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Hamiltonian nature of monopole dynamics

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ABSTRACT

Classical electromagnetism with magnetic monopoles is not a Hamiltonian field theory because the Jacobi identity for the Poisson bracket fails. The Jacobi identity is recovered only if all of the species have the same ratio of electric to magnetic charge or if an electron and a monopole can never collide. Without the Jacobi identity, there are no local canonical coordinates or Lagrangian action principle. To build a quantum field of magnetic monopoles, we either must explain why the positions of electrons and monopoles can never coincide or we must resort to new quantization techniques.

1. Introduction

This letter considers the classical gauge-free theory of electromagnetic fields interacting with electrically and magnetically charged matter as a Hamiltonian field theory. We begin with a brief history of magnetic monopoles and then describe why monopole theories are not Hamiltonian field theories.

The modern theory of magnetic monopoles was developed by Dirac [1,2]. He showed that an electron in the magnetic field of a monopole is equivalent to an electron whose wave function is zero along a semi-infinite 'string' extending from the location of the monopole. Along this string, the electromagnetic vector potential is undefined. The phase of the electron is no longer single valued along a loop encircling the Dirac string. In order for observables to be single valued, the phase shift must be an integer multiple of 2π, so the electric and magnetic charge must be quantized. The direction of the string is arbitrary. Changing it corresponds to a gauge transformation for the fields and a global phase shift for the wave function. To avoid the string entirely, we could instead define the vector potential for multiple patches around the monopole [3].

Magnetic monopoles can also be introduced in the hydrodynamic formulation of non-relativistic quantum mechanics [4]. Since this formulation involves fluid-like variables and the fields, it removes the ambiguity associated with the wave function and vector potential. Dirac strings are replaced by singular vortex lines.

Quantum field theories for magnetic monopoles and for dyons (particles with both electric and magnetic charge) were developed by Cabibbo & Ferrari [5] and Schwinger [6–8]. These theories use two nonsingular vector potentials that are related to the fields by a convolution with a string function. This string function allows for the derivation of the action and equations of motion for interacting electron and monopole fields.

Grand unified theories (GUTs) describe the strong, weak, and electromagnetic forces as a single theory whose symmetry is spontaneously broken at lower energies. If the symmetry is not broken in the same direction everywhere, then the fields will be zero at some locations. Around these locations, the fields resemble the fields of a magnetic monopole [9,10]. Magnetic monopoles are a generic feature of GUTs, including string theories [11].

If we assume that a GUT exists, then, shortly after the Big Bang, the expanding universe cooled through the critical temperature at which the symmetry is broken. There is no reason to assume that the symmetry would be broken in the same direction at causally disconnected locations. The boundaries between regions with differently broken symmetry would produce magnetic monopoles and strings [12]. Initial estimates of the number of monopoles produced this way were much too high [13], but the estimates are dramatically reduced by inflation [14].

The existence of astronomical magnetic fields produces a bound on the number of monopoles. If there were too many monopoles, they could move and screen out large magnetic fields, much like electrically charged matter screens out large electric fields in our universe [15].

Direct observations of magnetic monopoles remain inconclusive. Two early experiments detected candidate events [16,17], but

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one was immediately refuted [18] and the other has never been replicated. Extensive searches for monopoles have been done in matter, in cosmic rays, via catalyzing nucleon decays, and at colliders, all with negative results [19].

More information about magnetic monopoles can be found in one of the many relevant review articles [20,21] or textbooks [22,23].

The letter is organized as follows. In Sec. 2, we introduce a general and standard matter model for plasmas, the Vlasov-Maxwell equations, as a noncanonical Hamiltonian field theory, i.e. one without the standard Poisson bracket. We add monopoles to this theory in Sec. 3 and show that it no longer satisfies the Jacobi identity, a basic premise of Hamiltonian theory. Sec. 4 considers whether the interaction between a single electron and a single magnetic monopole is Hamiltonian. We discuss the importance of the Jacobi identity in Sec. 5 and consider the difficulties in quantization without the Jacobi identity in Sec. 6. We conclude in Sec. 7.

2. Vlasov-Maxwell equations

We approach the problem of magnetic monopoles from the perspective of plasma physics, although our conclusion is general and independent of the Vlasov-Maxwell matter model. In plasmas, the most important dynamics are the collective motions of the particles in collectively generated electromagnetic fields. The relevant dynamical variables are the distribution function $f_s(x, v, t)$ for each species s, which describes the probability density of the particles in phase space, and the electric and magnetic fields $E(x, t)$ and $B(x, t)$. The charge and mass of each species are e_s and m_s. The temperatures of plasmas are high enough and the densities are low enough that quantum effects are negligible.

The dynamics of the distribution function are governed by a mean-field transport equation. The phase space density is constant along particle trajectories:

$$\frac{df_s}{dt} = \frac{\partial f_s}{\partial t} + v \cdot \frac{\partial f_s}{\partial x} + \frac{e_s}{m_s}\left(E + \frac{v}{c} \times B\right) \cdot \frac{\partial f_s}{\partial v} = 0\,. \tag{1}$$

This is combined with Maxwell's equations for the electric and magnetic fields, with sources determined by the moments of the distribution function,

$$\rho = \sum_s e_s \int f_s\, dv\,, \quad j = \sum_s e_s \int f_s\, v\, dv\,. \tag{2}$$

The resulting Maxwell-Vlasov equations are a closed system of nonlinear partial integro-differential equations for f_s, E, and B. Many reductions have been developed to convert these equations to more manageable forms like gyrokinetics and magnetohydrodynamics. Since the Maxwell-Vlasov equations are more general than fluid equations, these results are generic for matter models without dissipation.

In 1931, Dirac wrote that "if we wish to put the equations of motion [of electromagnetism] in the Hamiltonian form, however, we have to introduce the electromagnetic potentials" [1]. Born and Infeld showed that this is not entirely true [24,25]. The Vlasov description of matter coupled with Maxwell's equations can be written as a Hamiltonian theory without introducing potentials if we allow for a noncanonical Poisson bracket (see Refs. [26–30] for review). The Hamiltonian functional and noncanonical Poisson bracket for this system are

$$\mathcal{H} = \sum_s \frac{m_s}{2}\int |v|^2 f_s\, d^3x\, d^3v + \frac{1}{8\pi}\int (|E|^2 + |B|^2)\, d^3x\,, \tag{3}$$

$$\begin{aligned}\{F, G\} = \sum_s \int \Big(&\frac{1}{m_s} f_s \left(\nabla F_{f_s} \cdot \partial_v G_{f_s} - \nabla G_{f_s} \cdot \partial_v F_{f_s}\right) \\ &+ \frac{e_s}{m_s^2 c} f_s\, B \cdot \left(\partial_v F_{f_s} \times \partial_v G_{f_s}\right) \\ &+ \frac{4\pi e_s}{m_s} f_s \left(G_E \cdot \partial_v F_{f_s} - F_E \cdot \partial_v G_{f_s}\right)\Big) d^3x\, d^3v \\ &+ 4\pi c \int (F_E \cdot \nabla \times G_B - G_E \cdot \nabla \times F_B)\, d^3x\,,\end{aligned} \tag{4}$$

where subscripts such as F_{f_s} refer to the functional derivative of F with respect to f_s.

The Hamiltonian, but not the Poisson bracket, needs to be modified to make this theory relativistic. Replace the $|v|^2$ in the kinetic energy term with $\gamma|v|^2 = c^2\sqrt{1 + |v|^2/c^2}$ (e.g. [29]).

It is straightforward to derive the Vlasov equation (1) and the dynamical Maxwell equations by setting

$$\frac{\partial f_s}{\partial t} = \{f_s, \mathcal{H}\}\,, \ \frac{\partial E}{\partial t} = \{E, \mathcal{H}\}\,, \ \frac{\partial B}{\partial t} = \{B, \mathcal{H}\}\,. \tag{5}$$

The constraints appear as Casimir invariants:

$$\mathcal{C}_E = \int h_E(x) \left(\nabla \cdot E - 4\pi \sum_s e_s \int f_s\, d^3v\right) d^3x\,, \tag{6}$$

$$\mathcal{C}_B = \int h_B(x)\, \nabla \cdot B\, d^3x\,, \tag{7}$$

where $h_E(x)$ and $h_B(x)$ are arbitrary functions. The Poisson bracket of $\mathcal{C}_E$ or $\mathcal{C}_B$ with anything is zero. Since the time dependence of anything is determined by its bracket with the Hamiltonian, Casimirs are conserved for any Hamiltonian. If the Casimirs are initially zero (as required by the divergence Maxwell's Equations), they will remain zero for all time.

There is an important subtlety to this formulation of electromagnetism. If a system is Hamiltonian, its Poisson bracket must satisfy the Jacobi identity for any functionals F, G, and H:

$$\{\{F, G\}, H\} + \{\{G, H\}, F\} + \{\{H, F\}, G\} = 0\,. \tag{8}$$

For the Vlasov-Maxwell system it was shown by direct calculation in [27,31] that

$$\begin{aligned}&\{F, \{G, H\}\} + \text{cyc} = \\ &\sum_s \int f_s\, \nabla \cdot B\, \left(\partial_v F_{f_s} \times \partial_v G_{f_s}\right) \cdot \partial_v H_{f_s} d^3x\, d^3v\,,\end{aligned} \tag{9}$$

which means the domain of functionals must be restricted to solenoidal vector fields, $\nabla \cdot B = 0$, or equivalently defined on closed but not necessary exact two-forms. Such a set of functionals is closed with respect to the bracket.

The two Casimirs mentioned above are not symmetric. The value of $\mathcal{C}_E$ could initially be chosen to be anything. If $\mathcal{C}_B \neq 0$, i.e. if $\nabla \cdot B \neq 0$, then the Vlasov-Maxwell system would cease to be a Hamiltonian field theory.

This is our first indication that the existence of magnetic monopoles is connected to the Hamiltonian nature of classical electromagnetism.

3. Vlasov-Maxwell with monopoles

What happens when we add magnetic monopoles?

We must change the Hamiltonian and/or the Poisson bracket so they produce the new equations of motion.

The appropriate Hamiltonian, Poisson bracket, and a detailed proof of the Jacobi identity were given in Section IV D and Appendix 3 of [31]. For species s with electric charge e_s and magnetic charge g_s the Hamiltonian is identical to (3) and the Poisson bracket is

$$
\begin{aligned}
\{F,G\} = \sum_s \int \Big(& \frac{1}{m_s} f_s \left(\nabla F_{f_s} \cdot \partial_v G_{f_s} - \nabla G_{f_s} \cdot \partial_v F_{f_s}\right) \\
& + \frac{e_s}{m_s^2 c} f_s \, B \cdot \left(\partial_v F_{f_s} \times \partial_v G_{f_s}\right) \\
& - \frac{g_s}{m_s^2 c} f_s \, E \cdot \left(\partial_v F_{f_s} \times \partial_v G_{f_s}\right) \\
& + \frac{4\pi e_s}{m_s} f_s \left(G_E \cdot \partial_v F_{f_s} - F_E \cdot \partial_v G_{f_s}\right) \\
& + \frac{4\pi g_s}{m_s} f_s \left(G_B \cdot \partial_v F_{f_s} - F_B \cdot \partial_v G_{f_s}\right) \Big) d^3x\, d^3v \\
& + 4\pi c \int \left(F_E \cdot \nabla \times G_B - G_E \cdot \nabla \times F_B\right) d^3x .
\end{aligned}
\tag{10}
$$

The Jacobi identity for this bracket is

$$
\begin{aligned}
& \{F,\{G,H\}\} + \text{cyc} = \\
& \quad \sum_s \frac{1}{m_s^2} \int \partial_v H_{f_s} \cdot \left(\partial_v F_{f_s} \times \partial_v G_{f_s}\right) \\
& \quad\quad \times f_s \left(e_s \nabla \cdot B - g_s \nabla \cdot E\right) d^3x\, d^3v .
\end{aligned}
\tag{11}
$$

For (11) to vanish for arbitrary F, G, H, and f_s, we must have

$$
e_s \nabla \cdot B = g_s \nabla \cdot E , \quad \forall s .
\tag{12}
$$

The case when every species has the same ratio of magnetic to electric charge is addressed in Section 6.11 of Jackson [32]. Using the duality transformation

$$
E' = E\cos\xi + B\sin\xi \ , \quad B' = -E\sin\xi + B\cos\xi \, ,
\tag{13}
$$

$$
e_s' = e_s\cos\xi + g_s\sin\xi \ , \quad g_s' = -e_s\sin\xi + g_s\cos\xi \, ,
\tag{14}
$$

with $\xi = \arctan(g_s/e_s)$, the magnetic charges are removed and the Jacobi identity is satisfied. These monopoles are trivial; this theory is equivalent to electromagnetism without monopoles. “The only meaningful question is whether *all* particles have the same ratio of magnetic to electric charge” [32].

If not all species have the same ratio of magnetic to electric charge, then the only way that the Jacobi identity could be satisfied is if $\nabla \cdot E = \nabla \cdot B = 0$. This is obviously not true in general.

When we add nontrivial magnetic monopoles to the Vlasov-Maxwell system, the Jacobi identity is not satisfied, so it is not a Hamiltonian field theory.

4. One electron and one monopole

Although we originally derived this result for the collective motion of many electrically and magnetically charged particles, it should also hold when there are only a small number of particles. Consider the interaction between a single electron with position X_e, velocity V_e, mass m_e, electric charge e, and magnetic charge 0 and a single monopole with position X_m, velocity V_m, mass m_m, electric charge 0, and magnetic charge g.

The Hamiltonian and Poisson bracket for this system follow from localizing on particles. Set

$$
f_s = \delta(x - X_s)\,\delta(v - V_s) \, , \quad s = e, m \, .
\tag{15}
$$

For the Poisson bracket, use the chain rule expressions

$$
\frac{\partial F}{\partial X_s} = \nabla \frac{\delta F}{\delta f}\bigg|_{(X_s,V_s)} \quad \text{and} \quad \frac{\partial F}{\partial V_s} = \partial_v \frac{\delta F}{\delta f}\bigg|_{(X_s,V_s)} ,
\tag{16}
$$

where on the left of each expression, F is the function of (X_s, V_s) obtained upon substituting (15) into the functional F on the right. This yields

$$
\mathcal{H} = \frac{1}{2} m_e V_e^2 + \frac{1}{2} m_m V_m^2 + \frac{1}{8\pi} \int \left(|E|^2 + |B|^2\right) d^3x ,
\tag{17}
$$

$$
\begin{aligned}
\{F,G\} = & \frac{1}{m_e} \left(\frac{\partial F}{\partial X_e} \cdot \frac{\partial G}{\partial V_e} - \frac{\partial G}{\partial X_e} \cdot \frac{\partial F}{\partial V_e} \right) \\
& + \frac{1}{m_m} \left(\frac{\partial F}{\partial X_m} \cdot \frac{\partial G}{\partial V_m} - \frac{\partial G}{\partial X_m} \cdot \frac{\partial F}{\partial V_m} \right) \\
& + \frac{e}{m_e^2 c} B(X_e) \cdot \left(\frac{\partial F}{\partial V_e} \times \frac{\partial G}{\partial V_e} \right) \\
& - \frac{g}{m_m^2 c} E(X_m) \cdot \left(\frac{\partial F}{\partial V_m} \times \frac{\partial G}{\partial V_m} \right) \\
& + \frac{4\pi e}{m_e} \left(\frac{\delta G}{\delta E}\bigg|_{X_e} \cdot \frac{\partial F}{\partial V_e} - \frac{\delta F}{\delta E}\bigg|_{X_e} \cdot \frac{\partial G}{\partial V_e} \right) \\
& + \frac{4\pi g}{m_m} \left(\frac{\delta G}{\delta B}\bigg|_{X_m} \cdot \frac{\partial F}{\partial V_m} - \frac{\delta F}{\delta B}\bigg|_{X_m} \cdot \frac{\partial G}{\partial V_m} \right) \\
& + 4\pi c \int \left(\frac{\delta F}{\delta E} \cdot \nabla \times \frac{\delta G}{\delta B} - \frac{\delta G}{\delta E} \cdot \nabla \times \frac{\delta F}{\delta B} \right) d^3x .
\end{aligned}
\tag{18}
$$

This Hamiltonian and Poisson bracket give the expected equations of motion: the Lorentz force laws and the dynamical Maxwell equations, with currents proportional to $V_s\,\delta(x - X_s)$. The divergence Maxwell equations, with delta function sources, appear in the Casimirs.

The Jacobi identity calculation for (18) can be done directly, but it follows easily upon substituting (15) and the second of (16) into (11), yielding

$$
\begin{aligned}
\{\{F,G\},H\} + \text{cyc} = & \frac{12\pi e g}{c} \delta(X_e - X_m) \\
& \times \left(\frac{1}{m_e^3} \frac{\partial F}{\partial V_e} \cdot \left(\frac{\partial G}{\partial V_e} \times \frac{\partial H}{\partial V_e} \right) \right. \\
& \quad \left. - \frac{1}{m_m^3} \frac{\partial F}{\partial V_m} \cdot \left(\frac{\partial G}{\partial V_m} \times \frac{\partial H}{\partial V_m} \right) \right) .
\end{aligned}
\tag{19}
$$

The Jacobi identity is not satisfied globally. There is a singularity when the positions of the electron and monopole coincide.

Classically, there is no reason why this coincidence couldn’t happen. A stationary monopole produces a radial magnetic field. An electron moving directly towards the monopole experiences a force $eV_e \times B/c = 0$. The electron passes through the monopole without experiencing any force at all.

This singularity is very different from the singularity for two electrically charged particles. That singularity comes from the Hamiltonian, can only be reached with infinite energy, and is removed if the point particles are replaced by continuous charge distributions. This singularity comes from the Jacobi identity, requires no energy to reach, and becomes worse if the point particles are replaced by continuous distributions because the Jacobi identity is violated at more locations.

The electromagnetic interaction between a single electron and a single magnetic monopole is not, in general, Hamiltonian.

5. Importance of the Jacobi identity

Electromagnetism with magnetic monopoles does not satisfy the Jacobi identity. Why should we care?

There is extensive literature on the algebraic and geometric nature of Hamiltonian mechanics (e.g. [33–37]) with phase space defined as a symplectic or Poisson manifold. The Jacobi identity is central to these results.

Darboux's theorem, when applied to Hamiltonian systems, says that the Jacobi identity implies the existence of a local transformation to canonical coordinates on a foliation parameterized by Casimir invariants [38]. For electromagnetism, this transformation occurs when we introduce the potentials: the Poisson bracket becomes simple and the Hamiltonian becomes more complicated.

If we apply an arbitrary coordinate transformation to a bracket that satisfies the Jacobi identity, the new bracket will also satisfy the Jacobi identity [39]. If the Jacobi identity is not satisfied, then no coordinate transform can turn it into a canonical bracket. Local canonical coordinates do not exist.

Most fundamental physical theories begin as a Lagrangian action principle. If you have a Lagrangian action principle and the Legendre transform exists, then you can transform it into a Hamiltonian system with a Poisson bracket that satisfies the Jacobi identity. Contrapose this. If your system has a Poisson bracket that doesn't satisfy the Jacobi identity, then no Lagrangian action principle exists.

6. Quantizing without the Jacobi identity

Standard methods of quantization fail without the Jacobi identity. Typically, we replace dynamical variables with operators whose commutation relation algebra matches the algebra of the Poisson bracket. However, commutators automatically satisfy the Jacobi identity, so it is impossible to match this algebra. Transforming to canonical coordinates first, then quantizing, is impossible since canonical coordinates do not exist. Even path integral quantization is impossible because there is no Lagrangian action principle [40,41].

If quantization without the Jacobi identity is so difficult, how did Dirac quantize magnetic monopoles [1,2]?

Dirac proceeds by locally transforming to canonical coordinates - by locally replacing fields with potentials. This comes at a cost. Dirac's theory has strings along which the electrons' wavefunctions are zero. Although the directions of the strings are arbitrary, the locations of the ends of the strings are not arbitrary because these are the locations of the monopoles. If we consider a monopole wave packet instead of a point monopole, the electron must avoid each volume element of the wave packet [42]. Dirac implicitly assumes that electrons' and monopoles' positions could never coincide. This claim needs to be justified. If it were true, it could restore the Hamiltonian nature of electromagnetism and remove the impediment to quantization.

New quantization techniques are needed to build a quantum field of magnetic monopoles. When expressed in terms of gauge group operators, a violation of the Jacobi identity corresponds to a nonzero 3-cocycle, which removes associativity of the operators [43–46]. A nonassociative star product for Wigner functions, doubling the size of the phase space to create a Hamiltonian structure on the extended space, and the geometric structure of a gerbe have been used to address this [47]. Another possible tool is beatification, which removes the explicit variable dependence from a Poisson bracket [48].

Electromagnetism is often considered to be a long wavelength limit of a more fundamental theory with a broken gauge symmetry [9,10]. Magnetic monopoles appear when the symmetry is broken in a topologically nontrivial way. In future work, we hope to write a classical SU(2) theory using an explicitly gauge invariant noncanonical Poisson bracket and check if it satisfies the Jacobi identity. We would then break the gauge symmetry in a topologically nontrivial way to determine which aspects of symmetry breaking are inherently quantum and which are inherited from the classical theory.

7. Conclusion

The existence of magnetic monopoles disrupts the Hamiltonian nature of classical electromagnetism. When the locations of an electron and a monopole coincide, the Jacobi identity for the Poisson bracket is violated. This result is most obviously seen in plasma physics, where the huge number of particles interacting collectively makes collisions between species almost guaranteed.

There are two ways to recover the Jacobi identity, but neither is satisfactory. All species could have the same ratio of magnetic to electric charge. This is a duality transformation away from the universe we currently observe without monopoles. Alternatively, we could insist that electrons' and monopoles' positions never coincide, as Dirac's theory implicitly assumed. Why can't monopoles collide with ordinary matter? How would this influence our attempts to detect them?

Traditional methods of quantization fail without the Jacobi identity, canonical coordinates, or a Lagrangian action principle. How should we quantize the interactions between arbitrary collections of electrically and magnetically charged particles?

Problems with the quantum theory of electromagnetism with magnetic monopoles have been known for decades [43]. In this letter, we showed that these problems are not inherently quantum. The quantum theory merely inherits the problems caused by the failure of the Jacobi identity for the classical Poisson bracket.

Since there is no experimental evidence, the argument for magnetic monopoles is aesthetic. The failure of the Jacobi identity taints this beauty. We should remain skeptical of any theory of magnetic monopoles that does not address the failure of the Jacobi identity.

Note added in proof

It has recently been shown that the bracket (10) not only violates the Jacobi identity, it also does not satisfy the weaker conditions for a twisted Poisson bracket [49].

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References

[1] P.A.M. Dirac, Quantised singularities in the electromagnetic field, Proc. R. Soc. A 133 (1931) 60–72.
[2] P.A.M. Dirac, The theory of magnetic poles, Phys. Rev. 74 (7) (1948) 817–830.
[3] T.T. Wu, C.N. Yang, Dirac's monopole without strings: classical Lagrangian theory, Phys. Rev. D 14 (2) (1976) 437–445.
[4] I. Bialynicki-Birula, Z. Bialynicki-Birula, Magnetic monopoles in the hydrodynamic formation of quantum mechanics, Phys. Rev. D 3 (10) (1971) 2410–2412.
[5] N. Cabibbo, E. Ferrari, Quantum electrodynamics with Dirac monopoles, Nuovo Cimento 23 (6) (1962) 1147–1154.
[6] J. Schwinger, Magnetic charge and quantum field theory, Phys. Rev. 144 (4) (1966) 1087–1093.
[7] J. Schwinger, Sources and magnetic charge, Phys. Rev. 173 (5) (1968) 1536–1544.
[8] J. Schwinger, A magnetic model of matter, Science 165 (3895) (1969) 757–760.
[9] G. 't Hooft, Magnetic monopoles in unified gauge theories, Nucl. Phys. B 79 (1974) 276–284.
[10] A.M. Polyakov, Particle spectrum in the quantum field theory, JETP Lett. 20 (1974) 194–195.
[11] J. Polchinski, Monopoles, duality, and string theory, Int. J. Mod. Phys. A 19 (supp01) (2004) 145–154.
[12] T. Kibble, Topology of cosmic domains and strings, J. Phys. A 9 (8) (1976) 1387–1398.
[13] J.P. Preskill, Cosmological production of superheavy magnetic monopoles, Phys. Rev. Lett. 43 (19) (1979) 1365–1368.
[14] A.H. Guth, Inflationary universe: a possible solution to the horizon and flatness problems, Phys. Rev. D 23 (2) (1981) 347–356.

[15] E.N. Parker, The origin of magnetic fields, Astrophys. J. 160 (1970) 383–404.
[16] P.B. Price, E.K. Shirk, W.Z. Osborne, L.S. Pinsky, Evidence for detection of a moving magnetic monopole, Phys. Rev. Lett. 35 (8) (1975) 487–490.
[17] B. Cabrera, First results from a superconductive detector for moving magnetic monopoles, Phys. Rev. Lett. 48 (20) (1982) 1378–1381.
[18] L.W. Alvarez, Analysis of a Reported Magnetic Monopole, 1975.
[19] C. Patrignani, et al., Particle Data Group, Review of particle physics, Chin. Phys. C 40 (100001) (2016) 1675–1682.
[20] K.A. Milton, Theoretical and experimental status of magnetic monopoles, Rep. Prog. Phys. 69 (2006) 1637–1711.
[21] A. Rajantie, Magnetic monopoles in field theory and cosmology, Philos. Trans. R. Soc. A 370 (2012) 5705–5717.
[22] M.F. Atiyah, N. Hitchin, The Geometry and Dynamics of Magnetic Monopoles, Princeton University Press, 1988.
[23] Y.M. Shnir, Magnetic Monopoles, Springer, 2005.
[24] M. Born, L. Infeld, Foundations of the new field theory, Proc. R. Soc. A 144 (1934) 425–451.
[25] M. Born, L. Infeld, On the quantization of the new field equations, Proc. R. Soc. A 147 (1934) 522–546.
[26] P.J. Morrison, The Maxwell-Vlasov equations as a continuous Hamiltonian system 80 (1980) 383–386.
[27] P.J. Morrison, Poisson brackets for fluids and plasmas, AIP Conf. Proc. 88 (1982) 13–46.
[28] J.E. Marsden, A. Weinstein, The Hamiltonian structure of the Maxwell-Vlasov equations, Physica D 4 (1982).
[29] I. Bialynicki-Birula, J.C. Hubbard, L.A. Turski, Gauge-independent canonical formulation of relativistic plasma theory, Physica A 128 (3) (1984) 509.
[30] P.J. Morrison, Hamiltonian description of the ideal fluid 70 (1998) 467–521.
[31] P.J. Morrison, A general theory for gauge-free lifting, Phys. Plasmas 20 (2013) 012104.
[32] J.D. Jackson, Classical Electrodynamics, 3rd edition, Wiley, 1998, pp. 273–275.
[33] E. Sudarshan, N. Mukunda, Classical Dynamics: A Modern Perspective, Wiley, New York, 1974.
[34] V.I. Arnold, V.V. Kozlov, A.I. Neishtadt, Mathematical Aspects of Classical and Celestial Mechanics, 3rd edition, Springer, 2006.
[35] J.-M. Souriau, Structure des systèmes dynamiques, Dunod, Paris, 1970.
[36] W. Thirring, Classical Mathematical Physics: Dynamical Systems and Field Theories, 3rd edition, Springer Verlag, New York, 1997.
[37] E.C. D'Avignon, Physical consequences of the Jacobi identity, arXiv:1510.06455.
[38] R.G. Littlejohn, Singular Poisson tensors, AIP Conf. Proc. 88 (1982) 47–66.
[39] L.P. Eisenhart, Continuous Groups of Transformations, 2nd edition, Dover, 1961, p. 250.
[40] F. Rohrlich, Classical theory of magnetic monopoles, Phys. Rev. 150 (4) (1966) 1104–1111.
[41] J. Godfrey, Do Dirac monopoles admit a Lagrangian?, Nuovo Cimento 71A (1) (1982) 134–146.
[42] G. Wentzel, Comments on Dirac's theory of magnetic monopoles, Prog. Theor. Phys. Suppl. 37 (1966) 163–174.
[43] H.J. Lipkin, W.I. Weisberger, M. Peshkin, Magnetic charge quantization and angular momentum, Ann. Phys. 53 (1969) 203–214.
[44] D.G. Boulware, S. Deser, B. Zumino, Absence of 2-cocycles in the Dirac monopole problem, Phys. Lett. B 153 (4,5) (1985) 307–310.
[45] R. Jackiw, Three-cocycle in mathematics and physics, Phys. Rev. Lett. 54 (3) (1985) 159–162.
[46] Y.-S. Wu, A. Zee, Cocycles and magnetic monopole, Phys. Lett. 152 (1,2) (1985) 98–102.
[47] R.J. Szabo, Higher quantum geometry and non-geometric string theory, in: 17th Hellenic School and Workshops on Elementary Particle Physics and Gravity, Corfu, Greece, 2018, arXiv:1803.08861.
[48] P.J. Morrison, J. Vanneste, Weakly nonlinear dynamics in noncanonical Hamiltonian systems with applications to fluids and plasmas, Ann. Phys. 368 (2016) 117–147.
[49] M. Lainz, C. Sardón, A. Weinstein, Plasma in monopole background is not twisted Poisson, Phys. Rev. D (2019), in press, arXiv:1908.03986.