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ABSTRACT

Helicity, a topological degree that measures the winding and linking of vortex lines, is preserved by ideal (barotropic) fluid dynamics. In the context of the Hamiltonian description, the helicity is a Casimir invariant characterizing a foliation of the associated Poisson manifold. Casimir invariants are special invariants that depend on the Poisson bracket, not on the particular choice of the Hamiltonian. The total mass (or particle number) is another Casimir invariant, whose invariance guarantees the mass (particle) conservation (independent of any specific choice of the Hamiltonian). In a kinetic description (e.g., that of the Vlasov equation), the helicity is no longer an invariant (although the total mass remains a Casimir of the Vlasov's Poisson algebra). The implication is that some "kinetic effect" can violate the constancy of the helicity. To elucidate how the helicity constraint emerges or submerges, we examine the fluid reduction of the Vlasov system; the fluid (macroscopic) system is a "sub-algebra" of the kinetic (microscopic) Vlasov system. In the Vlasov system, the helicity can be conserved if a special *helicity symmetry* condition holds. To put it another way, breaking helicity symmetry induces a change in the helicity. We delineate the geometrical meaning of helicity symmetry and show that for a special class of flows (the so-called epi-two-dimensional flows), the helicity symmetry is written as $\partial_\gamma = 0$ for a coordinate γ of the configuration space.

I. INTRODUCTION

The notion of helicity appears in both ideal fluid mechanics and ideal magnetohydrodynamics (MHD). In fluid mechanics, ideas pertaining to helicity and its conservation date back to the work of Lord Kelvin in the 19th century, while its recognition as an indicator of the topological linkage of vortex lines was given in Ref. 1. Similarly, that a kind of helicity is conserved by the ideal MHD equations was noted in Ref. 2, but its topological recognition in terms of magnetic field line linkage was given in Ref. 3. The study of helicity in a variety of fluid and magnetofluid contexts continues to fascinate researchers. For example, recently, the role helicity plays in the reconnection of vortex tubes has been experimentally observed in real fluids (e.g., Refs. 4 and 5) and the role it plays in cascades of turbulence has been investigated in a variety of numerical simulations (e.g., Refs. 6 and 7). However, to our knowledge, there have been no studies of how helicity emerges from the kinetic description that underlies the fluid description, which is the subject matter of the present paper.

Both ideal fluid mechanics and collisionless kinetic theory possess a noncanonical Hamiltonian structure,[8] with Poisson brackets that generate flows on Poisson manifolds.[9] Consequently, both fluids and kinetic theories possess Casimir invariants, universal invariants independent of any particular choice of the Hamiltonian for the respective theories. Therefore, these invariants represent types of *topological constraints* inherent to Poisson manifold phase spaces. Every orbit is constrained to Casimir leaves, level-sets of Casimirs, so the gradient of a Casimir is transverse to the leaf its constancy defines. This means that the gradient of the Casimir belongs to the kernel of the Poisson matrix (the 2-vector that maps the gradient of the Hamiltonian to the Hamiltonian vector field) that defines the Poisson bracket. By "degenerate," we mean that the Poisson matrix defined on the Poisson manifold has a nontrivial kernel. We note that the element of the kernel (co-vector) is not necessarily *integrable*, i.e., the gradient of some scalar. However, a Casimir is such an integral, yielding a foliation of the Poisson manifold by its level-sets.

When a Casimir is given, one can interpret it as an *adiabatic invariant* made "variable" by adding an *angle variable* to complete a conjugate action-angle pair.[10] After embedding the noncanonical Poisson manifold into the inflated phase space, the constancy of the Casimir can, then, be re-interpreted as arising from a symmetry with respect to the supplementary angle variable; hence, its constancy is now attributed to this specific symmetry of the Hamiltonian. The example of Sec. II B will delineate such a relation from the opposite viewpoint: starting from the canonical (symplectic) Poisson manifold $\mathfrak{sp}(6,\mathbb{R})$, we derive the noncanonical $\mathfrak{so}(3)$ Lie–Poisson algebra by *reduction*.[11] Restricting the six dimensions of $\mathfrak{sp}(6,\mathbb{R})$, represented by the position vector $\boldsymbol{q}$ and the momentum vector $\boldsymbol{p}$, to three dimensions determined by the angular momentum $\boldsymbol{\ell} = \boldsymbol{q} \times \boldsymbol{p}$, the system reduces to the three-dimensional $\mathfrak{so}(3)$ Lie–Poisson manifold with the magnitude $|\boldsymbol{\ell}|$ becoming the Casimir of the $\mathfrak{so}(3)$ Lie–Poisson bracket. Consequently, the effective available phase space shrinks to the two-dimensional spherical surface $|\boldsymbol{\ell}|$ = constant. A physical example of such a reduced angular momentum system is the *Euler top*, which is a point mass bound to the origin of the coordinate space by a rigid, mass-less rod. Then, the angle between the position $\boldsymbol{q}$ and the momentum $\boldsymbol{p}$ is fixed to be perpendicular. If this angle is allowed to vary (for example, if the rod is not sufficiently rigid), the constancy of the Casimir $|\boldsymbol{\ell}|$ is broken (see Sec. II B), i.e., "rigidity" is the root cause of the Casimir. More precisely, for the Casimir to be invariant, there must be a distinct separation of the time scale (or energy) between the dynamics of the top and the change of the angle variable. Consequently, from this point of view, we may interpret the Casimir as an adiabatic invariant.

In fluid mechanics, the *helicity* is a Casimir of the Hamiltonian formalism of the ideal (barotropic) fluid model,[12] which is a measure of the winding and linking of vortex lines.[1,3] Interestingly, in a kinetic description (e.g., the Vlasov equation; see Sec. III), the helicity is no longer an invariant. This implies that some "kinetic effect" can violate the topological constraint associated with the helicity. It is known that the ideal fluid system can be formulated as a reduction (subalgebra) of the Vlasov system.[8] In the Vlasov system, the helicity can still be conserved if a special *helicity symmetry* condition holds. To put it another way, braking of the helicity symmetry allows for a changing helicity. The aim of this work is to elucidate the geometrical meaning of the helicity symmetry and study how the topological constraint associated with the helicity invariance can be broken in kinetic theory. For a special class of flows (the so-called epi-two-dimensional flows[13]), we will show that the helicity symmetry is written as $\partial_y = 0$, with y being a configuration space coordinate.

II. CASIMIR AND GAUGE SYMMETRY IN A "REDUCED SYSTEM"—EXAMPLES

Because Casimirs play a central role in this work, we explain, by simple examples, how a Casimir is "created" by a *reduction* of some kind and how it is related to the *gauge symmetry* of the reduction.

A. Reduction of canonical variables

We start with the canonical Hamiltonian system of a point mass moving in $\mathbb{R}^n$ with the phase space being $M = \mathbb{R}^{2n}$. The coordinates of a point of M represent a state vector, $\boldsymbol{z} = (\boldsymbol{q},\boldsymbol{p})^{\mathrm{T}}$, with position $\boldsymbol{q}$ and momentum $\boldsymbol{p}$. On $C^\infty(M)$, the space of observables, we define the canonical Poisson bracket

$$[G,H] = \sum_{j=1}^{n} (\partial_{q^j} G)(\partial_{p_j} H) - (\partial_{q^j} H)(\partial_{p_j} G). \tag{1}$$

Denoting by $\partial_{\boldsymbol{z}} G \in T^*M$ the gradient of $G \in C^\infty(M)$ and by $\langle \boldsymbol{x}, \boldsymbol{y}\rangle$ the natural pairing of $\boldsymbol{x} \in T^*M$ and $\boldsymbol{y} \in TM$, we may rewrite (1) as

$$[G,H] = \langle \partial_{\boldsymbol{z}} G, J \partial_{\boldsymbol{z}} H\rangle, \tag{2}$$

with the Poisson operator (matrix)

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \quad \in \mathrm{Hom}(T^*M, TM). \tag{3}$$

The Hamiltonian vector $\dot{\boldsymbol{z}} \in TM$ is given by

$$\dot{\boldsymbol{z}} = J \partial_{\boldsymbol{z}} H.$$

We assume $n = 2$ and denote the corresponding symplectic manifold by M_4 $(= \mathbb{R}^4)$. As a trivial example of reduction, we suppose all observables are independent of q^2 and p_2. Then, the Poisson bracket evaluates as

$$[G,H]_{M_2} = (\partial_{q^1} G)(\partial_{p_1} H) - (\partial_{q^1} H)(\partial_{p_1} G), \tag{4}$$

which defines a canonical Poisson algebra on the submanifold $M_2 = \{\boldsymbol{z}_2 = (q^1, p_1)^{\mathrm{T}}\} = \mathbb{R}^2$, which is embedded in M_4 as a leaf $\{\boldsymbol{z} \in M_4;\ q^2 = c, p_2 = c'\}$, where c and c' are arbitrary constants.

An interesting situation occurs when we only suppress the coordinate q^2 in the set of observables: the reduced phase space is the three-dimensional submanifold $M_3 = \{\boldsymbol{z}_3 = (q^1, p_1, p_2)^{\mathrm{T}}\}$. For G and H satisfying $\partial_{q^2} G = \partial_{q^2} H = 0$, the Poisson bracket evaluates the same as (4) and we may write

$$[G, H]_{M_3} = \langle \partial_{\boldsymbol{z}_3} G, J \partial_{\boldsymbol{z}_3} H \rangle$$

with the Poisson operator (matrix)

$$J = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

whose rank is two. Therefore, M_3 is a degenerate Poisson manifold. The kernel of this J includes the vector $(0,0,1)^{\mathrm{T}}$, which can be integrated to define a Casimir $C = p_2$. Therefore, the effective dimension is further reduced down to two; the state vector $\boldsymbol{z}$ can only move on the two-dimensional leaf M_2. Evidently, the "freezing" of $C = p_2$ is due to the suppression of its conjugate variable q^2.

When we observe M_3 from M_4, the reduction (i.e., the suppression of the coordinate q^2 in the observables) means the *symmetry* $\partial_{q^2} = 0$. As usual, the integral of motion p_2 in M_4 arises because q^2 is ignorable, i.e., if the Hamiltonian has the symmetry $\partial_{q^2} H = 0$.

The variable q^2 conjugate to the Casimir $C = p_2$ can be regarded as a *gauge parameter*. The gauge group (denoted by Ad_C), which does not change the submanifold M_3 embedded in M_4, is generated by the adjoint action

$$\mathrm{ad}_C = [\circ, C] = \partial_{q^2},$$

implying that the *gauge symmetry* is written as $\partial_{q^2} = 0$. This is evident because the state vector $\boldsymbol{z}_3 = (q^1, p_1, p_2)^{\mathrm{T}} \in M_3$ is independent of q^2. Note that $\partial_{q^2} = 0$ is the symmetry producing the integral $C = p_2$ (which we may call *Casimir symmetry*) and, at the same time, the *gauge symmetry* of the submanifold M_3.

A similar reduction occurs when we consider the canonical pair,

$$\mu = \frac{1}{2}\left[(q^2)^2 + (p_2)^2\right], \quad \theta = \tan^{-1}\left(\frac{q^2}{p_2}\right).$$

If we suppress θ in the set of observables, μ becomes the Casimir of $M_3' = M_4/\{\theta\}$. The motion of a magnetized particle is an example, where μ corresponds to the magnetic moment and θ to the gyration angle. When the gyroperiod is negligibly shorter than the time scale of interest, μ can be dealt with as an adiabatic invariant. Such "coarse graining" means that we consider the average over $\theta \in [0, 2\pi)$ and put $\mathrm{ad}_\mu = \partial_\theta = 0$ for all observables. See Refs. 14 and 15 for in-depth treatments of magnetized charged particle dynamics and adiabatic invariance.

In Sec. II B, we consider another example that displays a less trivial relation between the Casimir and gauge symmetry.

B. Reduction of $\mathfrak{sp}(6, \mathbb{R})$ to the $\mathfrak{so}(3)$ Lie–Poisson manifold

In this next example, we examine the reduction that produces the $\mathfrak{so}(3)$ Lie–Poisson system and how its Casimir is related to the *gauge symmetry*, i.e., the invariance of the reduced variables with respect to the transformation (gauge group action) among the original variables.

We start with the canonical Hamiltonian system of $n = 3$ with the Poisson bracket given by (1). We let $\boldsymbol{z} = (\boldsymbol{q}, \boldsymbol{p})^{\mathrm{T}} \in M_6 = \mathbb{R}^6$ and consider the system where the observables are functions of only the angular momentum,

$$\boldsymbol{\ell} = \boldsymbol{q} \times \boldsymbol{p}. \tag{5}$$

The Euler top is such an example, where the Hamiltonian is $H(\boldsymbol{\ell}) = \sum_j \ell_j^2/(2I_j)$, with I_1, I_2, I_3 being the three moments of inertia. For such a system, the effective phase space is reduced to $M_{\boldsymbol{\ell}} \cong \mathbb{R}^3$. Let us evaluate $[\ ,\ ]$ for observables $\in C^\infty(M_\ell)$. The gradient of a functional $F \in C^\infty(M_\ell)$ is given by

$$\delta F = \langle \partial_{\boldsymbol{q}} F, \delta \boldsymbol{q} \rangle + \langle \partial_{\boldsymbol{p}} F, \delta \boldsymbol{p} \rangle = \langle \partial_{\boldsymbol{\ell}} F, \delta \boldsymbol{\ell} \rangle.$$

Inserting $\delta\boldsymbol{\ell} = (\delta\boldsymbol{q}) \times \boldsymbol{p} + \boldsymbol{q} \times (\delta\boldsymbol{p})$, we find

$$\partial_{\boldsymbol{q}} F = \boldsymbol{p} \times \partial_{\boldsymbol{\ell}} F \quad \text{and} \quad \partial_{\boldsymbol{p}} F = -\boldsymbol{q} \times \partial_{\boldsymbol{\ell}} F.$$

Therefore,

$$[G, H] = \langle \partial_{\boldsymbol{\ell}} G, \partial_{\boldsymbol{\ell}} H \times \boldsymbol{\ell} \rangle =: \{G, H\},$$

which is a Lie–Poisson bracket (see Remark 1) as follows:

$$\{G, H\} = \langle \partial_{\boldsymbol{\ell}} G, J(\boldsymbol{\ell}) \partial_{\boldsymbol{\ell}} H \rangle,$$

with the Poisson operator (matrix)

$$J(\boldsymbol{\ell}) := -\boldsymbol{\ell} \times \circ = \begin{pmatrix} 0 & \ell_3 & -\ell_2 \\ -\ell_3 & 0 & \ell_1 \\ \ell_2 & -\ell_1 & 0 \end{pmatrix}. \tag{6}$$

Note that this Poisson operator is a linear function of $\boldsymbol{\ell}$, the signature of a Lie–Poisson algebra (see Remark 1). Here, rank $J(\boldsymbol{\ell}) = 2$ [avoiding the point $\boldsymbol{\ell} = 0$ where rank $J(\boldsymbol{\ell}) = 0$], so we expect a single Casimir of the reduced Poisson algebra, which, evidently, is

$$C = \frac{1}{2}|\boldsymbol{\ell}|^2,$$

a function easily seen to satisfy $\{G, C\} = 0$ ($\forall G \in C^\infty(V_\ell)$), or $J(\boldsymbol{\ell})\partial_{\boldsymbol{\ell}} C = 0$.

When we take C as the Hamiltonian, the adjoint action

$$\begin{aligned} \mathrm{ad}_C = [\circ, C] &= \left(\sum_{j=1}^{3} \partial_{p_j} C \partial_{q^j} - \partial_{q^j} C \partial_{p_j} \right) \\ &= \boldsymbol{\ell} \times \boldsymbol{q} \cdot \partial_{\boldsymbol{q}} + \boldsymbol{\ell} \times \boldsymbol{p} \cdot \partial_{\boldsymbol{p}} \end{aligned} \tag{7}$$

generates the *gauge transformation* of the reduced variable $\boldsymbol{\ell}$; by direct calculation, it follows easily that $[\ell_j, C] = 0$ ($j = 1, 2, 3$).

This gauge transformation has the following geometrical meaning. By (7), the transformation $\boldsymbol{z} \mapsto \boldsymbol{z} + \varepsilon\tilde{\boldsymbol{z}}$ ($\tilde{z}_j = [z_j, C]$) gives a co-rotation of $\boldsymbol{q}$ and $\boldsymbol{p}$ around the axis $\boldsymbol{\ell}$ (note that this rotation is in the space M_6, not in the space $M_{\boldsymbol{\ell}}$); hence, $\boldsymbol{\ell} = \boldsymbol{q} \times \boldsymbol{p}$ does not change. The rotation angle can be written as

$$\theta = \frac{1}{2|\boldsymbol{\ell}|} \tan^{-1}\left(\frac{(\boldsymbol{\ell} \times \boldsymbol{q})_j}{q_j |\boldsymbol{\ell}|} \right)$$

(we choose the coordinate $q_j \neq 0$) and, evidently, $[\theta, C] = 1$. Let us embed $M_{\boldsymbol{\ell}}$ in the four-dimensional space $\widetilde{V_{\boldsymbol{\ell}}} = \{(\boldsymbol{\ell}, \theta);\ \boldsymbol{\ell} \in M_{\boldsymbol{\ell}},\ \theta \in [0, 2\pi)\}$. For $G(\boldsymbol{\ell}, \theta) \in C^\infty(\widetilde{M_{\boldsymbol{\ell}}})$, we obtain

$$[G, C] = \sum_{j=1}^{3} \partial_{\ell_j} G[\ell_j, C] + \partial_\theta G[\theta, C] = \partial_\theta G.$$

Therefore, the gauge symmetry $[\circ, C] = 0$ can be rewritten as $\partial_\theta = 0$. Reversing the view point, for every Hamiltonian $H(\boldsymbol{\ell}, \theta) \in C^\infty(\widetilde{M_{\boldsymbol{\ell}}})$ that has the symmetry $\partial_\theta H = 0$, C is invariant,

$$\dot{C} = [C, H] = -\partial_\theta H = 0.$$

Therefore, the conjugate variable θ dictates both the *gauge symmetry* $[\circ, C] = \partial_\theta = 0$ of the submanifold $M_\ell \subset M_6$ and the *Casimir symmetry* $\dot{C} = [C, H] = 0$ ($\forall H$ such that $\partial_\theta H = 0$). We can, further, embed $\widetilde{M_{\boldsymbol{\ell}}}$ in M_6 by identifying all canonical variables (see Remark 2).

Remark 1 (Lie–Poisson bracket). Given a Lie algebra $\mathfrak{g}$, we can construct a Poisson bracket on the dual space $\mathfrak{g}^*$; such brackets are called *Lie–Poisson brackets* because they were known to Lie in the 19th century. Let $[\,,]$ be the Lie bracket of $\mathfrak{g}$ and $\langle\,,\rangle$ be the pairing $\mathfrak{g}\times\mathfrak{g}^*\to\mathbb{K}$ (the field of scalars). We denote by $\boldsymbol{\mu}$ the vector of $\mathfrak{g}^*$. For $G(\boldsymbol{\mu})\in C^\infty(\mathfrak{g}^*)$, we define its *gradient* $\partial_{\boldsymbol{\mu}}G\in\mathfrak{g}$ by

$$\delta G = G(\boldsymbol{\mu}+\varepsilon\tilde{\boldsymbol{\mu}}) - G(\boldsymbol{\mu}) = \varepsilon\langle\partial_{\boldsymbol{\mu}}G,\tilde{\boldsymbol{\mu}}\rangle + O(\varepsilon^2) \quad (\forall\tilde{\boldsymbol{\mu}}\in\mathfrak{g}^*). \tag{8}$$

The dual space $\mathfrak{g}^*$ is made a Poisson manifold by endowing it with

$$\{G,H\} = \langle[\partial_{\boldsymbol{\mu}}G,\partial_{\boldsymbol{\mu}}H],\boldsymbol{\mu}\rangle = \langle\partial_{\boldsymbol{\mu}}G,[\partial_{\boldsymbol{\mu}}H,\boldsymbol{\mu}]^*\rangle, \tag{9}$$

where $[\ ,\ ]^*:\mathfrak{g}\times\mathfrak{g}^*\to\mathfrak{g}^*$ is the dual representation of $[\,,]$. Because of this construction, $\{\,,\}$ inherits bilinearity, anti-symmetry, and the Jacobi's identity from that of $[\,,]$. The Leibniz property is explicitly implemented by the derivation $\partial_{\boldsymbol{\mu}}$, so $\{\ ,\ \}$ is a Poisson bracket. The forgoing example of $\mathfrak{so}(3)$ and the Vlasov system's Poisson bracket to be formulated in Sec. III are examples of Lie–Poisson systems.

Remark 2 (complete set of canonical variables). Let us determine two other canonical variables (say ψ and φ) needed to embed $M_{\boldsymbol{\ell}}$ in M_6. These variables will determine the gauge freedom of $\boldsymbol{\ell}=\boldsymbol{q}\times\boldsymbol{p}$; we demand the canonical relations $[\ell_j,\psi]=[\ell_j,\varphi]=0$ (as well as commutations with C and θ), which implies that $\mathrm{ad}^*_\psi\ell_j=\mathrm{ad}^*_\varphi\ell_j=0$. On the surface transverse to $\boldsymbol{\ell}$, $\mathfrak{sl}(2;\mathbb{R})$ has the following two other actions:

$$\boldsymbol{z}\mapsto\boldsymbol{z}+\varepsilon(\ 0\ ,\boldsymbol{q}),\quad \boldsymbol{z}\mapsto\boldsymbol{z}+\varepsilon(\boldsymbol{q},\ -\boldsymbol{p}),$$

which correspond to twist and compression/extension deformations, respectively. These transformations can be generated by the following pair of conjugate variables:

$$\psi=\frac{|\boldsymbol{q}|^2}{2},\quad \varphi=\frac{\boldsymbol{q}\cdot\boldsymbol{p}}{|\boldsymbol{q}|^2}.$$

In summary, (C,θ,ψ,φ) span the complement of the symplectic leaves of the reduced system. Note that only C can be represented by the reduced variable $\boldsymbol{\ell}$, i.e., $C\in C^\infty(M_\ell)$. The other parameters ψ and φ inflate the phase space to recover M_6.

III. THE IDEAL FLUID SYSTEM AS A SUB-ALGEBRA OF THE VLASOV SYSTEM

A. Kinetic Lie–Poisson algebra for the Vlasov system

Let $\boldsymbol{z}=(\boldsymbol{x},\boldsymbol{v})=(x^1,\ldots,x^n,v_1,\ldots,v_n)$ be coordinates for a point of $M=X\times V=\mathbb{T}^n\times\mathbb{R}^n$, the phase space of a particle, which is the cotangent bundle T^*X of a configuration space X. For convenience, we call X the $\boldsymbol{x}$-space and V the $\boldsymbol{v}$-space.

We call a real-valued function $\psi(\boldsymbol{z})\in C^\infty(M)$ an observable, and the space $C^\infty(M)$ is endowed with the Poisson bracket

$$[\psi,\varphi]=\sum_{j=1}^n(\partial_{x^j}\psi)(\partial_{v_j}\varphi)-(\partial_{v_j}\psi)(\partial_{x^j}\varphi), \tag{10}$$

where we denote $\mathfrak{g}=C^\infty_{[\ ,\ ]}(M)$. The adjoint representation $\mathrm{ad}_h=[\circ,h]$ of this Lie algebra describes the Hamiltonian dynamics of a particle, i.e.,

$$\dot{\psi}=[\psi,h],$$

where h is the particle Hamiltonian.

The dual space $\mathfrak{g}^*$ is the set of *distribution functions*; for an observable $\psi\in\mathfrak{g}$ and a distribution function $f\in\mathfrak{g}^*$,

$$\langle\psi,f\rangle=\int_M\psi(z)f(z)\,\mathrm{d}z \tag{11}$$

evaluates the mean value of ψ over the distribution function f (see Remarks 3 and 4).

The function space $\mathfrak{g}^*$ of distributions will be the Poisson manifold with the following construction (corresponding here to the phase space M of the examples discussed in Sec. II). On the space $\mathfrak{V} = C^\infty(\mathfrak{g}^*)$ (the set of generalized observables defined for distributions = mixed states; see Remark 3), the Poisson–Vlasov Lie–Poisson bracket[16,17] (see also Refs. 18–21) is defined as follows:

$$\{G, H\} = \langle [\partial_f G, \partial_f H], f \rangle, \tag{12}$$

where $\partial_f H \in T^*\mathfrak{V} = \mathfrak{g}$ is the gradient of $H \in \mathfrak{V}$ (see Remark 1). Integrating by parts, we may rewrite (12) as

$$\{G, H\} = \langle \partial_f G, [\partial_f H, f]^* \rangle = \langle \partial_f G, J(f)\partial_f H \rangle, \tag{13}$$

where $[\ ,\ ]^* : \mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g}^*$ evaluates formally as $[a,b]^* = [a,b]$ (see Remark 4). We call $J(f) \circ = [\circ, f]^*$ the Poisson operator.

For $G(f) = \langle \delta(\boldsymbol{z} - \boldsymbol{\zeta}), f(\boldsymbol{z}) \rangle = f|_{z=\zeta}$, Hamilton's equation $\dot{G} = \{G, H\}$ evaluates the co-adjoint orbit; for every point $\boldsymbol{\zeta} \in M$,

$$\dot{f} = [\partial_f H, f]^*, \tag{14}$$

which is the Vlasov equation governing the evolution of the distribution function $f(\boldsymbol{z})$ under the action of the *particle Hamiltonian* $h = \partial_f H$. For example, let

$$h(\boldsymbol{z}) = \frac{1}{2}|\boldsymbol{v}|^2 + \Phi(\boldsymbol{x}), \quad H(f) = \frac{1}{2}\int_M \left(|\boldsymbol{v}|^2 + \Phi(\boldsymbol{x})\right) f(\boldsymbol{z})\, \mathrm{d}z,$$

where Φ functionally depends on f via Poisson's equation. The first term of h corresponds to the kinetic energy (we set the particle mass to unity), and the second term represents the potential energy (mean field). Then, (14) reads

$$\dot{f} = \sum_j -\partial_{v_j} h \partial_{x^j} f + \partial_{x^j} h \partial_{v_j} f = \sum_j -v_j \partial_{x^j} f + \partial_{x^j}\Phi\partial_{v_j} f.$$

Remark 3 (distribution function). The dual space $\mathfrak{g}^*$ may be identified as the set of n-forms on M. Then, it is better to say that $f\mathrm{d}z$ ($\mathrm{d}z$ is the phase space volume element), or, more generally, a measure on M, is the member of the dual space. However, regarding (11) as the definition of duality, we may identify the scalar part f as the member of the *dual space* $\mathfrak{g}^*$; see Remark 4 for the identification the dual space as the space of n-forms. The *pure state* $f = \delta(\boldsymbol{z} - \boldsymbol{\zeta}) \in \mathfrak{g}^*$ identifies a point in M and evaluates $\langle \psi, f \rangle = \psi(\boldsymbol{\zeta})$. A general f may be regarded as a mixed state.

Remark 4 (Hodge duality of $\mathfrak{g}$ and $\mathfrak{g}^*$). A distribution is rigorously a measure on the phase space M and is identified as an n-form $f^\star := \star f = f\mathrm{d}z$, where $\mathrm{d}z$ is the volume form (Lebesgue measure) of M, f is the scalar part of the distribution, and $\star$ is the Hodge star operator. As noted in Remark 3, however, it is often convenient to regard the scalar part f as the distribution function. Let us denote by $\mathfrak{g}^\star$ the Hodge-dual space of $\mathfrak{g}$. We may identify $\mathfrak{g}^* = \star\mathfrak{g}^\star$. For a scalar (0-form) $\varphi \in \mathfrak{g}$ and an n-form $f^\star \in \mathfrak{g}^\star$, we define $[\varphi, f^\star]^\star = \star[\varphi, \star f^\star] = [\varphi, f]^*\mathrm{d}z$. This $[\ ,\ ]^\star : \mathfrak{g} \times \mathfrak{g}^\star \to \mathfrak{g}^\star$ is the original form of the *dual* representation of $[\,,]$. Changing $\star$ to $*$ means that we take the scalar part (Hodge dual) of the distribution (a *distribution function* is the scalar part of a distribution).

B. Reduction to moment variables

As is well known, a "fluid model" is derived by taking the $\boldsymbol{v}$-space moments of a kinetic model. Here, we review how it works in the framework of Poisson algebras (Hamiltonian mechanics). For the distribution $f(\boldsymbol{z}) \in \mathfrak{g}^*$, we define

$$\rho(\boldsymbol{x}, t) = \int_V f(\boldsymbol{x}, \boldsymbol{v}, t)\, \mathrm{d}^n v, \tag{15}$$

$$P_j(\boldsymbol{x}, t) = \int_V v_j f(\boldsymbol{x}, \boldsymbol{v}, t)\, \mathrm{d}^n v \quad (j = 1, \ldots, n). \tag{16}$$

For convenience of notation, we subsume the density $\rho(\boldsymbol{x}, t)$ in $P_v(\boldsymbol{x}, t)$ as the zeroth component. Using $v_0 = 1$ as the zeroth component, we define the $n + 1$ dimensional co-vector (momentum) $\widehat{\boldsymbol{v}} = (v_0, \boldsymbol{v})^{\mathrm{T}}$; hence,

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$$\rho(\boldsymbol{x},t) = P_0(\boldsymbol{x},t) = \int_V v_0 f(\boldsymbol{x},\boldsymbol{v},t)\,\mathrm{d}^n v.$$

Therefore, using $\widehat{\boldsymbol{P}} = (P_0,\boldsymbol{P})^{\mathrm{T}} = (P_0,P_1,\ldots,P_n)^{\mathrm{T}}$, we get the unified representation

$$P_\nu(\boldsymbol{x},t) = \int_V v_\nu f(\boldsymbol{x},\boldsymbol{v},t)\,\mathrm{d}^n v \quad (\nu = 0,\ldots,n). \tag{17}$$

We will use a Greek letter (such as μ or ν) for an index that starts from zero and Roman letter (such as j or k) that starts from 1. In the vector notation, we will put $\widehat{\ }$ when we include a zeroth component.

For a functional $G(P_0,P_1,\ldots,P_n)$, the chain rule reads

$$\delta G = \int_M \partial_f G\,\delta f\,\mathrm{d}^n v\mathrm{d}^n x = \int_X \sum_{\nu=0}^{n} \partial_{P_\nu} G\,\delta P_\nu\,\mathrm{d}^n x. \tag{18}$$

By $\delta P_\nu = \int_V v_\nu \delta f\,\mathrm{d}^n v$ $(\nu = 0,1,\ldots,n)$, we obtain

$$\partial_f G = \sum_{\nu=0}^{n} (\partial_{P_\nu} G) v_\nu. \tag{19}$$

For $g^\nu := \partial_{P_\nu} G$ and $h^\nu := \partial_{P_\nu} H$, the kinetic Poisson bracket (10) evaluates as

$$\begin{aligned}[\partial_f G,\partial_f H] &= \sum_{j=1}^{n}\sum_{\nu=0}^{n} \partial_{x^j}(g^\nu v_\nu)\,h^j - g^j \partial_{x^j}(h^\nu v_\nu)\\ &= [(\boldsymbol{h}\cdot\nabla)\boldsymbol{g} - (\boldsymbol{g}\cdot\nabla)\boldsymbol{h}]\cdot\boldsymbol{v} + (\boldsymbol{h}\cdot\nabla g^0 - \boldsymbol{g}\cdot\nabla h^0),\end{aligned}$$

where $\boldsymbol{g} = (g^1,\ldots,g^n)^{\mathrm{T}}$ and $\boldsymbol{h} = (h^1,\ldots,h^n)^{\mathrm{T}}$. Hence, we obtain

$$\begin{aligned}\{G,H\} &= \langle[\partial_f G,\partial_f H], f\rangle\\ &= \int_X \sum_{\nu=0}^{n} [(\boldsymbol{h}\cdot\nabla)g^\nu - (\boldsymbol{g}\cdot\nabla)h^\nu]\cdot P_\nu\,\mathrm{d}^n x\\ &= \left(\partial_{\widehat{\boldsymbol{P}}} G, J_P(\widehat{\boldsymbol{P}})\partial_{\widehat{\boldsymbol{P}}} H\right) =: \{G,H\}_P,\end{aligned} \tag{20}$$

where the Poisson operator $J_P(\widehat{\boldsymbol{P}})$ for $n = 3$ is the Lie–Poisson form given in Ref. 12,

$$J_P(\widehat{\boldsymbol{P}}) = \begin{pmatrix} 0 & -\nabla\cdot(P_0\circ) \\ -P_0\nabla & -(\nabla\times\boldsymbol{P})\times\circ - \boldsymbol{P}(\nabla\cdot\circ) - \nabla(\boldsymbol{P}\cdot\circ) \end{pmatrix} \tag{21}$$

and

$$(\boldsymbol{a},\boldsymbol{b}) = \int \boldsymbol{a}(\boldsymbol{x})\cdot\boldsymbol{b}(\boldsymbol{x})\,\mathrm{d}^3 x. \tag{22}$$

C. Fluid variables

The bracket in terms of the usual fluid variables is derived by changing variables as follows:

$$\widehat{\boldsymbol{P}} = (P_0,P_1,\ldots,P_n) \;\leftrightarrow\; \widehat{\boldsymbol{U}} = (\rho,U_1,\ldots,U_n), \tag{23}$$

where

$$\rho(\boldsymbol{x}) = P_0(\boldsymbol{x}) = \int f(\boldsymbol{x},\boldsymbol{v})\,\mathrm{d}^3v, \tag{24}$$

$$U_j(\boldsymbol{x}) = \frac{P_j(\boldsymbol{x})}{P_0(\boldsymbol{x})} = \frac{\int v_j f(\boldsymbol{x},\boldsymbol{v})\,\mathrm{d}^3v}{\int f(\boldsymbol{x},\boldsymbol{v})\,\mathrm{d}^3v} \quad (j = 1,2,3). \tag{25}$$

The chain rule gives

$$\begin{aligned}\delta G &= \int_X \sum_\nu \partial_{P_\nu} G\,\delta P_\nu\,\mathrm{d}^n x\\ &= \int_X \partial_{U_0} G\delta P_0 + \sum_j \partial_{U_j} G\left(\frac{\delta P_j}{P_0} - \frac{P_j \delta P_0}{P_0^2}\right)\mathrm{d}^n x.\end{aligned}$$

Hence, we transform

$$\partial_{P_0} G = \partial_\rho G - \frac{1}{\rho}\boldsymbol{U}\cdot\partial_{\boldsymbol{U}} G, \quad \partial_{\boldsymbol{P}} G = \frac{1}{\rho}\partial_{\boldsymbol{U}} G,$$

by which we may calculate, for $G(\widehat{\boldsymbol{U}})$,

$$\partial_f G = \partial_\rho G + \sum_{j=1}^n \frac{v_j - U_j}{\rho}\partial_{U_j} G. \tag{26}$$

The Poisson bracket (20) transforms into the following *fluid Poisson bracket*: For $G(\widehat{\boldsymbol{U}}), H(\widehat{\boldsymbol{U}})$, the Vlasov Lie–Poisson bracket $\{\ ,\ \}$ evaluates as

$$\{G,H\} = \{G,H\}_F = (\partial_{\widehat{U}} G, J_F(\widehat{\boldsymbol{U}})\partial_{\widehat{U}} H), \tag{27}$$

where the Poisson operator $J_F(\widehat{\boldsymbol{U}})$ is, when $n = 3$, a form, also given in Ref. 12,

$$J_F(\widehat{\boldsymbol{U}}) = \begin{pmatrix} 0 & -\nabla\cdot \\ -\nabla & -\left(\dfrac{\nabla\times\boldsymbol{U}}{\rho}\right)\times \end{pmatrix}. \tag{28}$$

We call $\{G,H\}_F$ the fluid Poisson bracket.

In fact, bracket (28) gives the fluid mechanics equations, when we provide it with the Hamiltonian composed of the total fluid energy, i.e., assuming a barotropic internal energy $\mathscr{E}(\rho)$ and an external potential energy $\phi(\boldsymbol{x})$, we have

$$H(\widehat{\boldsymbol{U}}) = \int_X \rho\left(\frac{1}{2}|\boldsymbol{U}|^2 + \mathscr{E}(\rho) + \phi(\boldsymbol{x})\right)\mathrm{d}^3x. \tag{29}$$

Then,

$$\partial_{\widehat{U}} H = \begin{pmatrix} \frac{1}{2}|\boldsymbol{U}|^2 + h + \phi \\ \rho\boldsymbol{U} \end{pmatrix},$$

where $h = \partial_\rho(\rho\mathscr{E})$ is the enthalpy. Then, Hamilton's equations $\dot{\widehat{\boldsymbol{U}}} = J_U(\widehat{\boldsymbol{U}})\partial_{\widehat{U}} H$ are the same as the ideal fluid equations,

$$\begin{cases} \partial_t \rho = -\nabla \cdot (\boldsymbol{U}\rho), \\ \partial_t \boldsymbol{U} = -(\boldsymbol{U}\cdot\nabla)\boldsymbol{U} - \nabla(h+\phi). \end{cases} \tag{30}$$

By the thermodynamic definition of pressure, $\mathscr{P} = \rho^2 \partial_\rho \mathscr{E}$, we may rewrite $\nabla h = \nabla[\partial_\rho(\rho\mathscr{E})] = \rho^{-1}\nabla\mathscr{P}$.

In summary, by the *reduction* of the space of kinetic distributions $\mathfrak{g}^* = \{f(\boldsymbol{z})\}$ to the space of fluid variables $\mathfrak{g}_F^* = \{\widehat{\boldsymbol{U}} = (\rho, \boldsymbol{U})^{\mathrm{T}}\}$, the Vlasov–Lie–Poisson algebra $\mathfrak{V} = C^\infty_{\{\ ,\ \}}(\mathfrak{g}^*)$ is reduced to a sub-algebra $\mathfrak{V}_F = C^\infty_{\{\ ,\ \}_F}(\mathfrak{g}_F^*)$ dictated by the fluid Poisson bracket $\{G, H\}_F$. Sometimes, it is more convenient to use the equivalent moment variables $\widehat{\boldsymbol{P}}$; we denote the moment reduction by $\mathfrak{g}_P^* = \{\widehat{\boldsymbol{P}} = (P_0, \boldsymbol{P})^{\mathrm{T}}\}$ and the space of moment observables by $\mathfrak{V}_P = C^\infty_{\{\ ,\ \}_P}(\mathfrak{g}_P^*)$.

D. Sub-algebra consisting of linear functions of v_k

From (19), it is evident that $T^*\mathfrak{V}_P$ (or $T^*\mathfrak{V}_F$) consists of only linear functions of v_k. The following lemma guarantees that the moment system $\mathfrak{V}_P$ (or, equivalently, the fluid system $\mathfrak{V}_F$) is a sub-algebra of the Vlasov system $\mathfrak{V}$.

Lemma 1 (sub-algebra). *Let us consider a subset of observables such that*

$$\mathfrak{g}_L = \left\{ \sum_{\nu=0}^{n} \alpha^\nu(\boldsymbol{x}) v_\nu;\ \alpha^\nu(\boldsymbol{x}) \in C^\infty(X) \right\},$$

where $v_1, \ldots, v_n$ are the coordinates of the $\boldsymbol{v}$-space and $v_0 := 1$. This $\mathfrak{g}_L$ is a sub-algebra of $\mathfrak{g}$, i.e.,

$$[\psi, \phi] \in \mathfrak{g}_L \quad (\forall \psi, \phi \in \mathfrak{g}_L).$$

Proof. By direct calculation, we obtain for $\psi = \sum_\nu \alpha^\nu(\boldsymbol{x}) v_\nu$ and $\phi = \sum_\nu \beta^\nu(\boldsymbol{x}) v_\nu$,

$$[\psi, \phi] = \sum_{\nu=0}^{n} \left(\sum_{j=1}^{n} \beta^j \partial_{x^j} \alpha^\nu - \alpha_j \partial_{x^j} \beta^\nu \right) v_\nu.$$

(QED)

Note that $\psi \in \mathfrak{g}_L$ must be a linear function of v_ν, while it may be an arbitrary (smooth) function of $\boldsymbol{x}$. A similar kind of linear reduction was used to describe the Riemann reduction for self-gravitating ellipsoids in Ref. 22 and for two-dimensional vortices in Ref. 23.

IV. GAUGE SYMMETRY OF THE MOMENT (FLUID) REDUCTION

A. Casimirs and gauge symmetry

It is easy to see that the total particle number

$$C_0 = \int_X \rho(\boldsymbol{x})\, \mathrm{d}^3 x = \int_M f(\boldsymbol{x}, \boldsymbol{v})\, \mathrm{d}^3 x \mathrm{d}^3 v \tag{31}$$

is a Casimir of both kinetic and fluid systems (the first expression applies for $\mathfrak{V}_F$ and the second for $\mathfrak{V}$): because $\partial_\rho C_0 = 1$ and $\partial_f C_0 = 1$, evidently, $\{C_0, H\}_F = 0$ and $\{C_0, H\} = 0$ for every $H \in \mathfrak{V}_F$ and $H \in \mathfrak{V}$, respectively.

The *helicity*

$$\begin{aligned} C &= \frac{1}{2} \int \boldsymbol{U} \cdot (\nabla \times \boldsymbol{U})\, \mathrm{d}^3 x \\ &= \frac{1}{2} \int \varepsilon^{jk\ell} \left(\frac{\int v_j f\, \mathrm{d}^3 v}{\int f\, \mathrm{d}^3 v} \right) \partial_{x^k} \left(\frac{\int v_\ell f\, \mathrm{d}^3 v}{\int f\, \mathrm{d}^3 v} \right) \mathrm{d}^3 x \end{aligned} \tag{32}$$

is a Casimir of the fluid system but is not a Casimir of the kinetic system. In fact, $\partial_\rho C = 0$ and

$$\partial_U C = \nabla \times \boldsymbol{U} =: \boldsymbol{\Omega} \tag{33}$$

(we call $\boldsymbol{\Omega}$ the *vorticity*); hence, $\{C, H\}_F = 0$ for every $H \in \mathfrak{V}_F$. On the other hand, by (26), we obtain

$$\partial_f C = \frac{(\boldsymbol{v} - \boldsymbol{U}) \cdot \boldsymbol{\Omega}}{\rho}. \tag{34}$$

Hence, $\{C, H\} \neq 0$ for general $H \in \mathfrak{V}$.

The constancy of C in the fluid system is due to the *gauge symmetry* of the fluid variables, which is implemented through the fluid reduction.

Theorem 1 (gauge transformation generated by the Casimir invariant). *The co-adjoint action $f \mapsto f + \varepsilon[\partial_f C, f]^*$ generated by the Casimir (e.g., the helicity) C leaves the fluid variables unchanged, i.e.,*

$$\int_V v_\nu [\partial_f C, f]^* \mathrm{d}^n v = 0 \quad (\nu = 0, \ldots, n). \tag{35}$$

Proof. As the fluid system $\mathfrak{V}_F$ is a sub-algebra of the Vlasov system $\mathfrak{V}$ (Lemma 1), the Casimir C, being a constant in $\mathfrak{V}_F$, must also be a constant in $\mathfrak{V}$, given that the Hamiltonian is a function of only the fluid variables $(U_0, \ldots, U_n) = (\rho, \boldsymbol{U})$ or, equivalently, the moments $(P_0, \ldots, P_n)$. Therefore,

$$\dot{C} = -\{H, C\} = -\langle \partial_f H, [\partial_f C, f]^* \rangle = -\sum_\nu \langle v_\nu \partial_{P_\nu} H, [\partial_f C, f]^* \rangle$$

must vanish for all $H(P_0, \ldots, P_n)$. Since $\partial_{P_\nu} H$ $(\nu = 0, \ldots, n)$ only depend on $\boldsymbol{x}$, we can write

$$\langle v_\nu \partial_{P_\nu} H, [\partial_f C, f]^* \rangle = \int_X \partial_{P_\nu} H \left(\int_V v_\nu [\partial_f C, f]^* \mathrm{d}^n v \right) \mathrm{d}^n x.$$

Therefore, (35) holds.

(QED)

Note that the Proof of Theorem 1 only invokes the fact that C is a Casimir (invariant independent of the Hamiltonian) of the sub-algebra $\mathfrak{V}_F$; we did not use the explicit form of the helicity C. We can also demonstrate (35) by direct calculation using the relation (34) of the helicity C; let us see how that works out. Denoting the perturbation as $\tilde{f} = [\partial_f C, f]^*$ and putting $\boldsymbol{\omega} = \boldsymbol{\Omega}/\rho$, we observe

$$\begin{aligned}
\tilde{\rho} &= \int_V \tilde{f} \, \mathrm{d}^n v \\
&= \int_V [\partial_{\boldsymbol{x}}(\boldsymbol{\omega} \cdot \boldsymbol{v}) \cdot \partial_{\boldsymbol{v}} f - \partial_{\boldsymbol{x}}(\boldsymbol{\omega} \cdot \boldsymbol{U}) \cdot \partial_{\boldsymbol{v}} f - \boldsymbol{\omega} \cdot \partial_{\boldsymbol{x}} f] \, \mathrm{d}^n v \\
&= \int_V [-(\nabla \cdot \boldsymbol{\omega}) f - \boldsymbol{\omega} \cdot \nabla f] \, \mathrm{d}^n v \\
&= -(\nabla \cdot \boldsymbol{\omega})\rho - \boldsymbol{\omega} \cdot \nabla \rho = -\nabla \cdot (\boldsymbol{\omega}\rho) = -\nabla \cdot \boldsymbol{\Omega} = 0.
\end{aligned}$$

In addition, for $j = 1, 2, 3$,

$$\begin{aligned}
\tilde{P}_j &= \int_V v_j \tilde{f} \, \mathrm{d}^n v \\
&= \int_V v_j [\partial_{\boldsymbol{x}}(\boldsymbol{\omega} \cdot (\boldsymbol{v} - \boldsymbol{U})) \cdot \partial_{\boldsymbol{v}} f - \boldsymbol{\omega} \cdot \partial_{\boldsymbol{x}} f] \, \mathrm{d}^n v \\
&= \int_V - \sum_k [\partial_{v_k}(v_j \partial_{x^k} \boldsymbol{\omega} \cdot (\boldsymbol{v} - \boldsymbol{U})) f - \omega_k \partial_{x^k}(v_j f)] \, \mathrm{d}^n v \\
&= \int_V [-\partial_{x^j}(\boldsymbol{\omega} \cdot (\boldsymbol{v} - \boldsymbol{U})) - v_j \nabla \cdot \boldsymbol{\omega}] \, f \, \mathrm{d}^n v - \boldsymbol{\omega} \cdot \nabla P_j \\
&= -\boldsymbol{P} \cdot \partial_{x^j} \boldsymbol{\omega} + \rho \, \partial_{x^j}(\boldsymbol{U} \cdot \boldsymbol{\omega}) - P_j \nabla \cdot \boldsymbol{\omega} - \boldsymbol{\omega} \cdot \nabla P_j \\
&= \boldsymbol{\Omega} \cdot (\partial_{x^j} \boldsymbol{U} - \nabla U_j) = 0.
\end{aligned}$$

Remark 5 (baroclinic effect). The invariance of the fluid variables $(U_0,\ldots,U_3)$ under the gauge-group action $\mathrm{ad}^*_{\partial_f C} = [\partial_f C, \circ]^*$ is the reflection of the constancy of the helicity C in the barotropic fluid system $\mathfrak{V}_F$ (a sub-algebra of the Vlasov system $\mathfrak{V}$). As shown by the forgoing direct calculations, however, the gauge invariance of $(U_0,\ldots,U_3)$ is independent of the fluid model; even in a baroclinic fluid, in which C is not constant, the action of $\mathrm{ad}^*_{\partial_f C}$ on f does not change $(U_0,\ldots,U_3)$ (whereas it changes the entropy). To see more precisely how the helicity conservation and the gauge symmetry are related, let us look into the baroclinic effect. When the internal energy $\mathscr{E}$ not only depends on ρ but also on the specific entropy σ, the pressure term in the fluid equation (30) modifies as $\rho^{-1}\nabla\mathscr{P} = \nabla h - T\nabla\sigma$ to include the second non-exact term that causes the baroclinic effect [$T = (\partial h/\partial\sigma)_{\mathscr{P}}$ is the temperature]. Then, the helicity obeys

$$\frac{\mathrm{d}}{\mathrm{d}t}C = \int_X T\boldsymbol{\Omega}\cdot\nabla\sigma\,\mathrm{d}^3x.$$

On the other hand, the gauge transformation of $\sigma = -\int_V f\log f\,\mathrm{d}^3v/\rho$ yields (such as the foregoing calculations)

$$\tilde{\sigma} = \frac{-\int_V \tilde{f}(\log f + 1)\,\mathrm{d}^n v}{\rho} - \sigma\frac{\delta\rho}{\rho} = \rho^{-1}\boldsymbol{\Omega}\cdot\nabla\sigma.$$

Therefore, $\boldsymbol{\Omega}\cdot\nabla\sigma = 0$ is a generalization of the barotropic condition that makes the helicity C temporally invariant and, at the same time, the specific entropy σ gauge invariant [in addition to $(U_0,\ldots,U_3)$].

B. Casimir of two-dimensional system

As noted above, Theorem 1 applies to every Casimir of a sub-algebra. In a two-dimensional configuration space ($n = 2$), the fluid reduction works out differently, giving rise to a different Casimir.

Embedding $X\subset\mathbb{R}^2$ into $\mathbb{R}^3$, we define the unit normal vector $\boldsymbol{e}_\perp$ on X. For a two-dimensional co-vector $\boldsymbol{u} = (u_1,u_2)^{\mathrm{T}}$, we write $\boldsymbol{e}_\perp\times\boldsymbol{u} = (-u_2,u_1)^{\mathrm{T}}$. In the differential geometrical notation, $\boldsymbol{e}_\perp$ is the Hodge $*$ operator that maps a 1-form $\boldsymbol{u} = u_1\mathrm{d}x^1 + u_2\mathrm{d}x^2$ to the $(2-1)$-form $*\boldsymbol{u} = u_1\mathrm{d}x^2 - u_2\mathrm{d}x^1$. The *vorticity* is defined by

$$\boldsymbol{\Omega} = \nabla\times\boldsymbol{U} = (\partial_{x^1}U_2 - \partial_{x^2}U_1)\boldsymbol{e}_\perp =: W\boldsymbol{e}_\perp,$$

i.e., $\boldsymbol{\Omega} = (0,0,W)^{\mathrm{T}}$. Identifying $\boldsymbol{e}_\perp = \mathrm{d}x^1\wedge\mathrm{d}x^2$, W is the exact 2-form $W = \mathrm{d}\boldsymbol{U}$. Dividing it by the 2-form ρ, we define a scalar $\psi = W/\rho$.

The reduction to the fluid variables $\widehat{\boldsymbol{U}} = (\rho,U_1,U_2)^{\mathrm{T}}$ yields the fluid Poisson operator

$$J_F = \begin{pmatrix} 0 & -\nabla\cdot \\ -\nabla & -\psi\boldsymbol{e}_\perp\times \end{pmatrix}.$$

In the two-dimensional system, the helicity $\frac{1}{2}\int_X \boldsymbol{U}\cdot\boldsymbol{\Omega}\,\mathrm{d}^3x$ is identically zero, while its role is played by the following *cross-enstrophy*. For an arbitrary smooth scalar function g, we define

$$C(\widehat{\boldsymbol{U}}) = \int_X g(\psi)\rho\,\mathrm{d}^2x.$$

We easily find that $C(\widehat{\boldsymbol{U}})$ is a Casimir, i.e., $J_F\partial_{\widehat{U}}C = 0$. For example, let us take $g(\psi) = \psi^2/2$. Then,

$$\partial_\rho C = -\frac{\psi^2}{2},\quad \partial_U C = -\nabla_\perp\psi,$$

where $\nabla_\perp\psi = \boldsymbol{e}_\perp\times\nabla\psi$, which is identified as the exact $(n-1)$-form $*\mathrm{d}\psi$ (here, $n = 2$). By (26),

$$\partial_f C = \partial_\rho C + \frac{\boldsymbol{v}-\boldsymbol{U}}{\rho}\cdot\partial_U C = -\frac{\psi^2}{2} - \frac{\boldsymbol{v}-\boldsymbol{U}}{\rho}\cdot\nabla_\perp\psi.$$

Using this in Theorem 1, we obtain the following gauge transformation for the two-dimensional fluid variables:

$$\int_V v_\nu [\partial_f C, f]^* \mathrm{d}^3 v = 0 \quad (\nu = 0, \ldots, 2).$$

V. HELICITY FLOW AND ITS GEOMETRICAL MEANING

A. Characterization of $\partial_f C$

In Theorem 1, we have shown that the Casimir (helicity) C generates a Hamiltonian flow inducing the gauge transformation on the distribution function f that preserves the fluid variables $(\rho, \boldsymbol{U})$ (or the moments P_ν). We call the vector $[\circ, \partial_f C]$ the *helicity flow* in the phase space M; its co-adjoint action $[\partial_f C, \circ]^*$ on the distribution function f induces the gauge transformation. Since the fluid variables are integrals (moments) over the $\boldsymbol{v}$-space, it might be expected that the gauge symmetry pertains to some transformation in the $\boldsymbol{v}$-space that does not change the moments. However, it is not so; the following example shows that the helicity gauge is primarily about the $\boldsymbol{x}$-space transformation of f.

Example 1 (linear shear flow). Suppose that $\rho = 1$ and $\boldsymbol{U} = x^1 \boldsymbol{e}^2$ (a linear shear flow). Then, $\boldsymbol{\omega} = \boldsymbol{\Omega} = \boldsymbol{e}^3$, $\partial_f C = v_3$, and hence,

$$\tilde{f} = [\partial_f C, f]^* = -\partial_{x^3} f.$$

Evidently, the perturbation $\tilde{f}$ does not yield variations in the fluid variables ρ and $\boldsymbol{U}$ because they are independent of x^3.

This simple example suggests that $C' := \partial_f C$ is basically a momentum-like variable, which is conjugate to the coordinate parallel to $\boldsymbol{\omega}$. When the vector $\boldsymbol{\omega}$ is not constant, however, C' becomes a *generalized momentum*, mixing coordinates and momenta. It also contributes a spatial term $\boldsymbol{\omega} \cdot \boldsymbol{U}$ in C'; see (34). An interesting analogy of C' and "canonical momentum" of magnetized particle will be shown in Theorem 2. Let us study how such a C' generates a transformation in the phase space $M = X \times V$.

B. Helicity flow in the $\boldsymbol{v}$-space

Here, we study the geometrical meaning of the helicity flow. The adjoint operator $\mathrm{ad}_{C'} = [\circ, C']$, generated by $C' = \partial_f C \in \mathfrak{g}$, reads as the tangent vector (which we call the helicity flow) $\sum_{j=1}^n \tilde{x}^j \partial_{x^j} + \tilde{v}_j \partial_{v_j}$ with components

$$\tilde{\boldsymbol{x}} = \boldsymbol{\omega}, \tag{36}$$

$$\tilde{\boldsymbol{v}} = -\nabla(\boldsymbol{\omega} \cdot (\boldsymbol{v} - \boldsymbol{U})). \tag{37}$$

In order to elucidate the geometrical meaning of the transformation induced by $[C', \circ]$, let us invoke a differential geometrical notation. Note that $\boldsymbol{\omega} \in TX$ (vector in the $\boldsymbol{x}$-space) is defined as $i_{\boldsymbol{\omega}} \rho = \boldsymbol{\Omega}$ for the 3-form ρ and 2-form $\boldsymbol{\Omega}$ [formally, we write $\boldsymbol{\omega} = \boldsymbol{\Omega}/\rho$ to identify the tangent vector $\boldsymbol{\omega}$ as the $(2\text{–}3) = (-1)$-form]. Hence, let us call $\boldsymbol{\omega}$ the *vorticity vector*. In (37), $\boldsymbol{\omega} \cdot (\boldsymbol{v} - \boldsymbol{U})$ is the scalar $i_{\boldsymbol{\omega}}(\boldsymbol{v} - \boldsymbol{U})$, so $\nabla(\boldsymbol{\omega} \cdot (\boldsymbol{v} - \boldsymbol{U}))$ reads $\mathrm{d}(i_{\boldsymbol{\omega}}(\boldsymbol{v} - \boldsymbol{U})) \in T^*X$. By Cartan's formula, we may calculate

$$\mathrm{d}(i_{\boldsymbol{\omega}}(\boldsymbol{v} - \boldsymbol{U})) = \pounds_{\boldsymbol{\omega}}(\boldsymbol{v} - \boldsymbol{U}) - i_{\boldsymbol{\omega}} \mathrm{d}(\boldsymbol{v} - \boldsymbol{U}) = \pounds_{\boldsymbol{\omega}}(\boldsymbol{v} - \boldsymbol{U}) + i_{\boldsymbol{\omega}} \mathrm{d}\boldsymbol{U},$$

where $\pounds_{\boldsymbol{\omega}}$ is the Lie derivative. For the 2-form $\mathrm{d}\boldsymbol{U} = \boldsymbol{\Omega}$, we obtain $i_{\boldsymbol{\omega}} \mathrm{d}\boldsymbol{U} = -\boldsymbol{\omega} \times \boldsymbol{\Omega} = 0$. Therefore, we arrive at an illuminating expression

$$\tilde{\boldsymbol{v}} = -\pounds_{\boldsymbol{\omega}}(\boldsymbol{v} - \boldsymbol{U}). \tag{38}$$

Combined with (36), the adjoint action generated by the helicity is, therefore, primarily the flow $\boldsymbol{\omega}$ in X and its reaction $-\pounds_{\boldsymbol{\omega}}(\boldsymbol{v} - \boldsymbol{U})$ in V. Note that $\boldsymbol{v} - \boldsymbol{U}$ is the distance of $\boldsymbol{v}$ from the *average* $\boldsymbol{U}$.

Remark 6 (gauge transformation for two-dimensional fluid). Consider the two-dimensional case (see Sec. IV B). We define $\boldsymbol{\omega} = \nabla_\perp \psi/\rho$, which is identified as a vector such that $i_{\omega}\rho = \mathrm{d}\psi$, i.e.,

$$\boldsymbol{\omega} = \frac{1}{\rho}[(\partial_{x^2}\psi)\partial_{x^1} - (\partial_{x^1}\psi)\partial_{x^2}].$$

The adjoint action generated by the cross-enstrophy C is $[\partial_f C, \circ] = \sum_j \tilde{x}^j \partial_{x^j} + \tilde{v}_j \partial_{v_j}$ with

$$\tilde{\boldsymbol{x}} = \boldsymbol{\omega}, \tag{39}$$

$$\tilde{\boldsymbol{v}} = -\nabla\left[\boldsymbol{\omega}\cdot(\boldsymbol{v}-\boldsymbol{U}) - \frac{\psi^2}{2}\right] = -\mathrm{d}\left[i_{\omega}(\boldsymbol{v}-\boldsymbol{U}) - \frac{\psi^2}{2}\right]. \tag{40}$$

We find

$$i_{\omega}\mathrm{d}\boldsymbol{U} = \psi\mathrm{d}\psi.$$

Hence, we obtain

$$\tilde{\boldsymbol{v}} = -£_{\omega}(\boldsymbol{v}-\boldsymbol{U}), \tag{41}$$

which parallels (38) of the three-dimensional case.

C. Transformation in $\boldsymbol{v}$-space

To see how $-£_{\omega}(\boldsymbol{v}-\boldsymbol{U})$ works on each fiber $T^*_{\boldsymbol{x}}$, we first consider the case when ρ is constant $(=1)$. Then, $\boldsymbol{\omega} = \boldsymbol{\Omega}$ is simply the vector representation of the 2-form $\boldsymbol{\Omega}$, i.e., $i_{\omega}\mathrm{vol}_x = \boldsymbol{\Omega} = \mathrm{d}\boldsymbol{U}$ ($\mathrm{vol}_x = \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$ is the volume element of X). We calculate

$$\tilde{\boldsymbol{v}} = -(\nabla\boldsymbol{\Omega})\cdot(\boldsymbol{v}-\boldsymbol{U}) + (\boldsymbol{\Omega}\cdot\nabla)\boldsymbol{U}. \tag{42}$$

Since $\nabla\cdot\boldsymbol{\Omega} = 0$, we have $\mathrm{Tr}(\nabla\boldsymbol{\Omega}) = 0$; hence, $\nabla\boldsymbol{\Omega} \in \mathfrak{sl}(3,\mathbb{R})$. Therefore, the first term on the right-hand side of (42) represents a $\boldsymbol{v}$-space volume preserving map (epitomized by *rotation*) around the *center* $\boldsymbol{U}$. The second term $(\boldsymbol{\Omega}\cdot\nabla)\boldsymbol{U}$ is the displacement of the center $\boldsymbol{U}$ induced by the motion $\boldsymbol{\omega} = \boldsymbol{\Omega}$ in the $\boldsymbol{x}$-space.

Inhomogeneous ρ modifies (42) as

$$\tilde{\boldsymbol{v}} = -(\nabla\boldsymbol{\omega})\cdot(\boldsymbol{v}-\boldsymbol{U}) + (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{U} \tag{43}$$

with $\boldsymbol{\omega} = \boldsymbol{\Omega}/\rho$. The role of the second term is the same as the case of $\boldsymbol{\omega} = \boldsymbol{\Omega}$. However, the first term is no longer an $\mathfrak{sl}(3,\mathbb{R})$ action because $\mathrm{Tr}(\nabla\boldsymbol{\omega}) = \nabla\cdot\boldsymbol{\omega} = \boldsymbol{\Omega}\cdot\nabla\rho^{-1}$. We may decompose it as

$$\begin{aligned}-(\nabla\boldsymbol{\omega})\cdot(\boldsymbol{v}-\boldsymbol{U}) &= -\left[\frac{1}{\rho}(\nabla\boldsymbol{\Omega}) + \nabla\left(\frac{1}{\rho}\right)\otimes\boldsymbol{\Omega}\right]\cdot(\boldsymbol{v}-\boldsymbol{U}) \\ &= -\frac{1}{\rho}(\nabla\boldsymbol{\Omega})\cdot(\boldsymbol{v}-\boldsymbol{U}) - \boldsymbol{\Omega}\cdot(\boldsymbol{v}-\boldsymbol{U})\nabla\left(\frac{1}{\rho}\right),\end{aligned}$$

in which the first term is an $\mathfrak{sl}(3,\mathbb{R})$ action. The second term adjusts the variation of the density ρ induced by the $\boldsymbol{x}$-space motion $\boldsymbol{\omega}$; the $\boldsymbol{x}$-space divergence $\sum\partial_{x^j}\tilde{x}^i = \boldsymbol{\Omega}\cdot\nabla\rho^{-1}$ and the $\boldsymbol{v}$-space divergence $\sum\partial_{v_j}\tilde{v}_j = -\boldsymbol{\Omega}\cdot\nabla\rho^{-1}$ cancel each other.

D. Proper volume of $\boldsymbol{v}$-space

These observations guide us to the idea of a *proper metric* (or volume) of the fluid system. Let us return to the basic relation $i_{\omega}\rho = \boldsymbol{\Omega}$. We may assume that X is not Euclidean, but the metric is deformed by ρ so that

$$\mathrm{vol}_\rho = \rho\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$$

is the volume form (ρ may be viewed analogous to the $\sqrt{g}$ of a Riemannian metric). Then, we may evaluate the proper-volume divergence as

$$\operatorname{div}\boldsymbol{\omega} = (\mathrm{d}\, i_{\boldsymbol{\omega}}\mathrm{vol}_\rho)^* = \rho^{-1}\sum_j \partial_{x^j}(\rho\omega_j) = \rho^{-1}\sum_j \partial_{x^j}\Omega_j = 0,$$

implying that the first term $(\nabla\boldsymbol{\omega})\cdot(\boldsymbol{v}-\boldsymbol{U})$ of (43) is a "vol$_\rho$ preserving" map in V. Hence, the helicity generates a symplectic (thus M space volume preserving) and, at the same time, vol$_\rho$ preserving group.

VI. FOLIATION OF THE KINETIC PHASE SPACE BY THE HELICITY FLOW

A. Helicity symmetry in the phase space M

With $C' = \partial_f C$, the co-adjoint action

$$\mathrm{ad}^*_{C'} = [C', \circ]^* = -\boldsymbol{\omega}\cdot\partial_{\boldsymbol{x}} + \pounds_{\boldsymbol{\omega}}(\boldsymbol{v}-\boldsymbol{U})\cdot\partial_{\boldsymbol{v}}$$

generates the gauge group that keeps the fluid variables unchanged (Theorem 1). Conversely, if f satisfies

$$\mathrm{ad}^*_{C'} f = [C', f]^* = 0, \tag{44}$$

every Hamiltonian $H(f) \in \mathfrak{V}$ does not change the helicity C,

$$\dot{C} = \{C, H\} = -\{H, C\} = -\langle H', [C', f]^*\rangle = 0.$$

We say that f has the *helicity symmetry* if (44) holds. Then, even if the Hamiltonian H includes non-fluid variables [for instance, a higher-order moment such as $\int_V g(\boldsymbol{v}) f\,\mathrm{d}^n v$ with an arbitrary polynomial $g(\boldsymbol{v})$], the system behaves "fluid-like"—it being constrained to lie on the leaf of C (as well as on that of C_0), provided that f has the helicity symmetry $[C', f]^* = 0$. To put it another way, the symmetry breaking $[C', f]^* \neq 0$ is the necessary condition for the "kinetic effect" to manifest as the creation/annihilation of the helicity. We also note that helicity symmetry $[C', f]^* = 0$ is not a necessary condition for the helicity C to be conserved; if H only includes fluid variables P_ν, we obtain, by (35),

$$\dot{C} = -\{H, C\} = \sum_\nu \int_X \partial_{P_\nu} H\left(\int_V v_\nu [C', f]^* \mathrm{d}^n v\right)\mathrm{d}^n x = 0.$$

From the practice, following Theorem 1, it is evident that higher-moment variables, such as $\int_V g(\boldsymbol{v}) f\,\mathrm{d}^n v$, are not invariant under the helicity gauge transformation; hence, a non-fluid Hamiltonian, including higher moments, violates the helicity conservation if the helicity symmetry is broken. Hence, the helicity conservation can be caused by either the helicity symmetry or the fluid reduction. Seeing the helicity conservation as the litmus test, the fluid reduction (neglect of higher moments in the Hamiltonian) can be consistent with the kinetic model if the helicity symmetry holds for the distribution function.

The aim of this section is to characterize the helicity symmetry in terms of a set of canonical coordinates for the phase space M. For a limited class of $\boldsymbol{\omega}$, we can construct canonical variables $(\alpha, \beta, \gamma, \wp_\alpha, \wp_\beta, \wp_\gamma)$ such that $\wp_\gamma = C'$. Then, the helicity symmetry means $\mathrm{ad}^*_{C'} f = \partial_\gamma f = 0$. We call such a parameterization of M the *helicity foliation* (note the difference from the C = constant leaf in the function space $\mathfrak{g}^*$; cf. Remark 7).

B. Epi-2D flow

Suppose that the fluid velocity $\boldsymbol{U}$ (a 1-form in the three-dimensional configuration space) can be parameterized as

$$\boldsymbol{U} = \nabla\varphi + \alpha\nabla\beta. \tag{45}$$

Evidently, such velocity fields constitute a special class of flows, which we have called *epi-2D*[13] (see Remark 8). For these flows, the helicity is $C = \frac{1}{2}\int \nabla\varphi\cdot\nabla\alpha\times\nabla\beta\, d^3x$, which yields

$$C' = (\boldsymbol{v} - \boldsymbol{U}) \cdot \boldsymbol{\omega} = \frac{\boldsymbol{v} \cdot \nabla\alpha \times \nabla\beta}{\rho} - \frac{\nabla\varphi \cdot \nabla\alpha \times \nabla\beta}{\rho}$$
$$= \frac{\boldsymbol{v} \wedge \mathrm{d}\alpha \wedge \mathrm{d}\beta}{\rho} - \frac{\mathrm{d}\varphi \wedge \mathrm{d}\alpha \wedge \mathrm{d}\beta}{\rho}.$$

Let us denote an element of the Jacobian matrix by $\partial f^i/\partial x^j$, where $i, j = 1, 2, 3$, and the Jacobian determinant by $\partial(f^1, \ldots, f^n)/\partial(x^1, \ldots, x^n)$, $n \le 3$. For an epi-2D flow, we have the following.

Theorem 2 (parameterization by epi-2D fluid variables). *Suppose that, in an open set $W \subset X$,*

$$\frac{\partial(\alpha,\ \beta)}{\partial(x^j, x^k)} \neq 0 \quad (\exists j, k).$$

In a neighborhood $X_{\boldsymbol{x}}$ of $\boldsymbol{x} \in W$, there is a scalar γ such that

$$\mathrm{d}\alpha \wedge \mathrm{d}\beta \wedge \mathrm{d}\gamma = \frac{\partial(\alpha,\ \beta,\ \gamma)}{\partial(x^1, x^2, x^3)}\, \mathrm{vol}_x^3 = \rho, \tag{46}$$

by which we define three independent vectors ($\in TX$),

$$\boldsymbol{\omega}_\alpha = \frac{\mathrm{d}\beta \wedge \mathrm{d}\gamma}{\rho}, \quad \boldsymbol{\omega}_\beta = \frac{\mathrm{d}\gamma \wedge \mathrm{d}\alpha}{\rho}, \quad \boldsymbol{\omega}_\gamma = \boldsymbol{\omega} = \frac{\mathrm{d}\alpha \wedge \mathrm{d}\beta}{\rho}.$$

The variables α, β, γ, together with

$$\wp_\alpha = i_{\boldsymbol{\omega}_\alpha}\boldsymbol{v} - \partial_\alpha\varphi, \quad \wp_\beta = i_{\boldsymbol{\omega}_\beta}\boldsymbol{v} - \partial_\beta\varphi, \quad \wp_\gamma = i_{\boldsymbol{\omega}_\gamma}\boldsymbol{v} - \partial_\gamma\varphi, \tag{47}$$

constitute canonical coordinates in $W_{\boldsymbol{x}} \times V$. Among them, $\wp_\gamma = \partial_f C$; hence, the helicity symmetry is $\partial_\gamma = 0$.

Proof. The third coordinate γ can be constructed by solving (46) as a hyperbolic PDE. For instance, assume that $D_1 := \partial(\alpha, \beta)/\partial(x^2, x^3) \neq 0$ in an open set $W_{\boldsymbol{x}}$. Then, (46) can be cast into a first order PDE,

$$\partial_{x^1}\gamma + c_2\partial_{x^2}\gamma + c_3\partial_{x^3}\gamma = c_4, \tag{48}$$

where

$$c_2 = \frac{1}{D_1}\frac{\partial(\alpha,\ \beta)}{\partial(x^3, x^1)}, \quad c_3 = \frac{1}{D_1}\frac{\partial(\alpha,\ \beta)}{\partial(x^1, x^2)}, \quad c_4 = \frac{1}{D_1}\rho.$$

We can solve (48) for γ by the method of characteristics (see examples in Sec. VI C).

Let us evaluate the kinetic bracket $[\ ,\ ]$ explicitly. We may write

$$[\wp_\gamma, \circ\,] = \sum_j \left(\partial_{x^j}\left(\sum_k \omega_\gamma^k v_k - \partial_\gamma\varphi \right)\partial_{v_j} - \omega_\gamma^j \partial_{x^j} \right).$$

Evidently, by (46), we have

$$[\wp_\gamma, \gamma] = -\frac{\mathrm{d}\alpha \wedge \mathrm{d}\beta \wedge \mathrm{d}\gamma}{\rho} = -1$$

and

$$[\wp_\gamma, \alpha] = -\frac{\mathrm{d}\alpha \wedge \mathrm{d}\beta \wedge \mathrm{d}\alpha}{\rho} = 0, \quad [\wp_\gamma, \beta] = -\frac{\mathrm{d}\alpha \wedge \mathrm{d}\beta \wedge \mathrm{d}\beta}{\rho} = 0.$$

For the momentum-like variables, we observe

$$[\wp_\gamma, \wp_\alpha] = \sum_j v_j\Big((\boldsymbol{\omega}_\alpha \cdot \nabla)\omega_\gamma^j - (\boldsymbol{\omega}_\gamma \cdot \nabla)\omega_\alpha^j\Big) \\ -\boldsymbol{\omega}_\alpha \cdot \nabla(\partial_\gamma \varphi) + \boldsymbol{\omega}_\gamma \cdot \nabla(\partial_\alpha \varphi). \tag{49}$$

Using a vector calculus formula, let us calculate the Lie derivative $\pounds_{\boldsymbol{\omega}_\alpha}\boldsymbol{\omega}_\gamma$,

$$\begin{aligned}&(\boldsymbol{\omega}_\alpha \cdot \nabla)\boldsymbol{\omega}_\gamma - (\boldsymbol{\omega}_\gamma \cdot \nabla)\boldsymbol{\omega}_\alpha \\ &= \nabla \times (\boldsymbol{\omega}_\gamma \times \boldsymbol{\omega}_\alpha) + (\nabla \cdot \boldsymbol{\omega}_\gamma)\boldsymbol{\omega}_\alpha - (\nabla \cdot \boldsymbol{\omega}_\alpha)\boldsymbol{\omega}_\gamma \\ &= \nabla \times \left(\frac{1}{\rho}\nabla\beta\right) + \nabla\left(\frac{1}{\rho}\right) \times \left(\frac{(\nabla\beta \times \nabla\gamma) \times (\nabla\alpha \times \nabla\beta)}{\rho}\right) \\ &= \nabla\left(\frac{1}{\rho}\right) \times \nabla\beta - \nabla\left(\frac{1}{\rho}\right) \times \nabla\beta = 0.\end{aligned}$$

Therefore, in (49), $(\boldsymbol{\omega}_\alpha \cdot \nabla)\omega_\gamma^j - (\boldsymbol{\omega}_\gamma \cdot \nabla)\omega_\alpha^j = 0$ for every j. On the other hand, we observe

$$\boldsymbol{\omega}_\gamma \cdot \nabla\varphi = \frac{\nabla\alpha \times \nabla\beta}{\rho} \cdot (\partial_\alpha\varphi\nabla\alpha + \partial_\beta\varphi\nabla\beta + \partial_\gamma\varphi\nabla\gamma) = \partial_\gamma\varphi$$

and, similarly, $\boldsymbol{\omega}_\alpha \cdot \nabla\varphi = \partial_\alpha\varphi$. Therefore, the last two terms in (49) evaluate as $\boldsymbol{\omega}_\alpha \cdot \nabla(\partial_\gamma\varphi) + \boldsymbol{\omega}_\gamma \cdot \nabla(\partial_\alpha\varphi) = \partial_\alpha\partial_\gamma\varphi - \partial_\gamma\partial_\alpha\varphi = 0$. In summary, we find $[\wp_\gamma, \wp_\alpha] = 0$. The permutation $\alpha \to \beta \to \gamma \to \alpha$ yields all other canonical bracket relations.

Finally, note $\boldsymbol{\omega}_\gamma = \boldsymbol{\omega} = \nabla \times \boldsymbol{U}/\rho$ and

$$i_{\boldsymbol{\omega}_\gamma}\boldsymbol{U} = \frac{\mathrm{d}\alpha \wedge \mathrm{d}\beta \wedge \mathrm{d}\varphi}{\rho} = \frac{\partial_\gamma\varphi\,\mathrm{d}\alpha \wedge \mathrm{d}\beta \wedge \mathrm{d}\gamma}{\rho} = \partial_\gamma\varphi.$$

Hence, $\wp_\gamma = i_{\boldsymbol{\omega}}(\boldsymbol{v} - \boldsymbol{U}) = \partial_f C$.

(QED)

Remark 7 (fields vs coordinates). Some confusion may arise because the variables $(\alpha, \beta, \gamma, \wp_\alpha, \wp_\beta, \wp_\gamma)$ are at once fields (dependent dynamical variables) and coordinates on M. They are fields as is $\boldsymbol{U}$ in (45), but in Theorem 2, they are used as canonical coordinates, which is possible for any fixed value of the time variable.

Remembering the examples of reductions given in Sec. II, we see that the *helicity symmetry* $\partial_\gamma = 0$ yields the Casimir $\wp_\gamma = C'$ of the reduced (γ-suppressed) system ($\subset \mathfrak{g} = C^\infty(M)$). The helicity $C \in \mathfrak{V} = C^\infty(\mathfrak{g}^*)$ is the integral of $C' \in \mathfrak{g} = C^\infty(M)$ with respect to the distribution f, which inherits its invariance from the helicity symmetry in the phase space M. We note that C is a Casimir of the fluid subalgebra $\mathfrak{V}_F$ (Theorem 1), whose invariance is due to the wider reduction into the fluid variables, so the helicity symmetry in M is not a necessary condition for the constancy of C in the fluid system $\mathfrak{V}_F$. On the contrary, the helicity symmetry guarantees the constancy of C even in the general (non-reduced) Vlasov system $\mathfrak{V}$.

Remark 8 (Clebsch parameterization and topological charge). Representing a 1-form $\boldsymbol{U}$ as in (45) is called the *Clebsch parameterization*.

1. If $\boldsymbol{U}$ is written in the form of (45), the velocity $\boldsymbol{\omega} = (\nabla\alpha \times \nabla\beta)/\rho$ is *integrable* in the sense that two scalars α and β are the integrals of $\boldsymbol{\omega}$,

$$\boldsymbol{\omega} \cdot \nabla\alpha = 0, \quad \boldsymbol{\omega} \cdot \nabla\beta = 0.$$

   To represent a general 3-vector, however, we need another pair of parameters α' and β' to write[24]

$$\boldsymbol{U} = \nabla\varphi + \alpha\nabla\beta + \alpha'\nabla\beta'. \tag{50}$$

   Then, $\boldsymbol{\omega}$ is not necessarily integrable; the immersion of the orbits of $\mathrm{ad}_{C'}$ may not yield an embedded submanifold in X.
2. The Clebsch parameters α, β, φ (0-forms) and the density ρ (3-form) are dynamical. In the fluid system $\mathfrak{V}_F$, α, β, and ρ are Lie-dragged by the fluid velocity $\boldsymbol{U}^\dagger \in TX$ (the vector counterpart of $\boldsymbol{U} \in T^*X$), i.e.,

$$(\partial_t + \pounds_{\boldsymbol{U}^\dagger})\alpha = 0, \quad (\partial_t + \pounds_{\boldsymbol{U}^\dagger})\beta = 0, \quad (\partial_t + \pounds_{\boldsymbol{U}^\dagger})\rho = 0.$$

Therefore, the coordinate γ is also Lie-dragged, continuously representing the helicity symmetry. Only φ is modified by $(\partial_t + \pounds_{U^\dagger})\varphi = \frac{1}{2}U^2 - h - \phi$, where h is the specific enthalpy and ϕ is the potential energy.[13] In the general dynamics that is generated by $H(f) \in \mathfrak{V}$, however, the Clebsch parameters are no longer dictated only by the fluid variables.

3. With an arbitrary Lie-dragged scalar s (γ is a possible choice) and a fluid element $\Omega \subset X$ that moves with the velocity $\boldsymbol{U}^\dagger$, we can define a *charge*

$$Q = \int_\Omega \mathrm{d}\alpha \wedge \mathrm{d}\beta \wedge \mathrm{d}s,$$

which is a constant of motion.[13] This Q corresponds to the cross-enstrophy. While the invariance of the helicity C yields only one codimension for the possible dynamics in the function space $\mathfrak{g}^*$, the invariance of each charge evaluated for arbitrary Ω poses an infinite number of constraints.

4. While the invariants C and Q's belong to $\mathfrak{V}_F$, there are infinitely many codimensions that are separated from $\mathfrak{V}$ in the reduction to the subalgebra $\mathfrak{V}_F$; see Remark 2 for analogous examples of such variables in a finite-dimensional system.

C. Examples

Consider now some examples for which we can explicitly display the "symmetry coordinate" γ. As usual, we denote the three-dimensional Cartesian coordinates by x, y, and z. The essential part of construction is finding the γ that represents the helicity symmetry; φ appears only in the momentum-like variables, so it can be chosen arbitrarily. We assume $\rho = 1$ so that

$$\boldsymbol{\omega} = \nabla\alpha \times \nabla\beta = (0,\ \partial_z\alpha,\ -\partial_y\alpha)^{\mathrm{T}}$$

and find that α is the *Gauss potential* of the two-dimensional vector $(\omega_y, \omega_z)^{\mathrm{T}}$ on the surface β = constant.

Example 2 (elliptic vortex). A simple example is the ellipse: For positive a and b,

$$\alpha = a\frac{y^2}{2} + b\frac{z^2}{2},\ \beta = x \ \Rightarrow\ \boldsymbol{\omega} = (0,\ bz,\ -ay)^{\mathrm{T}}.$$

Solving $\boldsymbol{\omega} \cdot \nabla\gamma = 1$, we obtain

$$\gamma = \frac{1}{\sqrt{ab}}\tan^{-1}\left(\sqrt{\frac{a}{b}}\frac{y}{z}\right).$$

Figure 1 shows (a) the contours of α and the vector $\boldsymbol{\omega}$ and (b) the coordinates α (blue dotted lines) and $\gamma(y,z)$ (black straight lines). Only when $a = b$ (i.e., the circular vortex), α and γ are orthogonal to each other. The other coordinate $\beta = x$ is orthogonal to both α and γ.

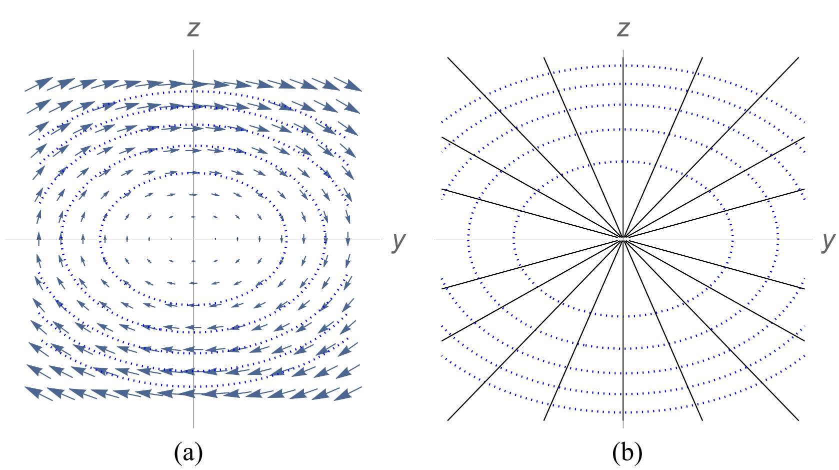

FIG. 1. (a) Elliptic vortex $\boldsymbol{\omega}$ and the corresponding Gauss potential α (dotted lines show the level-sets). (b) The relation between the coordinates α (blue dotted lines) and γ (black solid lines).

Example 3 (hyperbolic vortex). As the second example, let us consider the following hyperbola [see Fig. 2(a)]:

$$\alpha = -\frac{y^2}{2} + \frac{z^2}{2},\ \beta = x \ \Rightarrow\ \boldsymbol{\omega} = (0,\, z,\, y)^{\mathrm{T}}.$$

The determining equation for γ reads (taking z as the time-like variable)

$$\partial_z \gamma + \frac{z}{y}\,\partial_y \gamma = \frac{1}{y}. \tag{51}$$

Upon solving the characteristic equation

$$\frac{\mathrm{d}y}{\mathrm{d}z} = \frac{z}{y}, \quad y(0) = y_0,$$

we obtain $y(z) = \pm\sqrt{z^2 + y_0^2}$ or $y_0 = \pm\sqrt{y^2 - z^2}$. For an intermediate "time" ζ $(z \geq \zeta \geq 0)$, we have

$$y(\zeta) = \pm\sqrt{y^2 - z^2 + \zeta^2},$$

by which we can integrate (51) as

$$\gamma = \int_0^z \frac{\mathrm{d}\zeta}{y(\zeta)}.$$

Because the singularities of the integrand separate different branches of the solution, we first invoke the following indefinite integral:

$$\begin{aligned} g(y, z; \zeta) &= \int \frac{\mathrm{d}\zeta}{y(\zeta)} \\ &= \frac{1}{2}\left[\log\left(1 + \frac{\zeta}{y(\zeta)}\right) - \log\left(1 - \frac{\zeta}{y(\zeta)}\right)\right]. \end{aligned} \tag{52}$$

Evaluating $g(y, z;\ \zeta)$ at $\zeta = z$ and setting the "initial time" at $\zeta = 0$, we obtain

$$\gamma = \gamma_1 = g(y, z; \zeta)\big|_0^z = \pm\frac{1}{2}\left[\log\left(1 + \frac{z}{\sqrt{y^2}}\right) - \log\left(1 - \frac{z}{\sqrt{y^2}}\right)\right].$$

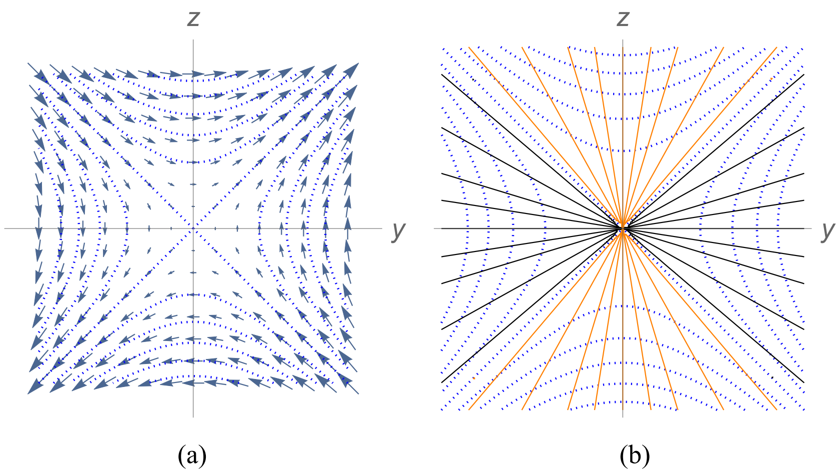

FIG. 2. (a) Hyperbolic vortex $\boldsymbol{\omega}$ and the corresponding Gauss potential α (dotted lines show the level-sets). (b) The relation between the coordinates α (blue dotted lines) and γ (γ_1: black solid lines. γ_2: orange solid lines).

The characteristic curves that start from $z = 0$ do not reach the domain $z^2 > y^2$; see Fig. 2(a). To construct a solution there, we reverse the roles of z and y and define characteristics for $z|_{y=0} = z_0$. By the same procedure, we obtain

$$\gamma = \gamma_2 = \pm\frac{1}{2}\left[\log\left(1+\frac{y}{\sqrt{z^2}}\right)-\log\left(1-\frac{y}{\sqrt{z^2}}\right)\right].$$

These two functions γ_1 and γ_2 define separate local coordinates in the $\boldsymbol{x}$-space. Figure 2 shows (a) the contours of α and the vector $\boldsymbol{\omega}$ and (b) the coordinates α (blue dotted lines) and $\gamma(y,z)$ (black and orange lines).

Here, we note that the solution γ of the determining equation $\nabla\alpha\times\nabla\beta\cdot\nabla\gamma = 1$ is not unique. Evidently, the transformation $\gamma\mapsto\gamma+f(\alpha)$ (f being an arbitrary C^1 function) produces an infinite set of solutions. Different choices of such transformations amount to changing the lower bound of the integral of (52) because $g(y,z;\ c)$ (c being an arbitrary constant) satisfies $\nabla\alpha\times\nabla\beta\cdot\nabla g(y,z;\ c) = 0$, i.e., $g(y,z;\ c) = f(\alpha)$. With the transformation, the boundaries of the coordinate patches move (see Fig. 3).

Example 4 (half an ABC vortex). A more complicated example is provided by considering "half" of the ABC flow (the example of Ref. 13; see Fig. 4): with three real constants a, b, and c, put $\varphi = az\sin x$ and

$$\alpha = b\,\sin\,y - c\,\cos\,z - az\,\cos\,x,\quad \beta = x \;\Rightarrow\; \boldsymbol{\omega} = \begin{pmatrix} 0 \\ c\,\sin\,z - a\,\cos\,x \\ -b\,\cos\,y \end{pmatrix}.$$

The determining equation for γ is, putting $A = a\cos x$,

$$\partial_z\gamma + \frac{A - c\,\sin\,z}{b\,\cos\,y}\partial_y\gamma = -\frac{1}{b\,\cos\,y}. \tag{53}$$

Upon solving the characteristic equation

$$\frac{\mathrm{d}y}{\mathrm{d}z} = \frac{A - c\,\sin\,z}{b\,\cos\,y},\quad y(0) = y_0,$$

we obtain, denoting $c' = c/b$ and $A' = A/b$,

$$y_0 = \sin^{-1}[\sin\,y + c'(1-\cos\,z) - A'z].$$

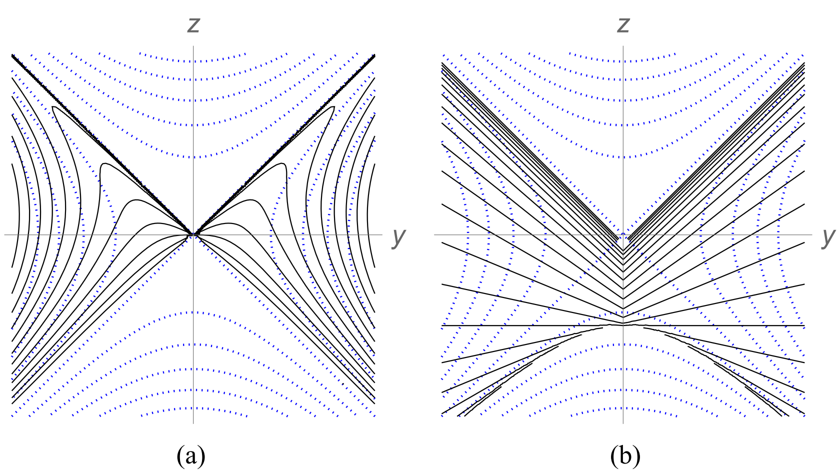

FIG. 3. (a) Transformed coordinate $g' = g + \alpha$. (b) Transformation obtained by shifting the lower bound of the integral of (52) to $z = -\pi/2$.

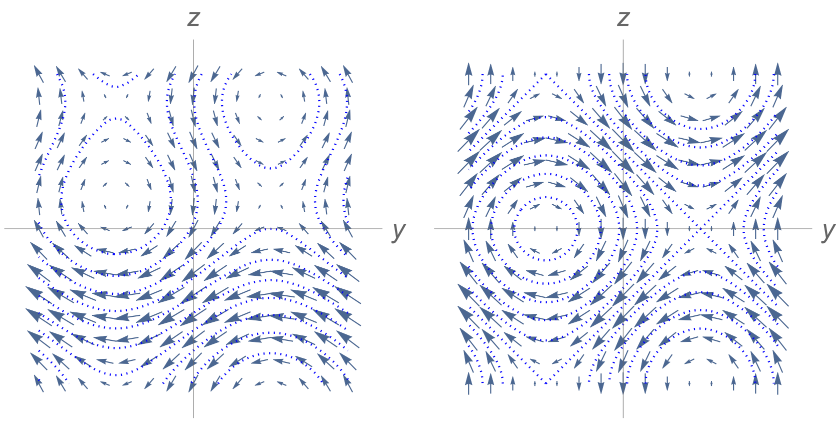

FIG. 4. "Half ABC" vortex $\boldsymbol{\omega}$ and the corresponding Gauss potential α (dotted lines show the level-sets) on (a) $\beta = x = 1$ surface and (b) $x = \pi/2$.

For an intermediate "time" ζ $(z > \zeta > 0)$, we have

$$y(\zeta) = \sin^{-1}[\sin y + c'(\cos\zeta - \cos z) + A'(\zeta - z)],$$

by which we integrate (53) to obtain

$$\begin{aligned} \gamma &= -\int_0^z \frac{\mathrm{d}\zeta}{b\cos y(\zeta)} \\ &= -\int_0^z \frac{\mathrm{d}\zeta}{\sqrt{b^2 - [b\sin y + c(\cos\zeta - \cos z) + a(\zeta - z)\cos x]^2}}. \end{aligned} \tag{54}$$

Although we cannot evaluate the integral of (54) in terms of elementary functions, it does represent the coordinate γ of the helicity symmetry.

For the characteristics curves that do not have an appreciable angle with respect to the y axis, we have to choose y as the independent (time-like) variable to rewrite the determining equation (53) as

$$\partial_y \gamma + \frac{b\cos y}{A - c\sin z}\,\partial_z \gamma = -\frac{1}{A - c\sin z}. \tag{55}$$

For $\eta(z) = Az + c\cos z$, we define its inverse function $z(\eta)$ [i.e., $z(\eta(z)) = z$]. Since $\eta(z)$ is not a monotonic function, $z(\eta)$ needs branch cuts. Solving the characteristic equation, we obtain

$$z(\eta) = z(b\sin\eta - b\sin y + Az + \cos z),$$

by which we can integrate (53) as

$$\gamma = -\int_0^y \frac{\mathrm{d}\eta}{A - c\sin z(\eta)}.$$

Figure 5 shows the coordinates α (blue dotted lines) and $\gamma(y, z)$ (solid lines; different colors indicate different coordinate patches).

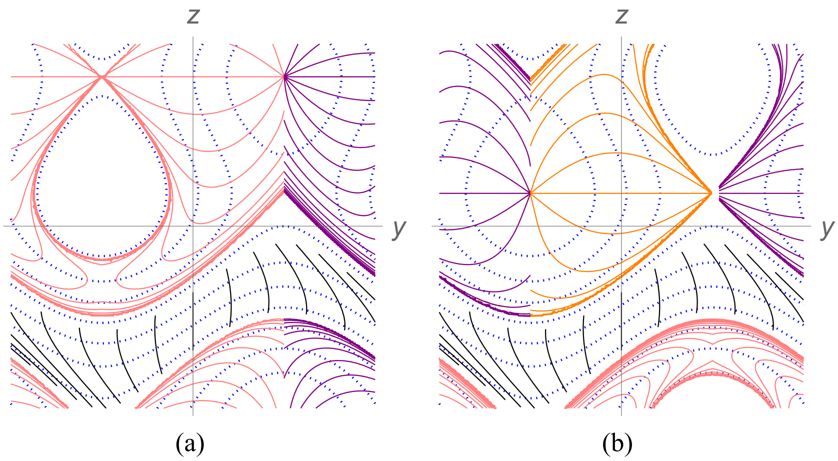

FIG. 5. The relation between the coordinates α (blue dotted lines) and the helicity symmetry coordinate γ (different colors indicate different coordinate patches) on the surface $\beta = x = 1$. [(a) and (b)] Differently patched coordinates.

VII. CONCLUSION

The helicity C is an invariant of the Hamiltonian system governing the fluid variables $(\rho, \boldsymbol{U})^{\mathrm{T}}$ (or, equivalently, the moments P_v). However, it is not an *a priori* invariant in the Vlasov system that dictates the dynamics of the distribution function f. It is the "reduction" $f \mapsto P_v$ that makes C the Casimir of the fluid system. Viewed from the Vlasov system, the helicity C represents the gauge symmetry of the fluid variables, i.e., by $C' = \partial_f C$, the co-adjoint action $\mathrm{ad}^*_{C'} = [C', \circ]^*$ on f generates the gauge group that keeps the fluid variables unchanged (Theorem 1).

The topological constraint on vortex lines, which is imposed by the helicity in the fluid system, can be extrapolated to the kinetic Vlasov system as the helicity symmetry. If f has the helicity symmetry $\mathrm{ad}^*_{C'} f = [C', f]^* = 0$, any Hamiltonian $H(f) \in \mathfrak{V}$ cannot change the helicity C, i.e., $\dot{C} = \{C, H\} = -\langle H', [C', f]^* \rangle = 0$. To put it another way, the non-symmetry $\mathrm{ad}^*_{C'} f \neq 0$ is the measure of the "kinetic effect" that can bring about a change in C, unfreezing the topological constraints on vortex lines. As delineated by Theorem 2, the helicity symmetry is primarily the homogeneity of f in the direction of the vorticity vector $\boldsymbol{\omega}$ in $\boldsymbol{x}$-space. If $\boldsymbol{\omega}$ is not integrable (the case for general 3D flow), the symplectic foliation by the helicity (immersion of the orbit of $\mathrm{ad}_{C'}$) may distribute densely in an open set $M_C \subseteq M$ (such as the Kronecker foliation). Then, any inhomogeneity of f in M_C may yield $\mathrm{ad}^*_{C'} f \neq 0$, violating the helicity symmetry. Even if $\boldsymbol{\omega}$ is integrable (epi-2D flow), C' is dynamical (see Remark 7), so it is difficult to maintain $\mathrm{ad}^*_{C'} f = 0$ in dynamics generated by an arbitrary Hamiltonian H, i.e., $[H', C']^* \neq 0$ for $H' = \partial_f H$. We note, however, that the helicity remains constant even when $\mathrm{ad}^*_{C'} f \neq 0$ if the Hamiltonian includes only the fluid variables P_v because $\int_V v_v \mathrm{ad}^*_{C'} f \, \mathrm{d}^3 v = 0$ for every f (Theorem 1).

Finally, we note the remarkable analogy between the Casimir C and the magnetic moment μ of a magnetized particle (see Sec. II A). The adiabatic invariance of the action μ is due to the separation of the microscopic gyration angle θ from the Hamiltonian. In the macroscopic model, the homogeneity of the distribution function with respect to θ justifies the separation of the action μ and angle θ variable, resulting in the macroscopic reduced system. Here, the homogenization of f with respect to the co-adjoint orbit of $\mathrm{ad}^*_{C'} = [C', \circ]^*$ yields C as an adiabatic invariant; the orbit is in the direction of the vorticity vector $\boldsymbol{\omega}$ in X, accompanied by $-\pounds_{\boldsymbol{\omega}}(\boldsymbol{v} - \boldsymbol{U})$ in V; see (36) and (37). For an epi-2D flow, we can write $\mathrm{ad}_{C'} = \partial_\gamma$ with conjugate variables $C' = \wp_\gamma$ and γ (Theorem 2); in the analogy of the magnetic moment, γ parallels the gyro-angle. As given in (47), the canonical momenta $\wp_j$ $(j = \alpha, \beta, \gamma)$ involve the spatial terms $-\partial_j \varphi = -i_{\omega_j} \boldsymbol{U}$, where $\boldsymbol{U}$ resembles the vector potential of the electromagnetic field.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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