (107) with (101) and (99) gives

$$\rho_1 = -\nabla \cdot (\rho \boldsymbol{\zeta}). \quad (110)$$

Equation (107) is merely the definition of the canonical momentum  $\boldsymbol{\pi}_\zeta$ . Exploiting the definitions (100) and (101), the relations (98) and (99), and also the stationary momentum equation and Ohm's law, which are given by

$$\mathbf{v} \cdot \nabla \mathbf{v} - \rho^{-1} \mathbf{J} \times \mathbf{B} + \rho^{-1} \nabla p = 0, \quad (111)$$

$$-\nabla \Phi + \left( \mathbf{v} - d_i \frac{\mathbf{J}}{\rho} \right) \times \mathbf{B} + \rho^{-1} \nabla p_e = 0, \quad (112)$$

we can corroborate that (108) and (109) give the perturbed Ohm's law and momentum equation, respectively. Note that  $\mathfrak{B}_{hmhd}$  is not yet fully expressed in terms of the displacement vectors  $\boldsymbol{\zeta}$  and  $\boldsymbol{\eta}$  due to  $\boldsymbol{\pi}_\eta$ , which appears explicitly in its expression. We can overcome this by combining the consistency condition (105) with the Hamiltonian equations (106) and (108) to find

is the canonical Poisson bracket and

$$\mathcal{H}_2 = \int d^3x \left[ \frac{1}{2\rho} |\boldsymbol{\pi}_\zeta - \rho \mathbf{v} \cdot \nabla \boldsymbol{\zeta} - \rho \boldsymbol{\eta} \cdot \nabla \mathbf{A}|^2 - \mathfrak{B}_{hmhd}(\boldsymbol{\zeta}, \boldsymbol{\eta}, \boldsymbol{\pi}_\eta) \right], \quad (104)$$

where  $\mathbf{A}_1$  has been expressed via Eq. (101). From (102)–(104), we find

$$\begin{aligned} - \frac{\delta \mathfrak{B}_{hmhd}}{\delta \boldsymbol{\eta}} &= d_i \nabla \mathbf{A} \cdot \left\{ \frac{\mathbf{J}}{\rho} \cdot \nabla (\rho \boldsymbol{\zeta}) + \rho \left[ (\boldsymbol{\zeta} - d_i \boldsymbol{\eta}) \cdot \nabla \frac{\mathbf{J}}{\rho} - \mathbf{v} \cdot \nabla \boldsymbol{\eta} \right. \right. \\ &\left. \left. + \boldsymbol{\eta} \cdot \nabla \mathbf{v} - \frac{\mathbf{J}}{\rho} \cdot \nabla \boldsymbol{\zeta} + d_i \frac{\mathbf{J}}{\rho} \cdot \nabla \boldsymbol{\eta} \right] + \nabla \times \nabla \right. \\ &\left. \times [\rho^{-1} \boldsymbol{\pi}_\eta - (\boldsymbol{\zeta} - d_i \boldsymbol{\eta}) \cdot \nabla \mathbf{A}] \right\}. \quad (105) \end{aligned}$$

Now, let us proceed by computing the Hamiltonian equations of motion

$$\begin{aligned} \partial_t \boldsymbol{\eta} &= \frac{\delta \mathcal{H}_2}{\delta \boldsymbol{\pi}_\eta} = -\mathbf{v} \cdot \nabla \boldsymbol{\eta} + \boldsymbol{\eta} \cdot \nabla \mathbf{v} - \frac{\mathbf{J}}{\rho} \cdot \nabla \boldsymbol{\zeta} + \boldsymbol{\zeta} \cdot \nabla \frac{\mathbf{J}}{\rho} \\ &+ d_i \left( \frac{\mathbf{J}}{\rho} \cdot \nabla \boldsymbol{\eta} - \boldsymbol{\eta} \cdot \nabla \frac{\mathbf{J}}{\rho} \right) + \frac{\mathbf{J}}{\rho^2} \cdot \nabla (\rho \boldsymbol{\zeta}) \\ &+ \rho^{-1} \nabla \times \nabla \times [\rho^{-1} \boldsymbol{\pi}_\eta - (\boldsymbol{\zeta} - d_i \boldsymbol{\eta}) \cdot \nabla \mathbf{A}], \quad (106) \end{aligned}$$

$$\partial_t \boldsymbol{\zeta} = \frac{\delta \mathcal{H}_2}{\delta \boldsymbol{\pi}_\zeta} = \rho^{-1} (\boldsymbol{\pi}_\zeta - \rho \mathbf{v} \cdot \nabla \boldsymbol{\zeta} - \rho \boldsymbol{\eta} \cdot \nabla \mathbf{A}), \quad (107)$$

$$\partial_t \boldsymbol{\pi}_\eta = - \frac{\delta \mathcal{H}_2}{\delta \boldsymbol{\eta}} = \nabla \mathbf{A} \cdot (\boldsymbol{\pi}_\zeta - \rho \mathbf{v} \cdot \nabla \boldsymbol{\zeta} - \rho \boldsymbol{\eta} \cdot \nabla \mathbf{A}) + \frac{\delta \mathfrak{B}_{hmhd}}{\delta \boldsymbol{\eta}}, \quad (108)$$

$$\partial_t \mathbf{A}_1 = \partial_t (\boldsymbol{\zeta} - d_i \boldsymbol{\eta}) \times \mathbf{B}_0. \quad (113)$$

Integrating in time would, in general, introduce a stationary vector; however, this should vanish because otherwise time independent terms would appear in the linearized dynamical equations. Therefore,  $\mathbf{A}_1 = (\boldsymbol{\zeta} - d_i \boldsymbol{\eta}) \times \mathbf{B}_0$  or  $\mathbf{B}_1 = \nabla \times [(\boldsymbol{\zeta} - d_i \boldsymbol{\eta}) \times \mathbf{B}_0]$ , which is the well-known solution of the perturbed induction equation (see Ref. 22). This expression is similar to the corresponding expression in ideal MHD. The difference is the appearance of the displacement vector  $\boldsymbol{\eta}$  multiplied by  $d_i$ ; so, the MHD result can be recovered in the limit  $d_i \rightarrow 0$ . This is an anticipated result, since the fluid velocity in the MHD induction equation is replaced by  $\mathbf{v} - d_i \mathbf{J}$  in the HMHD case. Finally, since (104) correctly describes the dynamics, we conclude that

$$- \int d^3x \mathfrak{B}_{hmhd}(\boldsymbol{\zeta}, \boldsymbol{\eta}) \geq 0, \quad (114)$$

where  $\mathfrak{B}_{hmhd}(\boldsymbol{\zeta}, \boldsymbol{\eta})$  is given by (97) with  $\mathbf{A}_1 = (\boldsymbol{\zeta} - d_i \boldsymbol{\eta}) \times \mathbf{B}_0$  is sufficient for stability. Note that the term containing  $\nabla \Phi_1$  can be neglected in view of  $\nabla \cdot (\rho \boldsymbol{\eta}) = 0$  and  $\boldsymbol{\eta} \cdot \hat{\mathbf{n}}|_{\partial D} = 0$ .

## VI. CONCLUSIONS

In this paper, we derived sufficient stability criteria exploiting the Hamiltonian structure of the XMHD model. The energy Casimir, dynamically accessible, and Lagrangian methods were used. Using the EC method, we ascertained that indefinite terms appear in the second variation of the EC functional occurring due to the vorticity-magnetic field coupling induced by the form of the Casimir invariants. We side-stepped this problem by considering equilibria with purely toroidal flow or special perturbations, assumptions that enable the removal of the indefiniteness. To study stability under three-dimensional perturbations, we employed the DA method, which allows the study of the stability of generic equilibria by restricting the perturbations to be tangent on the Casimir leaves. Such perturbations are consistent with the physics under consideration. Finally, we developed a Lagrangian stability analysis of the quasineutral two-fluid model written in MHD-like variables, namely, the Lagrangian counterparts of the center of mass velocity and current density. Subsequently, employing the Lagrange-Euler map, we jumped to the Eulerian viewpoint and upon performing a Legendre transformation, we found the Hamiltonian of the linear dynamics. Considering massless electrons, the definition of one of the two canonical momenta led to a relation between the perturbed magnetic potential and canonical variables. Requiring this relation to be preserved by the dynamics gave rise to a dynamical

constraint, whence we found the solution to the perturbed induction equation, namely,  $\mathbf{B}_1 = \nabla \times [(\boldsymbol{\zeta} - d_i \boldsymbol{\eta}) \times \mathbf{B}]$ . In addition, we generalized the HMHD energy principle so as to include the electron pressure contribution.

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## APPENDIX A: INTERMEDIATE RESULT FOR $\delta^2 \mathcal{H}_{da}$

Inserting expressions (60)–(62) into (57), we readily find

$$\begin{aligned} \delta^2 \mathcal{H}_{da} = & \int_V d^3x \left\{ \rho \left| -\nabla g_0 + \boldsymbol{\zeta} \times \boldsymbol{\omega} + \boldsymbol{\eta} \times \mathbf{B}^* - \frac{\mathbf{v}}{\rho} \cdot (\rho \boldsymbol{\zeta}) \right|^2 + |\delta \mathbf{B}_{da}|^2 + d_e^2 \rho^{-1} |\nabla \times \delta \mathbf{B}_{da}|^2 \right. \\ & + \rho^{-1} \left( c_s^2 - |\mathbf{v}|^2 - d_e^2 \frac{|\mathbf{J}|^2}{\rho^2} \right) [\nabla \cdot (\rho \boldsymbol{\zeta})]^2 - \boldsymbol{\zeta} \cdot \nabla \left( h + \frac{|\mathbf{v}|^2}{2} + d_e^2 \frac{|\mathbf{J}|^2}{2\rho^2} \right) \nabla \cdot (\rho \boldsymbol{\zeta}) - \boldsymbol{\eta} \cdot (\mathbf{v} \times \mathbf{B}^*) \nabla \cdot (\rho \boldsymbol{\zeta}) \\ & - \rho^{-1} \boldsymbol{\eta} \cdot (-d_i \mathbf{J} \times \mathbf{B}^* + d_e^2 \mathbf{J} \times \boldsymbol{\omega}) \nabla \cdot (\rho \boldsymbol{\zeta}) - \rho (\boldsymbol{\zeta} \times \mathbf{v}) \cdot \nabla \times (\boldsymbol{\zeta} \times \boldsymbol{\omega}) - \rho (\boldsymbol{\zeta} \times \mathbf{v}) \cdot \nabla \times (\boldsymbol{\eta} \times \mathbf{B}^*) \\ & - \rho (\boldsymbol{\eta} \times \mathbf{v}) \cdot \nabla \times [(\boldsymbol{\zeta} - d_i \boldsymbol{\eta}) \times \mathbf{B}^*] - [(\boldsymbol{\zeta} - d_i \boldsymbol{\eta}) \times \mathbf{J}] \cdot \nabla \times [(\boldsymbol{\zeta} - d_i \boldsymbol{\eta}) \times \mathbf{B}^*] - d_e^2 \rho (\boldsymbol{\eta} \times \mathbf{v}) \cdot \nabla \times (\boldsymbol{\eta} \times \boldsymbol{\omega}) \\ & \left. - d_e^2 [(\boldsymbol{\zeta} - d_i \boldsymbol{\eta}) \times \mathbf{J}] \cdot \nabla \times (\boldsymbol{\eta} \times \boldsymbol{\omega}) - d_e^2 (\boldsymbol{\eta} \times \mathbf{J}) \cdot \nabla \times (\boldsymbol{\zeta} \times \boldsymbol{\omega}) - d_e^2 (\boldsymbol{\eta} \times \mathbf{J}) \cdot \nabla \times (\boldsymbol{\eta} \times \mathbf{B}^*) \right\}. \end{aligned} \quad (\text{A1})$$

As a simple application, let us consider a stationary axisymmetric equilibrium with purely toroidal flow and variations with perturbation vectors that never leave the surfaces  $\psi^* = \text{const.}$ , i.e.,  $\boldsymbol{\zeta} \cdot \nabla \psi^* = 0$  and  $\boldsymbol{\eta} \cdot \nabla \psi^* = 0$ . To find the equilibrium conditions, we set  $\partial_t \rightarrow 0$  and  $\mathbf{v} = rv_\phi \nabla \phi$  in (5)–(6). Then, the XMHD equations reduce to

$$\begin{aligned} r^{-1} v_\phi \nabla (rv_\phi) - \nabla \left( h + \frac{|\mathbf{v}|^2}{2} + d_e^2 \frac{|\mathbf{J}|^2}{2\rho^2} \right) - \rho^{-1} \left[ \frac{\Delta^* \psi}{r^2} \nabla \psi^* \right. \\ \left. + \frac{B_\phi^*}{r} \nabla (rB_\phi) - \nabla (rB_\phi) \cdot (\nabla \psi^* \times \nabla \phi) \nabla \phi \right] = 0, \end{aligned} \quad (\text{A2})$$

$$\begin{aligned} r^{-1} v_\phi \nabla \psi^* - \rho^{-1} \left\{ d_i \left[ -\frac{\Delta^* \psi}{r^2} \nabla \psi^* - \frac{B_\phi^*}{r} \nabla (rB_\phi) \right. \right. \\ \left. \left. + \nabla (rB_\phi) \cdot (\nabla \psi^* \times \nabla \phi) \nabla \phi \right] + d_e^2 \left[ \frac{\Delta^* \psi}{r^2} \nabla (rv_\phi) \right. \right. \\ \left. \left. - \nabla (rB_\phi) \cdot (\nabla (rv_\phi) \times \nabla \phi) \nabla \phi \right] \right\} = \nabla \tilde{\Phi}, \end{aligned} \quad (\text{A3})$$

where  $\tilde{\Phi} = \Phi - d_i h_e + d_e^2 \rho^{-1} \mathbf{v} \cdot \mathbf{J} - d_i d_e^2 \rho^{-2} |\mathbf{J}|^2$ , with  $\Phi$  and  $h_e$  being the equilibrium electrostatic potential and electron specific enthalpy, respectively. Projecting Eqs. (A2) and (A3) along  $\nabla \phi$ , we find

$$\nabla (rB_\phi) \cdot (\nabla \psi^* \times \nabla \phi) = 0, \iff rB_\phi = F(\psi^*). \quad (\text{A4})$$

Similarly, projecting Eq. (A3) and using result (A4), we find

$$\nabla (rv_\phi) \cdot (\nabla \psi^* \times \nabla \phi) = 0, \iff rv_\phi = G(\psi^*). \quad (\text{A5})$$

Equations (A4) and (A5) imply  $\mathbf{J} = -\Delta^* \psi \nabla \phi + F'(\psi^*) \nabla \psi^* \times \nabla \phi$  and  $\boldsymbol{\omega} = G'(\psi^*) \nabla \psi^* \times \nabla \phi$ , respectively. Therefore,  $\mathbf{J} \cdot \nabla \psi^* = \boldsymbol{\omega} \cdot \nabla \psi^* = 0$ . This means that all three-vector fields  $\mathbf{v}$ ,  $\mathbf{B}^*$ , and  $\mathbf{J}$  lie on common flux surfaces labeled by  $\psi^*$ . This property of common flux surfaces was crucial for the derivation of a sufficient stability criterion in the context of MHD<sup>58</sup> for a three-dimensional incompressible displacement vector field. It is thus interesting to pursue the investigation of this possibility also in the context of

XMHD in the future. As regards the current application, we confine the perturbation vectors to be tangent to the characteristic surfaces. Also, note that using the result (A4) and projecting (A2) along  $\mathbf{B}^*$ , we find

$$\nabla \tilde{h} \cdot (\nabla \psi^* \times \nabla \phi) = 0, \iff \tilde{h} = \tilde{h}(\psi^*). \quad (\text{A6})$$

For equilibria with purely toroidal flows, subject to perturbations with displacement vectors tangent to the common surfaces, it is easy to understand that every product of the form  $\mathbf{b}_i \times \mathbf{c}_j$ , where  $\mathbf{b} = (\zeta, \boldsymbol{\eta})$  and  $\mathbf{c} = (\mathbf{v}, \mathbf{B}^*, \mathbf{J})$ , will be parallel to the vector  $\nabla \psi^*$  at each surface point, i.e.,  $\mathbf{b}_i \times \mathbf{c}_j = g_{ij}(r, z) \nabla \psi^*$ . Therefore, every vector of the form  $\nabla \times (\mathbf{b} \times \mathbf{c})$  will be  $\nabla g \times \nabla \psi^*$ , and consequently every term  $(\mathbf{b}_i \times \mathbf{c}_j) \cdot \nabla \times (\mathbf{b}_k \times \mathbf{c}_\ell)$  in (A1) will vanish. The same also holds for terms of the form  $\mathbf{b}_i \cdot (\mathbf{c}_j \times \mathbf{c}_k)$ , since  $(\mathbf{c}_j \times \mathbf{c}_k)$  is normal to the characteristic surfaces at each point, if not zero. In addition, the term containing  $\zeta \cdot \nabla \tilde{h}$  will vanish as well due to (A6). A rigorous proof can be carried out upon writing  $\zeta = r \zeta_\phi \nabla \phi + (\zeta \cdot \mathbf{B}_p^*) / |\mathbf{B}_p^*|^2 \mathbf{B}_p^*$ , which is a general representation of vectors tangent to  $\psi^* = \text{const.}$  surfaces, and similarly for  $\boldsymbol{\eta}$ ; then computing every single term in (A1), leads eventually to

$$\begin{aligned} \delta^2 \mathcal{H}_{da} = \int_V d^3x \left\{ \rho \left| \delta \mathbf{v}_{da} + \frac{r v_\phi}{\rho} \delta \rho_{da} \nabla \phi \right|^2 + |\delta \mathbf{B}_{da}|^2 + d_e^2 \rho^{-1} \right. \\ \left. \times |\nabla \times \delta \mathbf{B}_{da}|^2 + \rho^{-1} \left( c_s^2 - v_\phi^2 - d_e^2 \frac{|\mathbf{J}|^2}{\rho^2} \right) [\nabla \cdot (\rho \zeta)]^2 \right\}. \quad (\text{A7}) \end{aligned}$$

As a result,  $c_s^2 - v_\phi^2 - d_e^2 |\mathbf{J}|^2 / \rho^2 > 0$  is sufficient for stability and also for the ellipticity of the equilibrium Grad-Shafranov-Bernoulli equations. Actually for the ellipticity of the equilibrium system, the condition  $c_s^2 - d_e^2 |\mathbf{J}|^2 / \rho^2 > 0$  is sufficient, as was shown in Ref. 48.

## APPENDIX B: EXPANSION OF THE INTERNAL ENERGY

The difficulty in this expansion is that the Jacobians contain a dependence on the gradients of the fluid trajectories; therefore, we need to know how to differentiate the  $\mathcal{J}_s$ , because the expansion of the internal energy is effected through the expansion

$$\mathcal{J}_s = \mathcal{J}_{s0} + \frac{\partial \mathcal{J}_s}{\partial q_{s,j}^i} \frac{\partial \zeta_s^i}{\partial a_s^j} + \frac{1}{2} \frac{\partial^2 \mathcal{J}_s}{\partial q_{s,k}^i \partial q_{s,\ell}^j} \frac{\partial \zeta_s^i}{\partial a_s^k} \frac{\partial \zeta_s^j}{\partial a_s^\ell}, \quad (\text{B1})$$

where  $q_{s,j}^i := \frac{\partial q_s^i}{\partial a_s^j}$ . The derivatives of the Jacobian are  $\frac{\partial \mathcal{J}_s}{\partial q_{s,j}^i} = C_{si}^j$ , where  $C_{si}^j = \frac{1}{2} \epsilon_{i\ell k} \epsilon^{jmn} \frac{\partial q_s^\ell}{\partial a_s^m} \frac{\partial q_s^k}{\partial a_s^n}$  are the cofactors of  $\partial q_s^i / \partial a_s^j$  in  $\mathcal{J}_s$ . Following the procedure in Refs. 20 and 56, we find

$$\mathcal{J}_{s1} = \mathcal{J}_{s0} \frac{\partial v_s^i}{\partial q_s^i}, \quad \mathcal{J}_{s2} = \frac{\mathcal{J}_{s0}}{2} \left[ \left( \frac{\partial v_s^i}{\partial q_s^i} \right)^2 - \frac{\partial v_s^i}{\partial q_s^i} \frac{\partial v_s^j}{\partial q_s^j} \right]. \quad (\text{B2})$$

With these expressions at hand, we can find the second order perturbation of the internal energies in terms of the displacement vectors to be

$$\begin{aligned} U_{s2} = \frac{n_{s0}}{2 \mathcal{J}_{s0}} \left\{ \frac{\partial U_s}{\partial n_s} \left[ \left( \frac{\partial \zeta_s^i}{\partial q_s^i} + \alpha_s \frac{\partial \eta^i}{\partial q_s^i} \right)^2 + \left( \frac{\partial \zeta_s^i}{\partial q_s^i} + \alpha_s \frac{\partial \eta^i}{\partial q_s^i} \right) \right. \right. \\ \left. \left. \times \left( \frac{\partial \zeta_s^j}{\partial q_s^j} + \alpha_s \frac{\partial \eta^j}{\partial q_s^j} \right) \right] + \frac{n_{s0}}{\mathcal{J}_{s0}} \frac{\partial^2 U_s}{\partial n_s^2} \left( \frac{\partial \zeta_s^i}{\partial q_s^i} + \alpha_s \frac{\partial \eta^i}{\partial q_s^i} \right)^2 \right\}. \quad (\text{B3}) \end{aligned}$$

## APPENDIX C: EULERIAN DISPLACEMENT VECTORS

Let us begin with the Lagrange-Euler map and its inverse in order to understand how  $\mathbf{Q}$ ,  $\mathbf{D}$ , and the displacements  $\zeta$ ,  $\boldsymbol{\eta}$  are mapped into the Eulerian coordinates. From (66) and (71)–(73), we can effectively construct every map we need. For example,

$$\begin{aligned} \mathbf{Q}(\mathbf{a}_i, \mathbf{a}_e, t) = \frac{m_i}{m} \left( \mathbf{v} + \frac{m_e}{men} \mathbf{J} \right) \Big|_{\mathbf{x}=\mathbf{q}_i(\mathbf{a}_i, t)} + \frac{m_e}{m} \left( \mathbf{v} - \frac{m_i}{men} \mathbf{J} \right) \Big|_{\mathbf{x}=\mathbf{q}_e(\mathbf{a}_e, t)}, \\ \mathbf{D}(\mathbf{a}_i, \mathbf{a}_e, t) = \left( \mathbf{v} + \frac{m_e}{men} \mathbf{J} \right) \Big|_{\mathbf{x}=\mathbf{q}_i(\mathbf{a}_i, t)} - \left( \mathbf{v} - \frac{m_i}{men} \mathbf{J} \right) \Big|_{\mathbf{x}=\mathbf{q}_e(\mathbf{a}_e, t)}. \quad (\text{C1}) \end{aligned}$$

If these expressions are computed at  $\mathbf{a}_e = \mathbf{q}_e^{-1}(\mathbf{q}_i(\mathbf{a}_i, t), t)$  as in the Lagrangian (74) then at equilibrium we have  $\mathbf{Q}_0(\mathbf{a}_i, t) = \mathbf{v}(\mathbf{x})|_{\mathbf{x}=\mathbf{q}_{i0}(\mathbf{a}_i, t)}$  and  $\mathbf{D}_0(\mathbf{a}_i, t) = e^{-1} n^{-1}(\mathbf{x}) \mathbf{J}(\mathbf{x})|_{\mathbf{x}=\mathbf{q}_{i0}(\mathbf{a}_i, t)}$ . For the Eulerianization of the displacement vectors, we define their Eulerian displacements  $\tilde{\zeta}$  and  $\tilde{\boldsymbol{\eta}}$  by

$$\begin{aligned} \zeta(\mathbf{a}_i, \mathbf{a}_e, t) = \frac{m_i}{m} \left[ \tilde{\zeta}(\mathbf{x}, t) + \frac{m_e}{m} \tilde{\boldsymbol{\eta}}(\mathbf{x}, t) \right] \Big|_{\mathbf{x}=\mathbf{q}_{i0}(\mathbf{a}_i, t)} \\ + \frac{m_e}{m} \left[ \tilde{\zeta}(\mathbf{x}, t) - \frac{m_i}{m} \tilde{\boldsymbol{\eta}}(\mathbf{x}, t) \right] \Big|_{\mathbf{x}=\mathbf{q}_{e0}(\mathbf{a}_e, t)}, \\ \boldsymbol{\eta}(\mathbf{a}_i, \mathbf{a}_e, t) = \left[ \tilde{\zeta}(\mathbf{x}, t) + \frac{m_e}{m} \tilde{\boldsymbol{\eta}}(\mathbf{x}, t) \right] \Big|_{\mathbf{x}=\mathbf{q}_{i0}(\mathbf{a}_i, t)} \\ - \left[ \tilde{\zeta}(\mathbf{x}, t) - \frac{m_i}{m} \tilde{\boldsymbol{\eta}}(\mathbf{x}, t) \right] \Big|_{\mathbf{x}=\mathbf{q}_{e0}(\mathbf{a}_e, t)}. \quad (\text{C2}) \end{aligned}$$

Taking the time derivatives of (C2) with  $\mathbf{a}_i$  and  $\mathbf{a}_e$  held constant, we find

$$\begin{aligned} \dot{\zeta}(\mathbf{a}_i, \mathbf{a}_e, t) = \partial_t \tilde{\zeta}(\mathbf{x}, t) + \mathbf{v} \cdot \nabla \tilde{\zeta}(\mathbf{x}, t) + \frac{m_i m_e}{m^2} \mathbf{w} \cdot \nabla \tilde{\boldsymbol{\eta}}(\mathbf{x}, t), \\ \dot{\boldsymbol{\eta}}(\mathbf{a}_i, \mathbf{a}_e, t) = \partial_t \tilde{\boldsymbol{\eta}}(\mathbf{x}, t) + \mathbf{v} \cdot \nabla \tilde{\boldsymbol{\eta}}(\mathbf{x}, t) + \mathbf{w} \cdot \nabla \tilde{\zeta}(\mathbf{x}, t) \\ + \frac{m_e^2 - m_i^2}{m^2} \mathbf{w} \cdot \nabla \tilde{\boldsymbol{\eta}}(\mathbf{x}, t), \quad (\text{C3}) \end{aligned}$$

where  $\mathbf{w} := \mathbf{J}/(en)$ , and we have made use of  $\mathbf{v}_s(\mathbf{x}) = \mathbf{v}(\mathbf{x}) + \alpha_s \mathbf{J}(\mathbf{x}) / (en(\mathbf{x})) = \dot{\mathbf{q}}_{s0}(\mathbf{a}_s, t)|_{\mathbf{a}_s=\mathbf{q}_{s0}^{-1}(\mathbf{x}, t)}$ . This result, along with (C1), is used for the derivation of (84). Taking the first variation of (C1) and identifying

$$\delta \mathbf{Q} = \tilde{\zeta}, \quad \delta \mathbf{D} = \tilde{\boldsymbol{\eta}}, \quad \text{and} \quad \delta \mathbf{q}_s(\mathbf{a}_s, t)|_{\mathbf{a}_s=\mathbf{q}_s^{-1}(\mathbf{x}, t)} = \tilde{\zeta} + \alpha_s \tilde{\boldsymbol{\eta}},$$

after some manipulations, we find

$$\begin{aligned} \dot{\zeta} = \delta \mathbf{v} + \tilde{\zeta} \cdot \nabla \mathbf{v} + \frac{m_i m_e}{m^2} \tilde{\boldsymbol{\eta}} \cdot \nabla \mathbf{w}, \\ \dot{\boldsymbol{\eta}} = \delta \mathbf{w} + \tilde{\boldsymbol{\eta}} \cdot \nabla \mathbf{v} + \tilde{\zeta} \cdot \nabla \mathbf{w} + \frac{m_e^2 - m_i^2}{m^2} \tilde{\boldsymbol{\eta}} \cdot \nabla \mathbf{w}. \quad (\text{C4}) \end{aligned}$$

Combining (C4) with (C3), we arrive at (82) and (83).

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