CLEBSCH ANTI-REDUCTION OF LIE–POISSON SYSTEMS

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In honor of Anthony Bloch’s 65th birthday

Abstract. We propose a systematic procedure called the Clebsch anti-reduction for obtaining a canonical Hamiltonian system that is related to a given Lie–Poisson equation via a Poisson map. We describe both coordinate and geometric versions of the procedure, the latter apparently for the first time. The anti-reduction procedure gives rise to a collection of basic Lie algebra questions and leads to classes of invariants of the obtained canonical Hamiltonian system. An important product of anti-reduction is a means for numerically integrating Lie–Poisson systems while preserving their invariants, by utilizing symplectic integrators on the anti-reduced system. We give several numerical examples, including the Kida vortex, a model system for the rattleback toy, and the heavy top on a movable base with controls.

1. Introduction

1.1. The Lie–Poisson Dynamics. The formalization of mechanics by Lagrange and Hamilton evolved in the 19th century into the description of dynamical systems where the equations of motion are generated by canonical Poisson brackets, written in terms of canonical coordinates, position and momenta, with a Hamiltonian function. More modern differential geometric descriptions of Hamiltonian systems occurred well into the 20th century by, e.g., Mackey [30] and Jost [23], motivating present day symplectic geometry.

Less well-known is Poisson geometry, the origins of with date to Lie [28] in 1890 which, with contributions from Souriau [62] and others, but significantly from the seminal paper of Weinstein [68], was brought into modern geometric form. Like the canonical Poisson brackets of symplectic geometry, noncanonical Poisson brackets of Poisson geometry are closed binary operations on smooth phase space functions constituting a Lie algebra realization, but explicit reference to canonical coordinates is removed and degeneracy is allowed. A manifold with such a Poisson bracket is a generalization of the symplectic manifold called a Poisson manifold. Noncanonical Poisson brackets, including the present day coordinate free axioms, were present in the theoretical physics community in the mid 20th century in e.g. the works of Dirac [11], Martin [38], Pauli [58], Sudarshan [64].

A special kind of noncanonical Poisson bracket, the Lie–Poisson bracket, has explicit linear dependence on the phase space coordinate and is intimately related to a Lie algebra. Lie–Poisson dynamics, dynamics generated by Lie–Poisson brackets, is ubiquitous and is the structure of many basic equations of physics. It is this kind of dynamics that is the subject of the present paper.

An important example of Lie–Poisson dynamics is given by Euler’s equations for rigid body dynamics with a Lie–Poisson bracket based on the Lie algebra of infinitesimal rotations [38] (see
This example often serves as inspiration for generalization and exploration of new concepts. The Lie–Poisson bracket for the full ideal fluid including magnetohydrodynamics was given in Morrison and Greene [53]. This example was followed by the Lie–Poisson formulation of the Maxwell–Vlasov system of equations in Morrison [47], with a correction given in Weinstein and Morrison [69] and Marsden and Weinstein [33] and a limitation to the correction pointed out in Morrison [49], which was followed up more recently in Heninger and Morrison [20], Morrison [52] and then in Lainz et al. [26]. Another example from the mid 1980s is that given in Marsden et al. [35], where the Lie–Poisson bracket for general moment closures of the kinetic hierarchy were given. There is now a large literature with very many subsequent publications that can be found, e.g., in Morrison [50, 51] and Arnold and Khesin [3].

Given the ubiquity of the Lie–Poisson form, it is natural to inquire about its origin. One thread extends back to the quasi-coordinate description of Poincaré [59] (see also Hamel [19]), where Euler’s equations for rigid body dynamics, and its Lagrangian counterpart—the Euler–Poincaré equation—was first formulated on a general Lie algebra. This idea was applied to fluid dynamics in Arnold [1, 2] where Euler’s equations for the incompressible fluid are seen to be the Euler–Poincaré equations on the Lie algebra of a diffeomorphism group. This idea put into modern language results that extend back to Lagrange [25], who imposed the incompressibility constraint in the so-called Lagrangian variable fluid description using his method of multipliers to impose volume preservation in the Lagrangian (variational) formulation. (See Morrison et al. [56] for commentary.) Although these works did not explicitly give the Lie–Poisson bracket, the equations of motion for a reduced dynamics were obtained.

The main geometrical idea behind these works is now understood as a process of reduction from canonical to noncanonical Hamiltonian form as follows (see, e.g., Marsden and Ratiu [32, Chapter 13]): The configuration space of the systems is a Lie group $G$, and the basic equation of the system is a canonical Hamiltonian system defined on the cotangent bundle $T^*G$. However, the Hamiltonian has $G$-symmetry, and thus one may reduce the system to the dual $g^*$ of the Lie algebra $g$ of $G$. The resulting equation on $g^*$ has Lie–Poisson form.

There are also examples where the system is defined on a Lie group $G$, but the symmetry of the system is broken. A well-known example is the heavy top, where the symmetry is broken by the gravity; another is the compressible fluid, where density plays a role similar to gravity for the heavy top case. In either case, it is known that one can still recover the full symmetry by extending the configuration to a semidirect product extension $G \rtimes V$ using a $G$-representation on a vector space $V$; see, e.g., Holm et al. [21], Marsden et al. [36, 37]. As a result, one again obtains a Lie–Poisson system on the dual of the semidirect product Lie algebra $g \rtimes V$, which is a special case of Lie–Poisson brackets based on Lie algebra extensions [66] that occur in a variety of physical systems including magnetohydrodynamics (see Marsden and Morrison [31]).

1.2. Collectivization. Another class of Lie–Poisson systems arises as a result of so-called collectivization in the sense of Guillemin and Sternberg [15] (see also Holmes and Marsden [22] and Guillemin and Sternberg [16, Section 28]), where a noncanonical system is reduced to another noncanonical system. Given a Poisson manifold $P$ and an equivariant momentum map $M: P \to g^*$ associated with an action of a Lie group $G$ on $P$, one can show that $M$ is a Poisson map with respect to the Poisson bracket on $P$ and the Lie–Poisson bracket on $g^*$; see, e.g., Marsden and Ratiu [32, Theorem 12.4.1]. This implies that, under some condition, a Hamiltonian dynamics on
H (for which G is not a symmetry group) is mapped to a Lie–Poisson dynamics on \( \mathfrak{g}^* \). The term “collective” comes from the motivating examples of Guillemin and Sternberg [15, 16] such as the liquid drop model in nuclear physics, where one seeks a set of equations that describe aggregate motions of a number of particles “as if it were a rigid body or liquid drop”. The idea behind this is anticipated in Rosensteel [61] and dates to Riemann [60] (see Morrison et al. [55]).

1.3. Clebsch Anti-Reduction and Collectivization. What we refer to as Clebsch anti-reduction or just “anti-reduction” for short is the opposite of the collectivization described above: One first has a Lie–Poisson equation on \( \mathfrak{g}^* \), and then constructs a cotangent bundle \( T^*Q \) and an equivariant momentum map \( M: T^*Q \to \mathfrak{g}^* \) so that solutions of the new canonical Hamiltonian dynamics on \( T^*Q \) can be mapped by \( M \) to those of the Lie–Poisson dynamics on \( \mathfrak{g}^* \).

This theoretical concept is motivated by the early use of potentials for describing the velocity field of fluid mechanics: long before the introduction of the vector potential for representing a magnetic field, researchers considered various potential representations of velocity fields, the most famous of which is due to Clebsch [7, 8]. The connection between the Lie–Poisson brackets for fluid dynamics and the canonical Hamiltonian description in terms of the Clebsch representation was first given in Morrison [48, 49], Morrison and Greene [54], while two-dimensional vortex dynamics was considered later in Marsden and Weinstein [34]. A general theory for Lie-Poisson brackets, motivated by [48] was given in Morrison [50] and the present work places this in the geometric setting described above.

1.4. Lie–Poisson Integrators. Compared to symplectic integrators for canonical Hamiltonian systems (see, e.g., Hairer et al. [18] and Leimkuhler and Reich [27]), integrators for Lie–Poisson equations seem to be studied less extensively. Some earlier works include Ge and Marsden [14] and Channell and Scovel [6], and are based on generating functions. Engø and Faltinsen [13] used Lie group methods by exploiting the property that Lie–Poisson dynamics evolves on coadjoint orbits on \( \mathfrak{g}^* \). More recently, Ma and Rowley [29] developed a variational integrator for the Lie–Poisson equation by discretizing the corresponding variational principle. See also Martín de Diego [39] for a more recent survey on Lie–Poisson integrators.

Our work is an extension of a more recent series of works on the so-called collective integrators by McLachlan et al. [40, 41, 42]. Namely, in order to numerically solve a Lie–Poisson equation on \( \mathfrak{g}^* \), one solves the corresponding canonical Hamiltonian system on \( T^*Q \) using a symplectic integrator and then map the solution to \( \mathfrak{g}^* \) by \( M \).

The main advantage of collective integrators is that one can construct a Lie–Poisson integrator out of existing symplectic integrators. On the other hand, the main disadvantage is that it is not always clear how one can find a suitable cotangent bundle \( T^*Q \) and momentum map \( M \). Existing works of McLachlan et al. [40, 41, 42] and Ohsawa [57] are limited to Lie–Poisson equations on simple spaces such as \( \mathfrak{so}(3)^* \cong \mathfrak{su}(2)^* \) and \( \mathfrak{se}(3)^* \), and are based on ad-hoc constructions of such momentum maps \( M \).

1.5. Main Result and Outline. We propose a systematic anti-reduction that works for a wider class of Lie–Poisson equations by constructing a momentum map \( M: T^*\mathfrak{g} \to \mathfrak{g}^* \); hence the Lie–Poisson equation on \( \mathfrak{g}^* \) is “anti-reduced” to a canonical Hamiltonian system \( T^*\mathfrak{g} \cong T^*\mathbb{R}^n \) with \( n := \dim \mathfrak{g} \). We first show how this works in coordinate calculations in Section 2, followed by its geometric interpretation in Section 3.
We note in passing that the term *anti-reduction* is not the opposite of a common usage of symmetry reduction, where Lie–Poisson equations on $g^*$ are obtained from canonical Hamiltonian systems on the cotangent bundle $T^*G$ with $G$-symmetry; rather, our anti-reduced Hamiltonian system is defined on $T^*g$.

Section 4 addresses some theoretical questions regarding the anti-reduction. We first state a necessary condition for the anti-reduction to work, and then find some invariants possessed by the canonical Hamiltonian system obtained by anti-reduction (or the *anti-reduced system* for short). First two such invariants are possessed by *any* anti-reduced system; this is not surprising given that the anti-reduction doubles the dimension of the system. One is the momentum map associated with the intrinsic symmetry possessed by the anti-reduced system, and the other is an invariant associated with the Killing form on $g$. Additionally, if $g$ is semisimple, then there is another invariant associated with the Killing form. Furthermore, we show that if the Lie–Poisson bracket on $g^*$ possesses a Casimir then there is a corresponding Noether-type invariant (momentum map) in the anti-reduced system as well.

In Section 5, we briefly review the idea of the collective integrators. Although the anti-reduction doubles the number of equations, it has the advantage that the resulting system is a canonical Hamiltonian system on a vector space, and is amenable to a number of techniques developed for canonical Hamiltonian systems. For example, any symplectic integrator applied to the anti-reduced system yields an integrator for the Lie–Poisson equation that exhibits near-conservation of the Hamiltonian and the Casimirs. Since our anti-reduction works for a very wide class of Lie–Poisson equation, it widens the scope of collective integrators for Lie–Poisson equations.

Section 6 shows some examples of applications of anti-reduction, such as the Kida vortex [24] (see also Meacham et al. [43]), the rattleback [46, 70], and the heavy top on a movable base with a stabilizing control [9].

### 2. Clebsch Anti-Reduction

#### 2.1. Lie–Poisson Bracket

Let $g$ be an $n$-dimensional Lie algebra, and $\{E_i\}_{i=1}^n$ be a basis for it with the structure constants $\{c^k_{ij}\}_{1 \leq i,j,k \leq n}$, i.e., $[E_i, E_j] = c^k_{ij}E_k$; note that we use Einstein’s summation convention throughout the paper. We may then define the dual basis $\{E^*_i\}_{i=1}^n$ for $g^*$ by setting $\langle E^*_i, E_j \rangle = \delta^*_i_j$ under the standard dual pairing $\langle \cdot, \cdot \rangle: g^* \times g \rightarrow \mathbb{R}$.

For any smooth $f: g^* \rightarrow \mathbb{R}$, we define the derivative $Df(\mu) \in g$ evaluated at $\mu \in g^*$ so that, for any $\delta \mu \in g^*$,

$$\langle \delta \mu, Df(\mu) \rangle = \frac{d}{ds}f(\mu + s\delta \mu) \bigg|_{s=0}.$$ 

This results in the coordinate expression

$$Df(\mu) = \frac{\partial f}{\partial \mu_i} (\mu) E_i.$$ 

Then one defines the $(\pm)$-Lie–Poisson bracket (see Section 3.3 for the $(\pm)$-Lie–Poisson bracket) on $g^*$ as follows: For any $f, g: g^* \rightarrow \mathbb{R}$,

$$\{f, g\}_\pm (\mu) := \langle \mu, [Df(\mu), Dg(\mu)] \rangle = \mu_k c^k_{ij} \frac{\partial f}{\partial \mu_i} \frac{\partial g}{\partial \mu_j}. \quad (1)$$
The Lie–Poisson equation for a Hamiltonian \( h \): \( g^* \to \mathbb{R} \) is the Hamiltonian system defined using the above Poisson bracket, i.e.,

\[
\dot{\mu}_i = \{ \mu_i, h \} = \mu_k c_{ij}^k \frac{\partial h}{\partial \mu_j},
\]

or equivalently,

\[
\dot{\mu} = -\text{ad}_{\text{D}h(\mu)}^* \mu.
\]

2.2. Clebsch Anti-Reduction in Coordinates. The main idea of the Clebsch anti-reduction (see Morrison [48, 50]) is the following: Given an \( n \)-dimensional Lie–Poisson bracket (2), we would like to find a corresponding \( 2n \)-dimensional canonical Hamiltonian system. In other words, we would like to find a relationship between the Poisson bracket (1) and the canonical Poisson bracket of the form

\[
\{ F, G \} = \frac{\partial F}{\partial q} \frac{\partial G}{\partial p_i} - \frac{\partial G}{\partial q} \frac{\partial F}{\partial p_i},
\]

where \( F, G : T^* \mathbb{R}^n \to \mathbb{R} \).

Suppose that \( \mu = \mu_i E_i^j \in g^* \) and \( (q, p) \in T^* \mathbb{R}^n \) are related as follows:

\[
\mu_i = c^k_j q^j p_k.
\]

For any smooth \( f, g : g^* \to \mathbb{R} \), we may define \( F, G : T^* \mathbb{R}^n \to \mathbb{R} \) by setting \( F(q, p) := f(\mu) \) where \( \mu \) and \( (q, p) \) are related as above; similarly for \( G \) as well. Then, by the chain rule, we have

\[
\frac{\partial F}{\partial q^i} = \frac{\partial f}{\partial \mu_j} \frac{\partial \mu_j}{\partial q^i}, \quad \frac{\partial F}{\partial p_i} = \frac{\partial f}{\partial \mu_j} \frac{\partial \mu_j}{\partial p_i} = \frac{\partial f}{\partial \mu_j} c^j_{ik} q^k.
\]

As a result,

\[
\{ F, G \} = \frac{\partial F}{\partial q^i} \frac{\partial G}{\partial p_i} - \frac{\partial G}{\partial q^i} \frac{\partial F}{\partial p_i} = q^l p_m (c^m_{ji} c^l_{kj} - c^m_{jk} c^l_{ij}) \frac{\partial f}{\partial \mu_j} \frac{\partial g}{\partial \mu_k} = q^l p_m (c^m_{ji} c^l_{kj} - c^m_{jk} c^l_{ij}) \frac{\partial f}{\partial \mu_j} \frac{\partial g}{\partial \mu_k} = q^l p_m c^m_{jk} c^l_{ij} \frac{\partial f}{\partial \mu_j} \frac{\partial g}{\partial \mu_k} = \mu_i c^i_{jk} \frac{\partial f}{\partial \mu_j} \frac{\partial g}{\partial \mu_k} = \{ f, g \}_+ (\mu),
\]

where the fourth equality follows from the Jacobi identity for the structure constants.

Therefore, given any Lie–Poisson bracket in terms of \( \mu \), one can obtain an anti-reduced canonical bracket via (4). The Hamiltonian of the anti-reduced system will have the form \( H(q, p) = h(\mu) \) with \( \mu \) given as in (4). If the resulting equations of the canonical system are solved for \( t \mapsto (q(t), p(t)) \), then \( t \mapsto \mu(t) \) constructed according to (4) solves the Lie–Poisson equation (2).

3. Geometry of Clebsch Anti-Reduction

This section gives a geometric interpretation of the anti-reduction presented in Section 2.2. Particularly, we show that the map (4) is the momentum map associated with a natural \( g \)-action on the cotangent bundle \( T^*g \).
3.1. **Left g-action on T\(*g.** Let T\(*\g = \g \times \g^* be the cotangent bundle of \g and define

\[ \g \times T^*\g \to T^*\g; \quad (\xi, (q, p)) \mapsto (\text{ad}_\xi q, -\text{ad}^*_p) =: \xi_{T^*\g}(q, p). \]

In coordinates, we can write it as follows:

\[ \xi_{T^*\g}(q, p) = \xi^i c_{ij}^k q^j \frac{\partial}{\partial q^k} - \xi^i c_{ij}^k p^j \frac{\partial}{\partial p^j} \]

\[ = \xi^i c_{ij}^k \left( q^j \frac{\partial}{\partial q^k} - p^k \frac{\partial}{\partial p^j} \right). \]

Let us show that it is a left Lie algebra action, i.e., for any \( \xi, \eta \in \g \),

\[ [\xi, \eta]_{T^*\g} = -[\xi_{T^*\g}, \eta_{T^*\g}], \]

where the bracket on the left-hand side is the commutator in \( \g \) whereas the one on the right is the Jacobi–Lie bracket of vector fields on \( T^*\g \). In fact, in the coordinate representation with respect to the standard basis \( \{\partial/\partial q^i, \partial/\partial p^i\}_{i=1}^n \), we have,

\[ D\eta_{T^*\g} \cdot \xi_{T^*\g} = \left( \frac{\partial}{\partial q^i} (\text{ad}_\eta q) \cdot \text{ad}_\xi q, -\frac{\partial}{\partial p^i} (\text{ad}^*_\eta p) \cdot (-\text{ad}^*_\xi p) \right) \]

\[ = (\text{ad}_\eta \circ \text{ad}_\xi q, \text{ad}^*_\eta \circ \text{ad}^*_\xi p) \]

\[ = ([\eta, [\xi, q]], \text{ad}^*_\eta \circ \text{ad}^*_\xi p), \]

where the second line follows because \( q \mapsto \text{ad}_\eta q \) and \( p \mapsto \text{ad}^*_\eta p \) are linear. Therefore, we obtain

\[ [\xi_{T^*\g}, \eta_{T^*\g}] = D\eta_{T^*\g} \cdot \xi_{T^*\g} - D\xi_{T^*\g} \cdot \eta_{T^*\g} \]

\[ = ([\eta, [\xi, q]] - [\xi, [\eta, q]], \text{ad}^*_\eta \circ \text{ad}^*_\xi p - \text{ad}^*_\xi \circ \text{ad}^*_\eta p) \]

\[ = ([q, [\xi, \eta]], \text{ad}^*_\eta \circ \text{ad}^*_\xi p) \]

\[ = (-\text{ad}^*_\eta q, \text{ad}^*_\eta \circ \text{ad}^*_\xi p) \]

\[ = -[\xi, \eta]_{T^*\g}, \]

where we used the Jacobi identity of the commutator on \( \g \); note that it also gives the following dual version: for any \( \xi, \eta \in \g \),

\[ \text{ad}^*_\eta \circ \text{ad}^*_\xi - \text{ad}^*_\xi \circ \text{ad}^*_\eta = \text{ad}^*_\xi [\xi, \eta]. \quad (5) \]

If \( \g \) is the Lie algebra of a Lie group \( G \), then we may first consider the left \( G \)-action on \( T^*\g \) as follows:

\[ \Phi: G \times T^*\g \to T^*\g; \quad (g, (q, p)) \mapsto (\text{Ad}_g q, \text{Ad}^*_g p) := \Phi_g(q, p). \]

Clearly this is the cotangent lift of the adjoint action of \( G \) on \( \g \). Then its infinitesimal generator gives the above Lie algebra action:

\[ \frac{d}{ds} \Phi_{\exp(s\xi)}(q, p) \bigg|_{s=0} = (\text{ad}_\xi q, -\text{ad}^*_\xi p) = \xi_{T^*\g}(q, p). \]
3.2. Momentum Map. Let us find the momentum map associated with the above Lie algebra action. For any $\xi \in \mathfrak{g}$, define $M^\xi : T^*\mathfrak{g} \to \mathbb{R}$ by setting
\[ X_{M^\xi} = \xi_{T^*\mathfrak{g}}, \]
where $X_{M^\xi}$ is the Hamiltonian vector field for $M^\xi$ with respect to the canonical symplectic form on $T^*\mathfrak{g}$, i.e.,
\[ X_{M^\xi} = \frac{\partial M^\xi}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial M^\xi}{\partial q^j} \frac{\partial}{\partial p_j}. \]
It is a straightforward calculation to find
\[ M^\xi(q, p) = \langle p, \text{ad}_q \xi \rangle = -\langle \text{ad}^*_p \xi, \xi \rangle. \]
The momentum map $M_+ : T^*\mathfrak{g} \to \mathfrak{g}^*$ is then defined so that
\[ \langle M_+(q, p), \xi \rangle = M^\xi(q, p), \]
which yields
\[ M_+(q, p) = -\text{ad}^*_p \xi. \tag{6} \]
We can obtain a coordinate expression for $M_+$ using the dual basis $\{E^i_\ast\}_{i=1}^n$ for $\mathfrak{g}^*$ as follows:
\[ M_+(q, p) = -q^j c^k_{ij} p_k E^i_\ast = c^k_{ij} q^j p_k E^i_\ast, \]
which is nothing but \[4\] obtained earlier.

The above momentum map is infinitesimally equivariant: For any $\eta \in \mathfrak{g}$ and any $(q, p) \in T^*\mathfrak{g}$,
\[ T_{(q, p)}M_+ \cdot \eta_{T^*\mathfrak{g}}(q, p) = -\text{ad}_{[\eta, q]} \text{ad}_p \eta_p = -\text{ad}_{\eta} \text{ad}_p \xi = -\text{ad}_{\eta} M_+(q, p), \]
where we again used the dual version \[5\] of the Jacobi identity. The infinitesimal equivariance implies (see, e.g., Marsden and Ratiu [32, Theorem 12.4.1]) that $M_+$ is a Poisson map with respect to the canonical Poisson bracket \[3\] on $T^*\mathfrak{g} \cong T^*\mathbb{R}^n$ and the $(+)$-Lie–Poisson bracket \[1\] on $\mathfrak{g}^*$, i.e., for any smooth $f, g : \mathfrak{g}^* \to \mathbb{R}$,
\[ \{ f \circ M_+, g \circ M_+ \} = \{ f, g \} + \circ M_+. \tag{7} \]

3.3. Right Action and ($-$)-Lie–Poisson bracket. In order to find a Poisson map $M_-$ with respect to the ($-$)-Lie–Poisson bracket
\[ \{ f, g \} _- (\mu) = -\langle \mu, [ Df(\mu), Dg(\mu) ] \rangle = -\mu_k c^k_{ij} \frac{\partial f}{\partial \mu_i} \frac{\partial g}{\partial \mu_j}. \tag{8} \]
on $\mathfrak{g}^*$, one starts instead with the following right Lie algebra action:
\[ \mathfrak{g} \times T^*\mathfrak{g} \to T^*\mathfrak{g}; \quad (\xi, (q, p)) \mapsto (-\text{ad}_q \xi, \text{ad}_p \xi) =: \xi_{T^*\mathfrak{g}}(q, p), \]
which satisfies \[ [\xi, \eta]_{T^*\mathfrak{g}} = [\xi_{T^*\mathfrak{g}}, \eta_{T^*\mathfrak{g}}]. \]
Again, if $\mathfrak{g}$ is the Lie algebra of a Lie group $G$, then we may consider the following right $G$-action on $T^*\mathfrak{g}$:
\[ \Phi : G \times T^*\mathfrak{g} \to T^*\mathfrak{g}; \quad (g, (q, p)) \mapsto (\text{Ad}_g^{-1} q, \text{Ad}_g^* p) := \Phi_g(q, p). \]
Then we have
\[ \frac{d}{dt} \Phi_{\exp(t\xi)}(q, p) \bigg|_{t=0} = (-\text{ad}_q \xi, \text{ad}_p \xi) = \xi_{T^*\mathfrak{g}}(q, p). \]
The associated momentum map is
\[ M_-(q, p) = \text{ad}_q^* p = -c^k_{ij} q^i p^j E_k, \quad (9) \]
and satisfies, for any \( f, g: g^* \to \mathbb{R} \),
\[ \{ f \circ M_-, g \circ M_- \} = \{ f, g \}_- \circ M_. \quad (10) \]

3.4. Main Result. Summarizing the above arguments, our main result is the following:

**Theorem 1 (Anti-reduction of Lie–Poisson equations).** Given a smooth function \( h: g^* \to \mathbb{R} \), define
\[ H(q, p) := h \circ M_\pm(q, p) = h(\mp \text{ad}_q^* p), \quad (11) \]
using \( M_\pm: T^* g \to g^* \) defined in (6) or (9), respectively. Let \( t \mapsto (q(t), p(t)) \) be a solution to the canonical Hamiltonian system (referred to as the anti-reduced system)
\[ \dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q} \quad (12) \]
on \( T^* g \cong T^* \mathbb{R}^n \). Then \( t \mapsto \mu(t) := M_\pm(q(t), p(t)) \) gives a solution to the Lie–Poisson equation
\[ \dot{\mu} = \mp \text{ad}_{Dh(\mu)}^* \mu, \quad (13) \]
defined in terms of the \((\pm)-Lie–Poisson \) bracket, (1) or (8), respectively.

**Proof.** It easily follows from the property that \( M_\pm \) is Poisson (see (7) and (10)) with respect to the canonical Poisson bracket on \( T^* g \) and the \((\pm)-Lie–Poisson \) bracket on \( g^* \), respectively. \( \square \)

4. Properties of Anti-Reduction

4.1. Does Anti-Reduction Work for Any Lie Algebra? Theorem 1 itself does not impose any specific condition on the Lie algebra \( g \). However, for the anti-reduction to work effectively, one needs to make sure that the image \( M_\pm(T^* g) \) is the entire \( g^* \). Otherwise, one may not be able to find an initial condition for the canonical Hamiltonian system corresponding to an arbitrary initial condition for the Lie–Poisson equation.

One may certainly check directly whether \( M_\pm(T^* g) = g^* \) case by case. However, it is of theoretical interest to characterize it in terms of algebraic properties of \( g \). The following proposition gives a partial answer to the question by giving a necessary condition for it:

**Proposition 2.** If the center
\[ z(g) := \{ z \in g \mid [z, x] = 0 \ \forall x \in g \} \]
of \( g \) is non-trivial, i.e., \( z(g) \neq \{0\} \), then \( M_\pm(T^* g) \subseteq g^* \). In other words, a necessary condition for having \( M_\pm(T^* g) = g^* \) is that the center of \( g \) is trivial.

**Proof.** If \( z(g) \neq \{0\} \) then there exists \( z \in g \setminus \{0\} \) such that \( \text{ad}_q z = [q, z] = 0 \) for any \( q \in g \). Hence for any \( (q, p) \in T^* g \),
\[ \langle M_\pm(q, p), z \rangle = \mp \langle \text{ad}_q^* p, z \rangle = \mp \langle p, \text{ad}_q z \rangle = 0. \]
This implies that \( M_\pm(T^* g) \) is a proper subset of \( g^* \). \( \square \)
4.2. Intrinsic Symmetry and Conservation Law. The anti-reduced system (12) has some symmetry regardless of the type of the Lie algebra $\mathfrak{g}$ and of the form of the original Hamiltonian $h$. In fact, let $\mathbb{R}_{>0}$ be the group of positive real numbers under the standard multiplication, and consider the following $\mathbb{R}_{>0}$-action on $T^*\mathfrak{g}$:

$$\mathbb{R}_{>0} \times T^*\mathfrak{g} \rightarrow T^*\mathfrak{g}; \quad (e^s, (q, p)) \mapsto (e^s q, e^{-s} p).$$

This action is clearly symplectic because it is the cotangent lift of the $\mathbb{R}_{>0}$-action $(e^s, q) \mapsto e^s q$ on $\mathbb{R}^n$. The associated momentum map is then given by

$$F_0: T^*\mathfrak{g} \rightarrow \mathbb{R}; \quad F_0(q, p) = p \cdot q. \quad (14)$$

Clearly $M_\pm$ is invariant under the action, and hence so is the Hamiltonian $H$ defined above in (11). By Noether’s Theorem (see, e.g., Marsden and Ratiu [32, Theorem 11.4.1]), this implies that $F_0$ is an invariant of the Hamiltonian system (12).

4.3. Killing Form and Invariants. Let us write down the anti-reduced system (12) more explicitly. Setting

$$\eta(q, p) := Dh(-\text{ad}^*_q p)$$

for shorthand and taking derivatives of the Hamiltonian $H$ in (11), we have

$$\frac{\partial H}{\partial q^i} = \frac{\partial}{\partial q^i} h(-\text{ad}^*_q p) = \langle -\text{ad}^*_q E_i, Dh(-\text{ad}^*_q p) \rangle = -\langle p, \text{ad}_{E_i} \eta(q, p) \rangle = \langle p, \text{ad}_{\eta(q, p)} E_i \rangle = \langle \text{ad}^*_p \eta(q, p) E_i \rangle,$$

whereas

$$\frac{\partial H}{\partial p_i} = \frac{\partial}{\partial p_i} h(-\text{ad}^*_q p) = \langle -\text{ad}^*_q E_i, Dh(-\text{ad}^*_q p) \rangle = -\langle E_i, \text{ad}_q \eta(q, p) \rangle = \langle E_i, \text{ad}_{\eta(q, p)} q \rangle.$$

As a result, the Hamiltonian vector field $X_H$ on $T^*\mathfrak{g}$ is given by

$$X_H(q, p) = \left( \frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q} \right) = \left( \text{ad}_{\eta(q, p)} q, -\text{ad}^*_p \eta(q, p) p \right),$$

and hence (12) becomes

$$\dot{q} = \text{ad}_{\eta(q, p)} q, \quad \dot{p} = -\text{ad}^*_p \eta(q, p) p. \quad (15)$$

Remark 3. It follows easily from the above expression that $F_0$ from (14) is an invariant.

One can find another invariant from the above expression:
Proposition 4. Let \( \kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R} \) be the Killing form, i.e.,
\[
\kappa(x, y) := \mathrm{tr}(\text{ad}_x \circ \text{ad}_y) = \kappa_{ij} x^i y^j
\]
with \( \kappa_{ij} := c^l_{ik} c^k_{jl} \) in terms of the structure constants. Then \( \kappa(q, q) \) is an invariant of the anti-reduced system (15).

Proof. It follows from the following simple calculations:
\[
\frac{d}{dt} \kappa(q, q) = \kappa(\dot{q}, q) + \kappa(q, \dot{q}) = 2 \kappa(q, \dot{q}) = 2 \kappa(q, \text{ad}_{\eta(q,p)} q) = -2 \kappa(q, [q, \eta(q,p)]) = -2 \kappa([q,q], \eta(q,p)) = 0,
\]
where we used the associativity of the Killing form, i.e., for any \( x,y,z \in \mathfrak{g} \), \( \kappa(x, [y,z]) = \kappa([x,y], z) \). \( \square \)

Additionally, if \( \mathfrak{g} \) is semisimple, one can find yet another invariant. It is well known that the Killing form \( \kappa \) is non-degenerate if and only if \( \mathfrak{g} \) is semisimple. So if we define
\[
\kappa^\flat : \mathfrak{g} \to \mathfrak{g}^*; \quad x \mapsto \kappa(x, \cdot) =: \kappa^\flat(x),
\]
then it is an isomorphism, and thus one can define its inverse \( \kappa^\sharp := (\kappa^\flat)^{-1} : \mathfrak{g}^* \to \mathfrak{g} \). Then we can define an inner product on \( \mathfrak{g}^* \) as follows: For any \( \alpha, \beta \in \mathfrak{g}^* \),
\[
\kappa^\flat(\cdot, \alpha) = \kappa^\flat(\alpha, \cdot) = \kappa^\flat(\alpha) = \kappa^\flat(\beta) = \kappa^\flat(\alpha).
\]

Proposition 5. If \( \mathfrak{g} \) is semisimple, then \( \kappa^\flat(p, p) = \langle p, \kappa^\flat(p) \rangle \) is also an invariant of the anti-reduced system (15).

Proof. Let us first show that, for any \( x \in \mathfrak{g} \) and \( \alpha \in \mathfrak{g}^* \),
\[
\kappa^\flat(\text{ad}_x^* \alpha) = \text{ad}_x \kappa^\flat(\alpha).
\]
In fact, by setting \( a := \kappa^\flat(\alpha) \), we have, for any \( z \in \mathfrak{g} \),
\[
\langle \text{ad}_x^* \alpha, z \rangle = -\langle \alpha, [x, z] \rangle = -\langle \kappa^\flat(a), [x, z] \rangle = -\kappa(a, [x, z]) = -\kappa([a, x], z) = \kappa(\text{ad}_x a, z) = \langle \kappa^\flat(\text{ad}_x a), z \rangle.
\]
Hence $-\text{ad}_x^* \alpha = \kappa^\sharp (\text{ad}_x a)$, and so $\kappa^\sharp (-\text{ad}_x^* \alpha) = \text{ad}_x a$. Then, using the equality we have just proved, we have

$$
\frac{d}{dt} \kappa^\sharp (p, p) = \left\langle \dot{p}, \kappa^\sharp (p) \right\rangle + \left\langle p, \kappa^\sharp (\dot{p}) \right\rangle
= - \left\langle \text{ad}_{\eta(q,p)}^* p, \kappa^\sharp (p) \right\rangle + \left\langle p, \kappa^\sharp \left( - \text{ad}_{\eta(q,p)}^* p \right) \right\rangle
= - \left\langle \text{ad}_{\eta(q,p)}^* p, \kappa^\sharp (p) \right\rangle + \left\langle p, \text{ad}_{\eta(q,p)} \kappa^\sharp (p) \right\rangle
= 0.
\square
$$

4.4. Casimirs and Momentum Maps. If the Lie–Poisson bracket possesses a Casimir, then there must be a corresponding invariant for the the canonical Hamiltonian system (12). We would like to show that the invariant is indeed a Noether invariant (momentum map) of the anti-reduced system:

**Proposition 6.** Suppose that $f: \mathfrak{g}^* \to \mathbb{R}$ is a Casimir of the Lie–Poisson bracket (1) or (8), and define $F: T^* \mathfrak{g} \to \mathbb{R}$ by setting $F := f \circ M_\pm$, i.e.,

$$F(q, p) := f(\mp \text{ad}_q^* p).$$

Let us also define

$$\gamma: T^* \mathfrak{g} \to \mathfrak{g}; \quad \gamma(q, p) := Df(M_\pm(q, p)) = Df(-\text{ad}_q^* p),$$

and consider the following $\mathbb{R}$ (Lie algebra) action

$$\mathbb{R} \times T^* \mathfrak{g} \to \mathfrak{X}(T^* \mathfrak{g}); \quad (s, (q, p)) \mapsto (\text{ad}_{s \gamma(q, p)} q, -\text{ad}_s^* \gamma(q, p)) := s T^* \mathfrak{g}(q, p),$$

where $\mathfrak{X}(T^* \mathfrak{g})$ stands for the space of vector fields on $T^* \mathfrak{g}$. Then the momentum map corresponding to the action is $F$. Furthermore, the Hamiltonian $H$ is infinitesimally invariant under the action, and thus $F$ is an invariant of the anti-reduced system (12).

**Proof.** Performing the same calculation shown at the beginning of Section 4.3, we have

$$X_F(q, p) = \left( \frac{\partial F}{\partial p}, -\frac{\partial F}{\partial q} \right) = \left( \text{ad}_{\gamma(q, p)}^* q, -\text{ad}_q^* (\gamma(q, p)) p \right),$$

and so, for any $s \in \mathbb{R},$

$$X_{s F} = s T^* \mathfrak{g}.$$
for any \((q, p) \in T^* \mathfrak{g}\). Then, for any \(s \in \mathbb{R}\), the directional derivative of \(M_{\pm}\) along the vector field \(s_{T^* \mathfrak{g}}\) yields

\[
s_{T^* \mathfrak{g}}[M_{\pm}](q, p) = \mp s \left( \text{ad}^*_{\gamma(q,p)} p - \text{ad}^*_{q} \text{ad}^*_{\gamma(q,p)} p \right) = \mp s \left( \text{ad}^*_{\gamma(q,p)} p - \text{ad}^*_{q} \text{ad}^*_{\gamma(q,p)} p + \text{ad}^*_{q} \text{ad}^*_{\gamma(q,p)} p \right) = 0,
\]

where we used the dual version \([5]\) of the Jacobi identity in the second equality.

This implies that the Hamiltonian \(H := h \circ M_{\pm}\) is infinitesimally invariant under the \(\mathbb{R}\)-action as well. That \(F\) is an invariant of \([12]\) follows easily from either that \(M_{\pm}\) is Poisson or Noether’s Theorem (see, e.g., Marsden and Ratiu \([32]\) Theorem 11.4.1]).

5. Collective Integrators via Clebsch Anti-Reduction

5.1. Collective Lie–Poisson Integrators via Clebsch Anti-Reduction. Let \(\Psi_{\Delta t}: T^* \mathfrak{g} \to T^* \mathfrak{g}\) be a symplectic integrator with time step \(\Delta t\) for the anti-reduced system \([12]\). Assuming that \(\text{im } M_{\pm} = \mathfrak{g}^*\), for any given the initial condition \(\mu(0) = \mu_0 \in \mathfrak{g}^*\) for the Lie–Poisson equation \([13]\), one may find \((q_0, p_0) \in T^* \mathfrak{g}\) such that \(M_{\pm}(q_0, p_0) = \mu_0\), and then can obtain \((q_n, p_n)\) for \(n \geq 1\) using \(\Psi_{\Delta t}\) by setting \((q_{n+1}, p_{n+1}) = \Psi_{\Delta t}(q_n, p_n)\). Then we set \(\mu_n := M_{\pm}(q_n, p_n)\) for \(n \geq 1\) to construct a numerical solution for \([13]\). This is the basic idea of the collective Lie–Poisson integrators of McLachlan et al. \([40, 41]\).

One sees that the collective integrator inherits the near conservation of the Hamiltonian from the symplectic integrator \(\Psi_{\Delta t}\) because \(h(\mu_n) = H(q_n, p_n)\). Furthermore, Proposition \([5]\) shows that any Casimir \(f\) of the Lie–Poisson equation gives rise to the momentum map \(F := f \circ M_{\pm}\) of the anti-reduced system so that \(f(\mu_n) = F(q_n, p_n)\). However, since the symplectic integrator \(\Psi_{\Delta t}\) nearly conserves the momentum map \(F\), it follows that the collective integrator nearly conserves the Casimir \(f\) as well. These arguments follow from the backward error analysis of symplectic integrators; see, e.g., Leimkuhler and Reich \([27]\) Chapter 5 and Hairer et al. \([18]\) Chapter IX).

5.2. Gauss–Legendre Methods. For the symplectic integrator \(\Psi_{\Delta t}\) for the anti-reduced system \([12]\), we use the Gauss–Legendre methods—a family of implicit Runge–Kutta (RK) methods based on the points of Gauss–Legendre quadrature. The order of a Gauss–Legendre method is \(2s\) if it is based on \(s\) points \([17]\) Theorem 5.2]. The simplest Gauss–Legendre method is of order 2 and is the Implicit Midpoint Method. In this paper, we will use the \(4\)th order Gauss–Legendre method; see, e.g., Leimkuhler and Reich \([27]\) Table 6.4 on p. 154).

Since the Gauss–Legendre methods are known to be symplectic (see, e.g., Hairer et al. \([18]\) Theorem 4.2] and Leimkuhler and Reich \([27]\) Section 6.3.1]), they yield, combined with the map \(M_{\pm}\), collective integrators for the Lie–Poisson equation \([13]\).

One of the most important properties of the Gauss–Legendre methods is that they exactly conserve any quadratic invariant of the form

\[
I = z^T A z + b^T z, \quad z = \begin{bmatrix} q \\ p \end{bmatrix},
\]
where $A \in \mathbb{R}^{2d \times 2d}$ is a symmetric matrix and $b \in \mathbb{R}^{2d}$; see Cooper [10] and Hairer et al. [18, Theorems 2.1 and 2.2] for a proof. This property is particularly important for us because the momentum map $F_0$ and the two additional invariants $\kappa$ and $\kappa^*$ in terms of the killing form (see Propositions 4 and 5) are such quadratic invariants of the anti-reduced system.

6. Examples

6.1. Kida Vortex. The Kida vortex [24] is an elliptical vortex patch of constant vorticity in a two-dimensional flow. The equations of motion obtained by Kida describe the time evolution of the semi-major axis $a$ and semi-minor axis $b$ and of the angle $\phi$ of orientation of the ellipse in a steady shear background flow:

$$\dot{a} = \frac{\epsilon}{2} a \sin(2\phi), \quad \dot{b} = -\frac{\epsilon}{2} b \sin(2\phi), \quad \dot{\phi} = \frac{ab}{(a+b)^2} + \frac{\omega}{2} + \frac{\epsilon}{2} \frac{a^2 + b^2}{a^2 - b^2} \cos(2\phi),$$

where $\epsilon > 0$ is the constant rate of strain of the background shear flow. Defining the aspect ratio $\lambda := b/a$, the equations reduce to

$$\dot{\lambda} = -\epsilon \lambda \sin(2\phi), \quad \dot{\phi} = \frac{\lambda}{(1+\lambda)^2} + \frac{\omega}{2} + \frac{\epsilon}{2} \frac{1 + \lambda^2}{1 - \lambda^2} \cos(2\phi). \quad (16)$$

It is then not difficult to see that the above system of equations is Hamiltonian [44, 45]. Furthermore, Meacham et al. [43] showed that (16) follows from a Lie–Poisson equation on $\mathfrak{so}(2,1)^*$ obtained by projecting the Lie–Poisson structure for the 2D incompressible Euler equation onto quadratic moments of the vorticity. Specifically, let $\mathfrak{so}(2,1)$ be Lie algebra of the Lie group

$$\text{SO}(2,1) := \{ R \in \mathbb{R}^{3 \times 3} \mid R^T K R = K \} \text{ with } K := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$ 

A basis for $\mathfrak{so}(2,1)$ is given by $E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, for which the structure constants $\{c_{ij}^k\}_{i,j,k \in \{1,2,3\}}$ satisfy

$$\mu_k c_{ij}^k = \begin{bmatrix} 0 & \mu_3 & \mu_2 \\ -\mu_3 & 0 & -\mu_1 \\ -\mu_2 & \mu_1 & 0 \end{bmatrix},$$

for any $\mu \in \mathfrak{so}(2,1)^* \cong \mathbb{R}^3$. This is the Bianchi class B type VIII Lie algebra; see, e.g., Ellis and MacCallum [12] and Yoshida et al. [70]. The Casimir of the corresponding Lie–Poisson bracket (1)

is then

$$f_1(\mu) := \mu_1^2 + \mu_2^2 - \mu_3^2, \quad (17)$$

which is essentially the area of the ellipse.

The variables $(\mu_1, \mu_2, \mu_3)$ are related to the original variables $(\lambda, \phi)$ as follows:

$$\mu_2 = \frac{\pi}{16} \left( \lambda - \frac{1}{\lambda} \right) \cos(2\phi), \quad \mu_3 = -\frac{\pi}{16} \left( \lambda + \frac{1}{\lambda} \right) \cos(2\phi), \quad \mu_1^2 + \mu_2^2 - \mu_3^2 = -\frac{\pi^2}{64}. \quad (18)$$

With the Hamiltonian (the “excess energy” of the elliptical vortex patch [43]) $h:\mathfrak{so}(2,1)^* \cong \mathbb{R}^3 \to \mathbb{R}$ defined as

$$h(\mu) := \epsilon \mu_2 + \omega \mu_3 - \frac{\pi}{8} \ln \left( \frac{\pi}{8} - \mu_3 \right), \quad (19)$$
the Lie–Poisson equation $\dot{\mu} = -\text{ad}_{\mathbf{D}_H(\mu)}^* \mu$ from (2) yields
\[
\dot{\mu}_1 = \omega \mu_2 + \epsilon \mu_3 + \frac{\pi \mu_2}{\pi - 8 \mu_3}, \quad \dot{\mu}_2 = -\mu_1 \left( \omega + \frac{\pi}{\pi - 8 \mu_3} \right), \quad \dot{\mu}_3 = \epsilon \mu_1. \tag{20}
\]

One can then show that, using (18), the above Lie–Poisson equation gives rise to the original equation (16) of Kida.

The map (6) yields (lowering the indices for $q$ for simplicity),
\[
\mathbf{M}_+(q, p) = (q_2 p_3 + q_3 p_2, -q_3 p_1 - q_1 p_3, -q_1 p_2 + q_2 p_1),
\]
which clearly satisfies $\mathbf{M}_+(T^* \mathbf{g}) = \mathbf{g}^*$. We then have the Hamiltonian
\[
H(q, p) := h(\mathbf{M}_+(q, p)) = -\epsilon (q_3 p_1 + q_1 p_3 - \omega (q_1 p_2 - q_2 p_1) - \frac{\pi}{8} \ln \left( \frac{\pi}{8} + q_1 p_2 - q_2 p_1 \right).
\]

The anti-reduced system (12) is therefore
\[
\begin{align*}
\dot{q}_1 &= \omega q_2 - \epsilon q_3 + \frac{q_2}{8} q_1 p_2 - q_2 p_1 + \pi / 8, \\
\dot{q}_2 &= -\omega q_1 - \frac{q_1}{8} q_1 p_2 - q_2 p_1 + \pi / 8, \\
\dot{p}_1 &= \omega p_2 + \epsilon p_3 + \frac{p_1}{8} q_1 p_2 - q_2 p_1 + \pi / 8, \\
\dot{p}_2 &= -\omega p_1 - \frac{p_1}{8} q_1 p_2 - q_2 p_1 + \pi / 8, \\
\dot{p}_3 &= \epsilon p_1.
\end{align*}
\tag{22}
\]

Let us find additional invariants. The Killing form in this case is
\[
\kappa(x, y) = 2(x_1 y_1 + x_2 y_2 - x_3 y_3).
\]

One can see that it is non-degenerate, and thus there are two additional invariants (see Propositions 4 and 5):
\[
\kappa(q, q) = 2(q_1^2 + q_2^2 - q_3^2), \quad \kappa^*(p, p) = 2(p_1^2 + p_2^2 - p_3^2). \tag{23}
\]

Figure 1 shows numerical results with parameters $\epsilon = 1/2$ and $\omega = -1$ with initial condition determined by $\mu_1(0) = 1$, $f_1(\mu(0)) = -1/4$ and $h(\mu(0)) = 1$; this is a case from Meacham et al. [43, Fig. 2]. It shows the time evolution of the solution to (20) computed by the collective integrator as well as the trajectory of the solution in $\mathfrak{so}(2, 1) \times$ plotted with the level sets of the Hamiltonian $h$ and the Casimir $f_1$; see (19) and (17). We used the 4th order Gauss–Legendre method to solve the anti-reduced system (22) with the initial condition $(q(0), p(0))$ obtained by solving $\mathbf{M}_+(q(0), p(0)) = \mu(0)$; we additionally imposed $q(0) = (1, 0, 0)$ and $p_1(0) = 0$ to obtain the unique solution because $\mathbf{M}_+$ is not injective.

For comparison, we also solved the Lie–Poisson equation (20) directly using the 4th order explicit Runge–Kutta method. Figure 2 compares the time evolution of the relative errors in the Hamiltonian $h$ and the Casimir $f_1$ along these numerical solutions. Whereas the Runge–Kutta solution exhibits an excellent accuracy in preserving the Hamiltonian initially, it exhibits a drift that seems to be detrimental in the long run. Notice also that it exhibits a more significant drift in the Casimir. On the other hand, the solution of the collective integrator does not exhibit drifts in either the Hamiltonian or the Casimir.

Figure 3 shows how well the collective integrator preserves the momentum map $F_0$ from (14) and the two invariants (23) defined in terms of the Killing form. Recall that the Gauss–Legendre methods preserves these invariants exactly in theory. However, being an implicit method, it introduces an error in each step when solving nonlinear equations—the likely culprit of the small errors observed in the figures.
Figure 1. (a) Time evolution of $\mu$ computed using the anti-reduced system $\text{(22)}$. The solutions are shown for the time interval $0 \leq t \leq 100$ with time step $\Delta t = 0.1$. (b) The red curve is the Lie–Poisson dynamics of the Kida vortex in $\mathfrak{g}^* = \mathfrak{so}(2, 1)^* \cong \mathbb{R}^3$ computed using the anti-reduced system $\text{(22)}$ and mapped by $M_+$ in $\text{(21)}$. The green and orange surfaces are the level sets of the Hamiltonian $h$ and the Casimir $f_1$ from $\text{(19)}$ and $\text{(17)}$, respectively.

Figure 2. Time evolutions of relative errors in Hamiltonian $h$ and Casimir $f_1$ from the Kida system. The solid blue curve is the Runge–Kutta method directly applied to Lie–Poisson equation $\text{(20)}$ whereas the dashed red curve is the Gauss–Legendre method of order 4 applied to the anti-reduced system $\text{(22)}$. The solutions are shown for the time interval $0 \leq t \leq 1000$ with time step $\Delta t = 0.1$.

6.2. Rattleback. The rattleback (or celt) is a boat-shaped toy famous for its asymmetric preference in spin. Moffatt and Tokieda [46] derived the following simple equations for be the pitch, roll, and spin modes ($P, R, S$) of motion of the rattleback:

$$\dot{P} = \lambda PS, \quad \dot{R} = -RS, \quad \dot{S} = R^2 - \lambda P^2.$$  \hspace{1cm} (24)
Yoshida et al. [70] identified this set of equations as a Lie–Poisson equation as follows: Consider the three-dimensional Lie algebra $\mathfrak{g} = \text{span}\{E_1, E_2, E_3\}$ with

$$[E_1, E_2] = 0, \quad [E_1, E_3] = \lambda E_1, \quad [E_2, E_3] = -E_2,$$

where $\lambda > 1$ is the aspect ratio of the rattleback. Writing an arbitrary element in $\mathfrak{g}^*$ as $\mu = (P, R, S)$, the structure constants $\{c^k_{ij}\}_{i,j,k \in \{1, 2, 3\}}$ then satisfy

$$\mu_k c^k_{ij} = \begin{bmatrix} 0 & 0 & \lambda P \\ 0 & 0 & -R \\ -\lambda P & R & 0 \end{bmatrix}.$$

The Casimir for the corresponding Lie–Poisson bracket (see Yoshida et al. [70]) is then

$$f_1(P, R, S) := PR^\lambda. \quad (25)$$

Define the Hamiltonian $h : \mathfrak{g}^* \to \mathbb{R}$ as

$$h(P, R, S) := \frac{1}{2}(P^2 + R^2 + S^2). \quad (26)$$
Then the Lie–Poisson equation (2) with respect to the (+)-Lie–Poisson bracket gives (24).

Let us anti-reduce the above system. From (6), we have

\[ M_+(q,p) = (\lambda q_3 p_1, -q_3 p_2, q_2 p_2 - \lambda q_1 p_1), \]  

and so obtain the Hamiltonian \( H: T^*\mathfrak{g} \to \mathbb{R} \) as

\[
H(q,p) = h(M_+(q,p)) = \frac{1}{2}(\lambda^2 q_3^2 p_1^2 + q_3^2 p_2^2 + \lambda^2 q_1^2 p_1^2 - 2\lambda q_1 q_2 p_1 p_2 + q_2^2 p_2^2).
\]

As a result, the anti-reduced system (12) is

\[
\begin{align*}
\dot{q}_1 &= -\lambda q_1 q_2 p_2 + \lambda^2 (q_1^2 + q_3^2) p_1, \\
\dot{q}_2 &= -\lambda q_1 q_2 p_1 + \lambda^2 (q_2^2 + q_3^2) p_2, \\
\dot{q}_3 &= 0, \\
\dot{p}_1 &= \lambda q_2 p_1 p_2 - \lambda^2 q_1 p_1^2, \\
\dot{p}_2 &= \lambda q_1 p_1 p_2 - q_2 p_2^2, \\
\dot{p}_3 &= -q_3 (\lambda^2 p_1^2 + p_2^2).
\end{align*}
\]

The Killing form for this system is,

\[ \kappa(x,y) = (1 + \lambda^2)x_3y_3, \]

and thus, according to Proposition 4, one additional invariant can be found as

\[ \kappa(q,q) = (1 + \lambda^2)q_3^2. \]

This make sense since from the anti-reduced rattleback system (28), \( q_3 \) is a constant of motion.

Note that \( \kappa \) is degenerate and so one cannot define \( \kappa^* \) for this system.

We used the following parameters and initial condition taken from Yoshida et al. [70]: \( \lambda = 4, P(0) = R(0) = 0.01 \) and \( S(0) = 0.5 \). We determined the initial condition for the anti-reduced system (28) by imposing \( q_1(0) = q_3(0) = 0.1 \) and \( F_0(q(0), p(0)) = 1 \) and solving \( M_+(q(0), p(0)) = (P(0), R(0), S(0)) \). Figure 4 shows the results as we did for the Kida system using the same integrator.

![Figure 4](image.png)

**Figure 4.** (a) Time evolution of pitch \( P \), roll \( R \) and spin \( S \) computed using the anti-reduced system (28). The solutions are shown for the time interval \( 0 \leq t \leq 500 \) with time step \( \Delta t = 0.01 \). (b) The red curve is the Lie–Poisson dynamics of the rattleback in \( \mathfrak{g}^\ast \) computed using the anti-reduced system (28) and mapped by \( M_+ \) in (27). The green sphere and orange leaves are the level sets of the Hamiltonian \( h \) and the Casimir \( f_1 \) (see (26) and (25)), respectively.
As in the Kida case, we also compared our solution to that of the Lie–Poisson equation (24) obtained by 4th order explicit Runge–Kutta method. Figure 5(a) shows the time evolution of the relative errors in the Hamiltonian \( h \). Although the errors are quite small, both solutions accumulate errors picked up when the pitch \( P \) exhibits a sharp peak and the spin \( S \) changes the direction rapidly as shown in Fig. 4(a); see also the sharp corners in Fig. 4(b). The momentum map \( F_0 \) also exhibits a similar behavior, although it more or less oscillates around the exact value. However, as seen in Fig. 5(b), the collective integrator preserves the Casimir \( f_1 \) much better than the Runge–Kutta method.

**Figure 5.** Time evolutions of relative errors in Hamiltonian \( h \), Casimir \( f_1 \) and Momentum map \( F_0 \) from the rattleback system. The solid blue curve is the Runge–Kutta method directly applied to Lie–Poisson equation (24) whereas the dashed red curve is the Gauss–Legendre method of order 4 applied to the anti-reduced rattleback system. The solutions are shown for the time interval \( 0 \leq t \leq 500 \) with time step \( \Delta t = 0.01 \).

6.3. **Heavy Top on a Movable Base.** As a higher-dimensional and more practical example, consider the system shown in Fig. 6 from Contreras and Ohsawa [9]: It is a heavy top with mass \( m \) placed on a movable base—point mass \( M \) for simplicity—under gravity \( g \).
As the base is free to move, the system is defined by the rotational motion of the heavy top and
the linear motion of the base. Hence the natural configuration space is the matrix Lie group
\[ \text{SE}(3) = \left\{ (R, x) := \begin{bmatrix} R & x \\ 0 & 1 \end{bmatrix} \mid R \in \text{SO}(3), x \in \mathbb{R}^3 \right\}, \]
where position of the base defined by \( x \). The left translation of the tangent vector \((\dot{R}, \dot{x}) \in T_{(R,x)}\text{SE}(3)\) to the identity yields
\[
\begin{bmatrix}
\hat{\Omega} \\
\dot{v}
\end{bmatrix} := \begin{bmatrix} R & x \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \dot{R} \\
\dot{x} \end{bmatrix} = \begin{bmatrix} R^{-1} \dot{R} & R^{-1} \dot{x} \\ 0 & 0 \end{bmatrix} \in \mathfrak{se}(3),
\]
which are the angular velocity of the top and the base velocity with respect to the body frame of
the top. Note that we identify \( \mathfrak{se}(3) = \mathfrak{so}(3) \ltimes \mathbb{R}^3 \) with \( \mathbb{R}^3 \times \mathbb{R}^3 \) via the hat map \( \hat{\cdot} : \mathbb{R}^3 \to \mathfrak{so}(3) \); see, e.g., \cite{equation} Eq. (9.2.7) on p. 289.

Let \( \bar{m} := m + M \) be the total mass of the system, \( \mathbb{I} := \text{diag}(I_1, I_2, I_3) \) the inertia mass matrix of
the heavy top (generally, for a heavy top \( I_1 = I_2 \)), \( l \) the length from the point of the heavy top that
touches the base to the center of mass of the heavy top, and \( \chi \) the unit vector in that direction in
the body frame; see Fig. 6.

Using the body angular momentum \( \Pi \) and the linear impulse \( P \) related to \( \Omega \) and \( v \) as
\[ \Pi = \mathbb{I} \Omega + m \chi \times v, \quad P = -m \chi \times \Omega + \bar{m} v, \]
the Hamiltonian of the system is
\[ h(\Pi, P, \Gamma, x_3) := \frac{1}{2} (\Pi \cdot (\mathbb{J}^{-1} \Pi) + 2km/\Pi \cdot (P \times \chi) + P \cdot (\mathbb{M}^{-1} P)) + mgl \chi \cdot \Gamma + \bar{m}g x_3 \]
with
\[ \mathbb{J} := \text{diag} \left( I_1 - \frac{m^2 l^2}{\bar{m}}, I_1 - \frac{m^2 l^2}{\bar{m}}, I_3 \right), \quad \mathbb{M} := \text{diag} \left( \bar{m} - \frac{m^2 l^2}{I_1}, \bar{m} - \frac{m^2 l^2}{I_1}, \bar{m} \right) \]
Then the equations of motion are written as the Lie–Poisson equation on \( (\mathfrak{se}(3) \ltimes \mathbb{R}^4)^* \):
\[
\begin{align*}
\dot{\Pi} &= \Pi \times \frac{\partial h}{\partial \Pi} + P \times \frac{\partial h}{\partial P} + \Gamma \times \frac{\partial h}{\partial \Gamma}, \\
\dot{P} &= P \times \frac{\partial h}{\partial P} - \frac{\partial h}{\partial x_3} \Gamma, \\
\dot{\Gamma} &= \Gamma \times \frac{\partial h}{\partial \Pi}, \\
\dot{x}_3 &= \Gamma \cdot \frac{\partial h}{\partial P},
\end{align*}
\]
The main goal of [9] is to stabilize the upright position of the heavy top by applying control \( u \) to the base, i.e., the second equation of \( (29) \) is replaced by
\[
\hat{P} = P \times \frac{\partial h}{\partial \Pi} - \frac{\partial h}{\partial x_3} \Gamma + u,
\]
Specifically, the control \( u \) was broken into two as \( u = u^p + u^k \), corresponding to the potential and kinetic shaping, with the potential part being \( u^p = \frac{\partial h}{\partial x_3} \Gamma = \bar{m} g \), so that the Lie–Poisson equation \( (29) \) now becomes
\[
\dot{\Pi} = \Pi \times \frac{\partial h}{\partial \Pi} + P \times \frac{\partial h}{\partial P} + \Gamma \times \frac{\partial h}{\partial \Gamma}, \quad \dot{P} = P \times \frac{\partial h}{\partial \Pi} + u^k, \quad \dot{\Gamma} = \Gamma \times \frac{\partial h}{\partial \Pi}, \quad (30)
\]
where we dropped the equation for \( x_3 \) because it is now decoupled from the rest. In [9], it is found, via the method of controlled Lagrangians [4, 5], applying the control
\[
u^k = (\rho - \bar{m})(\tilde{v} - v \times \Omega)\]
with \( \rho \in \mathbb{R} \) renders the system \( (30) \) the Lie–Poisson equation on \( (se(3) \times \mathbb{R}^3)^* \) with a new control Hamiltonian \( h_c: (se(3) \times \mathbb{R}^3)^* \to \mathbb{R} \) given by
\[
h_c(\Pi, P, \Gamma) = \frac{1}{2}(\Pi \cdot (\mathbb{J}_c^{-1} \Pi) + 2k_m l_m \cdot (P \times \Pi) + P \cdot (M_c^{-1} P)) + mgl \cdot \Gamma \quad (31)
\]
with
\[
\mathbb{J}_c := \text{diag}\left(I_1 - \frac{m^2 l^2}{\rho}, I_1 - \frac{m^2 l^2}{\rho}, I_3\right), \quad M_c := \text{diag}\left(\rho - \frac{m^2 l^2}{I_1}, \rho - \frac{m^2 l^2}{I_1}\right).
\]
Then the equations of motion are given by the Lie–Poisson equation
\[
\dot{\mu} = \{\mu, h_c\}_-
\]
with \( \mu = (\Pi, P, \Gamma) \in (se(3) \times \mathbb{R}^3)^* \) and the following (−)-Lie–Poisson bracket on \( (se(3) \times \mathbb{R}^3)^* \):
\[
\{f, g\}_- (\Pi, P, \Gamma) = -\left\langle \Pi, \frac{\partial f}{\partial \Pi} \times \frac{\partial g}{\partial \Pi} \right\rangle - \left\langle P, \frac{\partial f}{\partial \Pi} \times \frac{\partial g}{\partial P} - \frac{\partial g}{\partial \Pi} \times \frac{\partial f}{\partial P} \right\rangle - \left\langle \Gamma, \frac{\partial f}{\partial \Pi} \times \frac{\partial g}{\partial \Gamma} - \frac{\partial g}{\partial \Pi} \times \frac{\partial f}{\partial \Gamma} \right\rangle, \quad (32)
\]
which, incidentally, is identical to the Lie–Poisson bracket given in Thiffeault and Morrison [65] for a rigid body insulator that is acted on by an electric field as well as gravity (see also Thiffeault and Morrison [66, 67]). More explicitly, we have
\[
\dot{\Pi} = \Pi \times \frac{\partial h_c}{\partial \Pi} + P \times \frac{\partial h_c}{\partial P} + \Gamma \times \frac{\partial h_c}{\partial \Gamma}, \quad \dot{P} = P \times \frac{\partial h_c}{\partial \Pi}, \quad \dot{\Gamma} = \Gamma \times \frac{\partial h_c}{\partial \Pi}. \quad (33)
\]
Noting that this is the (−)-Lie–Poisson bracket [8], we find that the corresponding structure constants \( \{e^k_{ij}\}_{i,j,k \in \{1,...,9\}} \) satisfy
\[
\mu_k e^k_{ij} = -\begin{bmatrix}
\hat{\Pi} & \hat{P} & \hat{\Gamma} \\
\hat{P} & 0 & 0 \\
\hat{\Gamma} & 0 & 0
\end{bmatrix}.
\]
The Lie–Poisson bracket \( (32) \) possesses the following Casimirs:
\[
f_1 = \|P\|^2, \quad f_2 = \|P \times \Gamma\|^2, \quad f_3 = \|\Gamma\|^2. \quad (34)
\]
Let us anti-reduce the above Lie–Poisson system using the momentum map $M_-$ from (9). Noting that $g = \mathfrak{se}(3) \times \mathbb{R}^3 \cong \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$, let us write $q = (a_1, a_2, a_3) \in g$ and $p = (b_1, b_2, b_3) \in g^*$ with $a_i, b_i \in \mathbb{R}^3$ for $i \in \{1, 2, 3\}$. Then we can write the momentum map $M_-$ as

$$M_-(q, p) = -(a_1 \times b_1 + a_2 \times b_2 + a_3 \times b_3, a_1 \times b_2, a_1 \times b_3).$$

Therefore, defining the Hamiltonian $H: T^*(\mathfrak{se}(3) \times \mathbb{R}^3) \to \mathbb{R}$ as $H(q, p) = h_c(M_-(q, p))$, the anti-reduced system is given as

$$\frac{\dot{q}}{\partial H} = -\partial H = -\frac{\partial H}{\partial q}.$$

Using the structured constants from above, we find the Killing form

$$\kappa(x, y) = -6(x_1y_1 + x_2y_2 + x_3y_3),$$

which is degenerate. Therefore, by Proposition 4, we have an additional invariant for the anti-reduced system:

$$\kappa(q, p) = -6(q_1^2 + q_2^2 + q_3^2).$$

Following [9], the parameters are chosen as follows: $M = 0.44 \text{ [kg]}$, $m = 0.7 \text{ [kg]}$, $I_1 = I_2 = 0.2 \text{ kg} \cdot \text{m}^2$, $I_3 = 0.24 \text{ kg} \cdot \text{m}^2$, $l = 0.215 \text{ [m]}$, $g = 9.8 \text{ [m/s}^2\text{]}$. The parameter $\rho$ was chosen such that $\rho = 0.9m^2l^2/I_1$ to ensure stability of the upright position. The initial condition is $\Omega(0) = (0.1, 0.2, 0.1), v(0) = 0$, and $\Gamma(0) = (\cos \theta_0 \sin \varphi_0, \sin \theta_0 \sin \varphi_0, \cos \varphi_0)$ with $\theta_0 = \pi/3$ and $\varphi_0 = \pi/20$.

To get the initial conditions for the anti-reduced system (35), we set $a_1(0) = \Gamma(0) \times P(0), b_1(0) = (0, 0, 0)$ and solved $M_-(q(0), p(0)) = (\Pi(0), P(0), \Gamma(0))$ for the remaining values $a_2(0), a_3(0), b_2(0), b_3(0)$ of $(q(0), p(0))$.

We solved the anti-reduced system (35) using the 4th order Gauss–Legendre method, and also solved the Lie–Poisson system (33) directly using the 4th order Runge–Kutta method for comparison.

Figure 7 shows the time evolutions of the relative errors of the Hamiltonian $h_c$ and the Casimirs $f_1, f_2, f_3$. Just as in the Kida vortex case, we observe drifts in addition to oscillations in all the invariants for the Runge–Kutta solution, whereas we see that the proposed collective integrator preserves these invariants while oscillating in a relatively thin band. Relative errors for the momentum map $F_0$ and the invariant $F_0$ are shown in Figure 8. We observe that $F_0$ is oscillating around the exact value with an increasingly amplitude, whereas the Killing form invariant $F_1$ is oscillating with more or less the same amplitude. Since both of them are quadratic in $(q, p)$, they are invariants of the Gauss–Legendre method, any error must be due to roundoff and/or the nonlinear solver used in each step.

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References

Figure 7. Time evolutions of relative errors in Hamiltonian \(h\) and three Casimirs \(f_1, f_2, f_3\) from the Heavy Top on a movable base system. The solid blue curve is the Runge–Kutta method directly applied to Lie–Poisson equation \((33)\) whereas the dashed red curve is the Gauss–Legendre method of order 4 applied to the anti-reduced system. The solutions are shown for the time interval \(0 \leq t \leq 30\) with time step \(\Delta t = 0.01\).


Figure 8. Time evolutions of relative errors in momentum map $F_0$ and the additional invariant $\kappa(q,q)$ computed by the Gauss–Legendre method of order 4 applied to the anti-reduced system for heavy top on a movable base. The solutions are shown for the time interval $0 \leq t \leq 30$ with time step $\Delta t = 0.01$.


