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ABSTRACT

The classical equations of motion of a charged particle in a spherically symmetric distribution of magnetic monopoles can be transformed into a system of linear equations, thereby providing a type of integrability. In the case of a single monopole, the solution was given long ago by Poincaré. In the case of a uniform distribution of monopoles, the solution can be expressed in terms of parabolic cylinder functions (essentially the eigenfunctions of an inverted harmonic oscillator). This solution is relevant to recent studies of nonassociative star products, symplectic lifts of twisted Poisson structures, and fluids and plasmas of electric and magnetic charges.

I. INTRODUCTION

Recent studies of nonassociative star products by Bakas and Lüst[1] have led to models involving the classical motion of charged particles in distributions of magnetic monopoles. Such models are nonassociative because the obvious definition of the Poisson bracket fails to satisfy the Jacobi identity.[2,3] In these models, this failure can be expressed in terms of a closed three-form, essentially $\nabla \cdot \mathbf{B}$, which qualifies the structure as twisted Poisson. Such systems pose a challenge to quantization since all the usual methods depend in one way or another on the Jacobi identity. Star quantization of monopole systems has been of interest for some time; see, for example, the work of Cariñena *et al.*[4] and Soloviev,[5] but the use of nonassociative star products for distributions of monopoles is more recent.

In a different approach, Kupriyanov and Szabo[6] tackled such systems directly and provided a symplectic lift in a phase space of doubled dimensionality, that is, in a fully associative framework in which the usual Jacobi identity is valid. In addition, some work has been done on the continuum Poisson structures associated with fluids and plasmas containing distributions of monopoles. It turns out that in most interesting circumstances, these (continuum) Poisson structures are also nonassociative.[3] In fact, Lainz, Sardón, and Weinstein[7] have shown that twisted Poisson structures for particle motion correspond to fluid structures that are not only not Poisson, they are not even twisted Poisson. We ourselves have had a long-standing interest in Poisson structures in fluids and plasmas[2,8–10] and in charged particle motion in magnetic fields.[11]

The case of spherically symmetric distributions of monopoles is especially interesting because the magnetic field is rotationally invariant, and in a symplectic setting, this would be enough to produce a conserved angular momentum that would lead to complete integrability. Indeed, if the distribution consists of a single monopole, this is exactly what happens; the problem of a single charged particle moving in the field of a single monopole was first solved by Poincaré,[12] who exhibited the conserved angular momentum and showed that the motion lies on a cone whose apex is the monopole. Particle motion in other spherically symmetric distributions of monopoles, including the uniform distribution, has resisted a complete solution. Bakas and Lüst[1] produced some first integrals and derived some quantitative constraints on the motion but did not obtain a complete solution, and Kupriyanov and Szabo,[6] with their doubled symplectic structure, found some integrals in involution, but not enough to produce complete integrability (they needed twice the usual number because of their doubled symplectic structure, and in particular, they did not find an angular momentum vector).

In this article, we will exhibit a transformation that makes the equations of motion of a charged particle in the field of a spherically symmetric distribution of magnetic monopoles a linear system. In the case of a uniform distribution of monopoles, we will show that the solution can be given in terms of parabolic cylinder functions, that is, essentially the energy eigenfunctions of an inverted harmonic oscillator. In addition, we find an S-matrix connecting asymptotic states as $t \longrightarrow \pm\infty$. We do not call this complete integrability because the usual definition of integrability[13] requires a symplectic structure, but the system is completely integrable in most ordinary senses.

II. THE SOLUTION

A. The setup and some first integrals

Before specializing to the spherically symmetric case, we note that the general nonrelativistic equation of motion for a particle of mass m and electric charge e in any magnetic field $\mathbf{B}$ is assumed to be

$$\ddot{\mathbf{r}} = \frac{e}{mc}\mathbf{v} \times \mathbf{B}, \tag{1}$$

regardless of whether $\nabla \cdot \mathbf{B} = 0$ or not. Here, $\mathbf{v} = \dot{\mathbf{r}}$ is the velocity and $\ddot{\mathbf{r}} = \dot{\mathbf{v}}$ is the acceleration of the particle. In the symplectic setting, that is, when $\nabla \cdot \mathbf{B} = 0$, the equations of motion (1) preserve the volume form $d^3\mathbf{r}\, d^3\mathbf{p}$ in phase space, according to the usual Liouville theorem, and moreover, this form is proportional to $d^3\mathbf{r}\, d^3\mathbf{v}$. In the nonsymplectic setting ($\nabla \cdot \mathbf{B} \neq 0$), the vector potential and, hence, the canonical momentum $\mathbf{p} = m\mathbf{v} - (e/c)\mathbf{A}$ are not defined. The form $d^3\mathbf{r}\, d^3\mathbf{v}$, however, is defined and is preserved by the flow, as shown in the following calculation:

$$\frac{\partial}{\partial \mathbf{r}} \cdot \dot{\mathbf{r}} + \frac{\partial}{\partial \mathbf{v}} \cdot \dot{\mathbf{v}} = 0. \tag{2}$$

Thus, the system possesses a preserved volume form in phase space even in the nonsymplectic setting.

We now specialize to a spherically symmetric distribution of magnetic monopoles with density $\rho(r)$ so that Maxwell's equation is $\nabla \cdot \mathbf{B} = 4\pi\rho(r)$. We denote the magnetic charge inside radius r by $g(r)$,

$$g(r) = 4\pi \int_0^r r'^2\, dr'\, \rho(r'), \tag{3}$$

so that Gauss's law gives

$$\mathbf{B}(\mathbf{r}) = g(r)\frac{\mathbf{r}}{r^3}. \tag{4}$$

Then, the equation of motion in this magnetic field is

$$\ddot{\mathbf{r}} = h(r)\mathbf{v} \times \mathbf{r} \tag{5}$$

with

$$h(r) = \frac{e}{mc}\frac{g(r)}{r^3}. \tag{6}$$

Parts of the solution to this system have been given by Bakas and Lüst.[1]

It follows from (5) that $\mathbf{r} \cdot \dot{\mathbf{v}} = \mathbf{v} \cdot \dot{\mathbf{v}} = 0$, which implies

$$\frac{dv^2}{dt} = 2\mathbf{v} \cdot \dot{\mathbf{v}} = 0 \tag{7}$$

or

$$\mathbf{v}^2 = v^2 = v_0^2 = \text{ const.} \tag{8}$$

This is the first integral of conservation of energy, where the 0-subscript indicates initial conditions at $t = 0$.

As a first special case, we take $\mathbf{v}_0 = 0$, which by (8) implies $\mathbf{v} = 0$ and $\mathbf{r} = \text{const.}$ for all t. We, henceforth, dispense with this case by assuming $\mathbf{v}_0 \neq 0$, which implies that $\mathbf{v} \neq 0$ for all t.

We obtain a second first integral by noting that

$$\frac{d}{dt}|\mathbf{v} \times \mathbf{r}|^2 = 2(\dot{\mathbf{v}} \times \mathbf{r}) \cdot (\mathbf{v} \times \mathbf{r}) = 2\left\|\begin{matrix} \dot{\mathbf{v}} \cdot \mathbf{v} & \dot{\mathbf{v}} \cdot \mathbf{r} \\ \mathbf{r} \cdot \mathbf{v} & \mathbf{r} \cdot \mathbf{r} \end{matrix}\right\| = 0 \tag{9}$$

so that $|\mathbf{v} \times \mathbf{r}|^2 = |\mathbf{v}_0 \times \mathbf{r}_0|^2$. This leads to a second special case, in which $\mathbf{v}_0 \times \mathbf{r}_0 = 0$, which implies that $\mathbf{v} \times \mathbf{r} = 0$ for all t. This in turn implies that $\mathbf{r}$ and $\mathbf{v}$ are parallel with $\dot{\mathbf{v}} = 0$ so that $\mathbf{r}(t) = \mathbf{v}_0 t + \text{const.}$, and the particle moves on a radial line with constant velocity. We, henceforth, dispense with this special case by assuming that $\mathbf{v}_0 \times \mathbf{r}_0 \neq 0$, which implies that $\mathbf{v} \times \mathbf{r} \neq 0$ for all t.

Since $(d/dt)r^2 = 2\mathbf{r}\cdot\mathbf{v}$ and $(d/dt)(\mathbf{r}\cdot\mathbf{v}) = \mathbf{v}^2 = v_0^2$, we have $\mathbf{r}\cdot\mathbf{v} = \mathbf{r}_0\cdot\mathbf{v}_0 + v_0^2 t$, and we see that there exists a unique time t such that $\mathbf{r}\cdot\mathbf{v} = 0$. We, henceforth, take this time to be $t = 0$ so that $\mathbf{r}_0\cdot\mathbf{v}_0 = 0$ and

$$\mathbf{r}\cdot\mathbf{v} = v_0^2 t. \tag{10}$$

Using $(d/dt)r^2 = 2\mathbf{r}\cdot\mathbf{v}$ together with (10) implies

$$r^2 = r_0^2 + v_0^2 t^2, \tag{11}$$

and we see that the particle reaches a minimum distance from the origin r_0 at $t = 0$. This minimum distance cannot be zero since we are assuming that $\mathbf{r}_0\times\mathbf{v}_0 \neq 0$. In addition, note that with this choice of initial time,

$$|\mathbf{v}\times\mathbf{r}|^2 = v_0^2 r_0^2. \tag{12}$$

B. The frame

We also note that

$$|\mathbf{v}\times(\mathbf{v}\times\mathbf{r})|^2 = v^2|\mathbf{v}\times\mathbf{r}|^2 = v_0^2|\mathbf{v}_0\times\mathbf{r}_0|^2 = v_0^4 r_0^2 = \text{ const}, \tag{13}$$

and that under our assumptions, this constant is nonzero. Thus, we have three nonvanishing, mutually orthogonal vectors,

$$\mathbf{F}_1 = \mathbf{v}\times(\mathbf{v}\times\mathbf{r}), \qquad \mathbf{F}_2 = \mathbf{v}\times\mathbf{r}, \qquad \mathbf{F}_3 = -\mathbf{v}, \tag{14}$$

whose magnitudes are constants. These vectors must evolve by means of a time-dependent rotation, much as in rigid body theory.

To find this rotation, we differentiate the three vectors with respect to time. First, $d\mathbf{v}/dt$ is given by (5). Next, we have

$$\frac{d}{dt}(\mathbf{v}\times\mathbf{r}) = \dot{\mathbf{v}}\times\mathbf{r} = h(r)[(\mathbf{r}\cdot\mathbf{v})\mathbf{r} - r^2\mathbf{v}]. \tag{15}$$

We use the identity

$$\mathbf{r} = \frac{1}{v^2}[\mathbf{v}(\mathbf{v}\cdot\mathbf{r}) - \mathbf{v}\times(\mathbf{v}\times\mathbf{r})] \tag{16}$$

in this to express the result as a linear combination of the three $\mathbf{F}$'s, obtaining

$$\frac{d}{dt}(\mathbf{v}\times\mathbf{r}) = -\frac{h(r)}{v^2}[(\mathbf{v}\times\mathbf{r})^2\,\mathbf{v} + (\mathbf{v}\cdot\mathbf{r})\,\mathbf{v}\times(\mathbf{v}\times\mathbf{r})] \tag{17}$$

or, with the help of (8), (10), and (12),

$$\frac{d}{dt}(\mathbf{v}\times\mathbf{r}) = -h(r)[r_0^2\,\mathbf{v} + t\mathbf{v}\times(\mathbf{v}\times\mathbf{r})]. \tag{18}$$

As for the third vector, we have

$$\frac{d}{dt}[\mathbf{v}\times(\mathbf{v}\times\mathbf{r})] = \mathbf{v}\times\frac{d}{dt}(\mathbf{v}\times\mathbf{r}) = -h(r)t\mathbf{v}\times[\mathbf{v}\times(\mathbf{v}\times\mathbf{r})] = h(r)tv_0^2\,\mathbf{v}\times\mathbf{r}. \tag{19}$$

We summarize (5), (17), and (19) by writing

$$\frac{d}{dt}\begin{pmatrix}\mathbf{F}_1 & \mathbf{F}_2 & \mathbf{F}_3\end{pmatrix} = \begin{pmatrix}\mathbf{F}_1 & \mathbf{F}_2 & \mathbf{F}_3\end{pmatrix} h(r)\begin{pmatrix} 0 & -t & 0 \\ tv_0^2 & 0 & -1 \\ 0 & r_0^2 & 0\end{pmatrix}. \tag{20}$$

We now define orthonormal unit vectors of a "body frame,"

$$\hat{\mathbf{f}}_1 = \frac{\mathbf{F}_1}{v_0^2 r_0}, \qquad \hat{\mathbf{f}}_2 = \frac{\mathbf{F}_2}{v_0 r_0}, \qquad \hat{\mathbf{f}}_3 = \frac{\mathbf{F}_3}{v_0}. \tag{21}$$

This frame is right-handed,

$$\hat{\mathbf{f}}_i\times\hat{\mathbf{f}}_j = \epsilon_{ijk}\,\hat{\mathbf{f}}_k, \tag{22}$$

where we use the summation convention. In terms of these vectors, (20) becomes

$$\frac{d}{dt}\begin{pmatrix}\hat{\mathbf{f}}_1 & \hat{\mathbf{f}}_2 & \hat{\mathbf{f}}_3\end{pmatrix} = \begin{pmatrix}\hat{\mathbf{f}}_1 & \hat{\mathbf{f}}_2 & \hat{\mathbf{f}}_3\end{pmatrix} h(r)\begin{pmatrix} 0 & -v_0 t & 0 \\ v_0 t & 0 & -r_0 \\ 0 & r_0 & 0\end{pmatrix}. \tag{23}$$

We write this equivalently as

$$\frac{d\hat{\mathbf{f}}_i}{dt} = \hat{\mathbf{f}}_j\,\Omega_{ji}, \tag{24}$$

where the matrix Ω is defined by (23) [including the factor $h(r)$].

Note that since r is a known function of time [see (11)], Ω is too. Thus, (23) or (24) is a system of linear differential equations with time-dependent coefficients. Given its solution, one can use (16), with (8) and (10), to write our solution of (5) as

$$\mathbf{r} = -v_0 t\,\hat{\mathbf{f}}_3 - r_0\,\hat{\mathbf{f}}_1. \tag{25}$$

Equation (24) defines the Frenet–Serret apparatus for a space curve, with $s = v_0 t$ being the arc length. Because $\hat{\mathbf{f}}_3$ is anti-tangent to the curve, the curvature is $\kappa(s) = h(r)r_0/v_0$ and $\hat{\mathbf{f}}_2$ is the unit normal. Similarly, $\hat{\mathbf{f}}_1$ is the binormal and the torsion $\tau(s) = -h(r)t = -h(r)s/v_0$. Hence, $\tau/\kappa = -v_0 t/r_0 = -s/r_0$, from which it follows by an observation due to Enneper that the orbits are geodesics on a cone (see Ref. 14, p. 47). This generalizes Poincaré's result for a single monopole.[12] Rather than going through the details of Enneper's analysis, we will provide a simpler and more direct derivation of these conclusions in Sec. II I.

We will now put (24) into a more convenient form with some ideas from rigid body theory. We define a "space frame" $\hat{\mathbf{e}}_i$ as the unit vectors of the coordinate axes (which are time-independent), and we define the angular velocity $\boldsymbol{\omega}$ of the body frame relative to the space frame by

$$\frac{d\hat{\mathbf{f}}_i}{dt} = \boldsymbol{\omega}\times\hat{\mathbf{f}}_i. \tag{26}$$

Now, writing $\boldsymbol{\omega} = \omega_k\hat{\mathbf{f}}_k$ and using (22), we obtain $d\hat{\mathbf{f}}_i/dt = \omega_k\,\epsilon_{kij}\,\hat{\mathbf{f}}_j$, which, combined with (24), gives

$$\Omega_{ij} = -\epsilon_{ijk}\,\omega_k, \qquad \omega_i = -\frac{1}{2}\epsilon_{ijk}\,\Omega_{jk}, \tag{27}$$

the usual isomorphism between $\mathfrak{so}(3)$ and $\mathbb{R}^3$. Finally, comparing this with (23), we find

$$\boldsymbol{\omega} = h(r)(r_0\,\hat{\mathbf{f}}_1 + v_0 t\,\hat{\mathbf{f}}_3). \tag{28}$$

Now, by combining (25) and (28), we see that

$$\boldsymbol{\omega} = -h(r)\mathbf{r} = -\frac{e\mathbf{B}(r)}{mc}, \tag{29}$$

so the instantaneous axis of rotation of the body frame, apart from sign, is in the direction of the magnetic field, that is, in the radial direction, while its magnitude is the instantaneous gyrofrequency at the particle position. This would obviously be true in a uniform magnetic field, but the fact that it generalizes to the nonuniform fields considered here was not obvious to us.

Now, we break the velocity of the particle into its parallel and perpendicular components along the radial direction so that $v^2 = v_\parallel^2 + v_\perp^2$. We note that (10) implies $rv_\parallel = v_0^2 t$ so that

$$v_\perp = \sqrt{v_0^2 - v_\parallel^2} = \frac{v_0 r_0}{r}, \tag{30}$$

where we use (11). This gives us the effective gyroradius,

$$r_g = \frac{v_\perp}{|\boldsymbol{\omega}|} = \frac{v_0 r_0}{r^2 h(r)}. \tag{31}$$

Now, let us define a rotation operator R by $\hat{\mathbf{f}}_i = R\hat{\mathbf{e}}_i$, that is, R maps the space frame into the body frame. We write the matrix elements of R in the space frame as $R_{ij} = \hat{\mathbf{e}}_i\cdot(R\hat{\mathbf{e}}_j) = \hat{\mathbf{e}}_i\cdot\hat{\mathbf{f}}_j$. Then, by inserting a resolution of the identity, we have

$$\frac{d\hat{\mathbf{f}}_i}{dt} = \dot{R}\hat{\mathbf{e}}_i = \hat{\mathbf{f}}_j[\hat{\mathbf{f}}_j\cdot(\dot{R}\hat{\mathbf{e}}_i)] = \hat{\mathbf{f}}_j[(R\hat{\mathbf{e}}_j)\cdot(\dot{R}\hat{\mathbf{e}}_i)] = \hat{\mathbf{f}}_j[\hat{\mathbf{e}}_j\cdot(R^{-1}\dot{R}\hat{\mathbf{e}}_i)] \tag{32}$$

so that $\Omega = R^{-1}\dot{R}$ or

$$\dot{R} = R\Omega, \tag{33}$$

where we now write R for the matrix with components R_{ij}.

This is a convenient form for the linearized equations; the solution is a curve $R(t) \in SO(3)$. The matrix Ω belongs to the Lie algebra, a convenient basis for which is the set of matrices J_i, defined by $(J_i)_{jk} = -\epsilon_{ijk}$. These satisfy

$$[J_i, J_j] = \epsilon_{ijk} J_k, \tag{34}$$

and we have

$$\Omega = h(r)(r_0 J_1 + v_0 t J_3), \tag{35}$$

which has the same content as (28).

C. The lift into $SU(2)$

To solve (33) with Ω given by (35), we lift the curve $R(t) \in SO(3)$ to a curve $u(t) \in SU(2)$. The projection: $SU(2) \longrightarrow SO(3)$ is

$$R_{ij} = \frac{1}{2}\mathrm{tr}(u^\dagger \sigma_i u \sigma_j), \tag{36}$$

where $u \in SU(2)$ and σ_i are the Pauli matrices. The map: $u \mapsto R(u)$ is a homomorphism. In addition, the map $J_i \mapsto -(i/2)\sigma_i$ is the isomorphism between the Lie algebras. Thus, the lifted equation of motion is

$$\dot{u} = -\frac{i}{2}h(r)\, u(r_0\, \sigma_1 + v_0 t\, \sigma_3). \tag{37}$$

We call the solution $u(t)$ that satisfies $u(0) = 1$ "the fundamental solution," where 1 is the identity matrix. It suffices to determine this solution since any other that satisfies $u(0) \neq 1$ is given by $u(t) = u(0)u_f(t)$, where $u_f(t)$ is the fundamental solution. In the fundamental solution, the space and body frames coincide at $t = 0$.

It is slightly more convenient to work with $w = u^\dagger$, a substitution that converts right-actions into left-actions. In addition, we note that the second column of an element of $SU(2)$ is the time-reversed image of the first column, that is, $w_{12} = -\bar{w}_{21}$, $w_{22} = \bar{w}_{11}$, where the overbar indicates complex conjugation, so it suffices to solve for the first column of w. Thus, the equations of motion become

$$\frac{d}{dt}\begin{pmatrix} w_{11} \\ w_{21} \end{pmatrix} = \frac{i}{2}h(r)\begin{pmatrix} v_0 t & r_0 \\ r_0 & -v_0 t \end{pmatrix}\begin{pmatrix} w_{11} \\ w_{21} \end{pmatrix}. \tag{38}$$

D. Uniform sphere of monopolium

We now specialize to the case $\rho(r) = \rho_0 = \text{const.}$, which implies $g(r) = (4\pi/3)\rho_0 r^3$ and $h(r) = (4\pi/3)(\rho_0 e/mc) = h_0 = \text{const.}$ In the following, we assume $h_0 > 0$; the case $h_0 < 0$ is handled similarly. The problem has one dimensionless parameter, which we denote by

$$p = \sqrt{h_0 r_0^2/4v_0}. \tag{39}$$

An interpretation of this parameter is given below.

It is interesting that the equations of motion (38) now become the normal form for Landau–Zener transitions in one dimension.[15–17] Taking the second derivative causes these equations to decouple, and we obtain

$$\frac{d^2 w_{11}}{dt^2} + \left[\frac{h_0^2}{4}(r_0^2 + v_0^2 t^2) - \frac{i h_0 v_0}{2}\right] w_{11} = 0, \tag{40a}$$

$$\frac{d^2 w_{21}}{dt^2} + \left[\frac{h_0^2}{4}(r_0^2 + v_0^2 t^2) + \frac{i h_0 v_0}{2}\right] w_{21} = 0. \tag{40b}$$

The solution can be expressed in terms of parabolic cylinder functions, essentially the energy eigenfunctions for an inverted harmonic oscillator. The properties of these functions that are needed for this paper are summarized in the Appendix.

By the substitution

$$z = e^{-i\pi/4}\omega_0 t, \tag{41}$$

where $\omega_0 = \sqrt{h_0 v_0}$, Eqs. (40) become

$$\frac{d^2 w_{11}}{dz^2} + \left(ip^2 + \frac{1}{2} - \frac{z^2}{4}\right) w_{11} = 0, \tag{42a}$$

$$\frac{d^2 w_{21}}{dz^2} + \left(ip^2 - \frac{1}{2} - \frac{z^2}{4}\right) w_{21} = 0. \tag{42b}$$

According to (A1), the solutions can be written as

$$w_{11}(t) = A_1 D_\nu(z) + B_1 D_\nu(-z), \tag{43a}$$

$$w_{21}(t) = A_2 D_{\nu-1}(z) + B_2 D_{\nu-1}(-z), \tag{43b}$$

where D_ν is a parabolic cylinder function, where $A_{1,2}$ and $B_{1,2}$ are constants, and where

$$\nu = ip^2. \tag{44}$$

The solutions (43) must satisfy (38) with $h(r) = h_0$. By the substitution (41), the latter equations can be written as

$$\frac{d}{dz}\begin{pmatrix} w_{11} \\ w_{21} \end{pmatrix} = \begin{pmatrix} -z/2 & -e^{-i\pi/4} p \\ -e^{-i\pi/4} p & z/2 \end{pmatrix}\begin{pmatrix} w_{11} \\ w_{21} \end{pmatrix}. \tag{45}$$

Now, substituting (43) into these and using the recursion relations (A3), we find that the four constants are related by

$$A_2 = pe^{-i\pi/4} A_1, \qquad B_2 = -pe^{-i\pi/4} B_1. \tag{46}$$

Finally, by imposing the initial conditions $w_{11}(0) = 1$, $w_{21}(0) = 0$ of the fundamental solution, we find the first column of w (and the second row of u) in the form

$$w_{11}(t) = u_{22}(t) = A[D_\nu(z) + D_\nu(-z)], \tag{47a}$$

$$w_{21}(t) = -u_{21}(t) = Ape^{-i\pi/4}[D_{\nu-1}(z) - D_{\nu-1}(-z)], \tag{47b}$$

where

$$A = \frac{\Gamma((1-\nu)/2)}{2^{(\nu+1)/2}\sqrt{2\pi}} = \frac{1}{2D_\nu(0)}. \tag{48}$$

Then, by using $w_{22} = \bar{w}_{11}$, $w_{12} = -\bar{w}_{21}$, we find the second column of w (and the first row of u),

$$w_{12}(t) = -u_{12}(t) = \frac{B}{pe^{-i\pi/4}}[D_\nu(z) - D_\nu(-z)], \tag{49a}$$

$$w_{22}(t) = u_{11}(t) = B[D_{\nu-1}(z) + D_{\nu-1}(-z)], \tag{49b}$$

where

$$B = \frac{\Gamma(1-\nu/2)}{2^{\nu/2}\sqrt{2\pi}} = \frac{1}{2D_{\nu-1}(0)}. \tag{50}$$

The constants A and B satisfy the identities,

$$\frac{1}{|A|^2} \pm \frac{p^2}{|B|^2} = 4e^{\pm p^2\pi/2}, \qquad AB = \frac{\Gamma(1-\nu)}{2\sqrt{2\pi}}. \tag{51}$$

In taking the complex conjugates of w_{11} and w_{21}, it helps to note that $\overline{D_\nu(z)} = D_{\bar{\nu}}(\bar{z}) = D_{-\nu}(iz)$, and one must work with the linear dependencies (A2) satisfied by the functions D_ν to bring the results into the form shown. One must also use various identities satisfied by the Γ-function. To check the results, we can show that $\det w = 1$ for all t; in this calculation, $\det w$ turns out to be proportional to the Wronskian of the solutions $D_\nu(\pm z)$.

The matrix $u(t) \in SU(2)$, given by (47) and (49), can be projected onto the matrix $R(t) \in SO(3)$ via (36). Actually, it is easier to write u in axis-angle form and then to use the same axis and angle for R. The matrix $R(t)$, so determined, is the fundamental solution; from it, the time evolution of the vectors $\hat{\mathbf{f}}_i$ follows from the definition of R, that is, $\hat{\mathbf{f}}_i(t) = R(t)\hat{\mathbf{e}}_i$. To put this in matrix form, we insert a resolution of the identity,

$$\hat{\mathbf{f}}_i(t) = \hat{\mathbf{e}}_j[\hat{\mathbf{e}}_j \cdot (R(t)\hat{\mathbf{e}}_i)] = \hat{\mathbf{e}}_j R_{ji}(t). \tag{52}$$

From the evolution of $\hat{\mathbf{f}}_i$, we obtain that of the $\mathbf{F}_i$ by (32). The position as a function of time $\mathbf{r}(t)$ is given in terms of these vectors by (25). We omit the details and focus, instead, on the asymptotic properties of the solution.

E. Qualitative features of the asymptotic motion

It is easy to derive some qualitative and semi-quantitative features of the motion when t is large. In a uniform magnetic field, the motion has gyrofrequency $\omega_g = e|\mathbf{B}|/mc$ and gyroradius $r_g = v_\perp/\omega_g$. In the actual motion for which $\mathbf{B}$ is not constant, the motion will be qualitatively similar to that in a uniform field if $\mathbf{B}$ does not change much over the distance r_g. We will call such motion "adiabatic" (a word usually defined in a symplectic context). For the sphere of monopolium, (29) gives $\omega_g = h_0 r$ and (31) gives $r_g = r_0 v_0/h_0 r^2$. Thus, as the particle moves to larger radii, the gyrofrequency goes to infinity as r and the gyroradius goes to zero as $1/r^2$. The condition that $|\mathbf{B}|$ does not change much in direction over a distance r_g is $r_g \ll r$, or

$$\left(\frac{r}{r_0}\right)^3 \gg \frac{v_0}{h_0 r_0^2} = \frac{1}{4p^2}. \tag{53}$$

Thus, if $p \ll 1$, the motion does not become adiabatic until r has reached a large multiple of r_0, while if $p \gg 1$, the motion is adiabatic for all t. A similar conclusion is reached regarding the change in the direction of $\mathbf{B}$.

From (11), we have

$$r(t) = v_0 t\sqrt{1 + \frac{r_0^2}{v_0^2 t^2}} \approx v_0 t + \frac{r_0^2}{2v_0 t} + \text{const.} + \cdots. \tag{54}$$

Then, we can integrate the gyrofrequency $\omega_g(t) = h_0 r(t)$ to find

$$\int^t dt'\, \omega_g(t') = \frac{h_0 v_0 t^2}{2} + \frac{h_0 r_0^2}{2v_0} \ln\, t + \text{other terms}. \tag{55}$$

We will see this phase $\phi(t)$ again soon in (56). It can be interpreted as the accumulated gyrophase in the asymptotic region.

F. The asymptotic behavior

To be more quantitative about the asymptotic behavior, we invoke the asymptotic forms of the parabolic cylinder functions (A6) and (A7). After some work, we find the asymptotic form of the matrix u when $t \longrightarrow \infty$, including corrections that go as $1/t$. We use the frequency $\omega_0 = \sqrt{h_0 v_0}$ and a dimensionless time $\tau = \omega_0 t$. We also define

$$\phi(t) = 2p^2\, \ln\, \tau + \tau^2/2. \tag{56}$$

Then, we find for $t \longrightarrow +\infty$,

$$u_{11}(t) = \frac{e^{-\pi p^2/4}}{2}\left(\frac{e^{-i\phi/2}}{A} + \frac{p^2 e^{i\pi/4}}{\bar{B}}\frac{e^{i\phi/2}}{\tau}\right), \tag{57a}$$

$$u_{12}(t) = \frac{e^{-\pi p^2/4}}{2}p\left(-\frac{e^{i\pi/4}}{\bar{B}}e^{i\phi/2} + \frac{e^{-i\phi/2}}{A\tau}\right), \tag{57b}$$

$$u_{21}(t) = \frac{e^{-\pi p^2/4}}{2}p\left(\frac{e^{-i\pi/4}}{B}e^{-i\phi/2} - \frac{e^{i\phi/2}}{\bar{A}\tau}\right), \tag{57c}$$

$$u_{22}(t) = \frac{e^{-\pi p^2/4}}{2}\left(\frac{e^{i\phi/2}}{\bar{A}} + \frac{p^2 e^{-i\pi/4}}{B}\frac{e^{-i\phi/2}}{\tau}\right). \tag{57d}$$

The forms shown make it easy to check that $u_{21} = -\bar{u}_{12}$ and $u_{22} = \bar{u}_{11}$. It is also easy to check that $\det u = 1$ to order $1/\tau$, with the help of the identities (51).

For negative times, we may use the identities

$$\begin{pmatrix} u_{11}(-t) & u_{12}(-t) \\ u_{21}(-t) & u_{22}(-t) \end{pmatrix} = \begin{pmatrix} u_{11}(t) & -u_{12}(t) \\ -u_{21}(t) & u_{22}(t) \end{pmatrix}, \tag{58}$$

which follow from (47) and (49). These are exact (not just asymptotic), but when $t \longrightarrow -\infty$, we can use these in conjunction with (57) to obtain the asymptotic forms for $t \longrightarrow -\infty$.

G. The S-matrix

The matrices $u(t)$ and $u(-t)$ do not have a limit as $t \longrightarrow \infty$ because of the rapidly varying phase $\phi(t)$, but their leading asymptotic forms (that is, neglecting terms that go as $1/t$) can be factored,

$$u(t) = m_+ \, q(\hat{\mathbf{z}}, \phi(t)), \qquad u(-t) = m_- \, q(\hat{\mathbf{z}}, \phi(t)), \tag{59}$$

in which the ϕ-dependence appears only in the second factor. Here, $q(\hat{\mathbf{n}}, \theta) \in SU(2)$ is notation for a spin rotation in axis-angle form,

$$q(\hat{\mathbf{n}}, \theta) = \cos\frac{\theta}{2} - i(\hat{\mathbf{n}} \cdot \boldsymbol{\sigma}) \sin\frac{\theta}{2}, \tag{60}$$

and the m-matrices are given by

$$m_+ = \frac{e^{-p^2\pi/4}}{2}\begin{pmatrix} 1/A & -pe^{i\pi/4}/\bar{B} \\ pe^{-i\pi/4}/B & 1/\bar{A} \end{pmatrix}, \qquad m_- = \frac{e^{-p^2\pi/4}}{2}\begin{pmatrix} 1/A & pe^{i\pi/4}/\bar{B} \\ -pe^{-i\pi/4}/B & 1/\bar{A} \end{pmatrix}. \tag{61}$$

The notation m is a mnemonic for Møller since these matrices are analogous to the Møller wave operator in scattering theory.[19] We can define $m_\pm(\pm t)$ for any $t > 0$ by $u(\pm t) = m_\pm(\pm t) q(\hat{\mathbf{z}}, \phi(t))$, whereupon the limits $m_\pm(\pm\infty)$ exist and are given by (61) (and denoted simply as $m_\pm$).

The quantities $u(t)$, $m_\pm(t)$, and $q(\hat{\mathbf{n}}, \phi)$ are elements of $SU(2)$. To see the effects of this in three dimensions, we write $R(t)$, $M_\pm(t)$, and $Q(\hat{\mathbf{n}}, \theta)$ for the corresponding elements of $SO(3)$. As for $Q(\hat{\mathbf{n}}, \theta)$, it is a rotation in axis-angle form with the same axis and angle as $q(\hat{\mathbf{n}}, \theta)$. Then, for $t > 0$, we have

$$R(t) = M_+(t) Q(\hat{\mathbf{z}}, \phi(t)), \qquad R(-t) = M_-(-t) Q(\hat{\mathbf{z}}, \phi(t)), \tag{62}$$

and (52) implies

$$\hat{\mathbf{f}}_i(t) = \hat{\mathbf{e}}_j [M_+(t) Q(\hat{\mathbf{z}}, \phi(t))]_{ji}, \qquad \hat{\mathbf{f}}_i(-t) = \hat{\mathbf{e}}_j [M_-(-t) Q(\hat{\mathbf{z}}, \phi(t))]_{ji}. \tag{63}$$

The vectors $\hat{\mathbf{f}}_i(\pm t)$ do not approach a limit as $t \longrightarrow \infty$; instead, $\hat{\mathbf{f}}_1$ and $\hat{\mathbf{f}}_2$ spin ever more rapidly about the direction $\hat{\mathbf{f}}_3$ as t gets larger. However, we can strip off this rapid evolution by defining for $t > 0$,

$$\hat{\mathbf{a}}_{+i}(t) = \hat{\mathbf{f}}_j(t) [Q(\hat{\mathbf{z}}, -\phi(t))]_{ji}, \qquad \hat{\mathbf{a}}_{-i}(-t) = \hat{\mathbf{f}}_j(-t) [Q(\hat{\mathbf{z}}, -\phi(t))]_{ji}. \tag{64}$$

Then, the limits $\hat{\mathbf{a}}_{\pm i}(\pm t)$ exist as $t \longrightarrow \infty$, what we will call simply $\hat{\mathbf{a}}_{\pm i}$. These are analogous to the asymptotic states in the interaction picture in scattering theory (a version of the "in states" and "out states").

Finally, we define the S-matrix by

$$\hat{\mathbf{a}}_{+i} = \hat{\mathbf{a}}_{-j} \, S_{ji}, \tag{65}$$

which implies

$$S = M_-^{-1} \, M_+. \tag{66}$$

We write s (in lower case) for the corresponding element of $SU(2)$; it is $m_-^{-1} m_+$. From (61), we find

$$s_{11} = s_{22} = e^{-p^2\pi}, \tag{67}$$

$$s_{21} = -\bar{s}_{12} = \frac{e^{-p^2\pi/2} p e^{-i\pi/4} \sqrt{2\pi}}{\Gamma(1-\nu)} = e^{i(\eta - \pi/4)} \sqrt{1 - e^{-2\pi p^2}}, \tag{68}$$

where $\eta = \arg\Gamma(1+\nu)$. By comparing this with (36), we see that s is a rotation by an angle θ about an axis $\hat{\mathbf{b}}$ in the x–y plane, $s = q(\hat{\mathbf{b}}, \theta)$, where

$$\cos\frac{\theta}{2} = e^{-p^2\pi}, \qquad \hat{\mathbf{b}} = \hat{\mathbf{x}} \cos\alpha + \hat{\mathbf{y}} \sin\alpha, \tag{69}$$

where $\alpha = \eta + \pi/4$.

Thus, the asymptotic frame vectors are related by

$$\hat{\mathbf{a}}_{+i} = \hat{\mathbf{a}}_{-j} \, Q(\hat{\mathbf{b}}, \theta)_{ji}. \tag{70}$$

The particle comes in asymptotically along a radial line and goes out asymptotically along a different radial line, and the scattering angle is θ given by (69). However, the azimuthal angle of the scattering, measured with respect to the incoming direction depends ever more sensitively on the azimuthal angle of the angular component of the velocity in the remote past. The asymptotic states are not free, and a cross section in the usual sense does not exist.

H. Intuitive picture of the trajectory

When the particle is far from the origin, the magnetic field is strong. The particle's motion can be decomposed into radial motion (aligned with the magnetic field) and rapid gyration about the magnetic field. Far from the origin, the motion is nearly in the radial direction. The more interesting dynamics occurs when the particle "scatters" off the weaker field near the origin. The behavior depends on the magnitude of the dimensionless parameter p.

For large p, the particle never gets close enough to the origin to experience a weak magnetic field. The particle remains "adiabatic" for all time and must stay close to a magnetic field line in one radial direction. The particle approaches the origin, then reflects, and returns the way it came. The scattering angle is close to π.

For small p, the particle does get close enough to the origin to experience a weak magnetic field. Near the origin, the motion is approximately force free, so the particle goes past the origin and continues on the other side. The scattering angle is close to zero.

Figure 1 illustrates an orbit with initial conditions $r_0 = v_0 = 1$ and $p = 0.553$, which, according to (69), corresponds to a scattering angle of $\theta = 135^\circ$. For purposes of illustration, it is convenient to project the orbit onto the unit sphere, as is done in Fig. 1. The asymptotic spirals as $t \longrightarrow \pm\infty$ can be seen, corresponding to the incoming and outgoing directions. The point of symmetry on the orbit is $t = 0$, the point of minimum approach where $r = r_0$. The orbit illustrated is not the fundamental solution, but rather for the purpose of the presentation, it has been rotated to place the orbit entirely within the principal octant of the sphere. More extensive numerical work confirms formula (69) for the scattering angle.

I. Cones and geodesics

The case $\rho(\mathbf{r}) = g_0\delta^3(\mathbf{r})$, $\mathbf{B} = g_0\mathbf{r}/r^3$ is the point monopole at the origin, for which the solution is well known. In this case, Bakas and Lüst[1] have provided an illustration of the orbit on the cone. We offer another way of visualizing this solution.

Poincaré[12] pointed out that the orbit is a geodesic on the cone. This is because vectors $\mathbf{r}$ and $\mathbf{v}$, which are linearly independent and which span the tangent plane to the cone, are orthogonal to the force vector. Therefore, the magnetic force is a force of constraint on the cone, and the motion is a geodesic.

Since, however, a cone is isometric to a plane, the orbit may be visualized by drawing a straight line on a piece of paper, which is then rolled into a cone. One can clearly see the effective repulsion of the particle from regions of high magnetic field strength (the mirror effect). One line suffices to visualize a continuum of possible orbits, that is, for various initial conditions, as the paper is rolled into a cone in different ways.

It turns out that Poincaré's construction generalizes to arbitrary, spherically symmetric distributions of monopoles. To do this, we define a generalized cone as the union of a smooth family of radial half-lines, emanating from the origin. The cone that applies to motion in a spherically symmetric distribution of monopoles is the one swept out by the half-line joining the origin with the particle position $\mathbf{r}(t)$ and extending out to infinity, as t goes from $-\infty$ to $+\infty$. Such a cone is conveniently visualized by its intersection with the unit sphere, which is a curve. This curve may self-intersect (although not in our examples), and as t goes from $-\infty$ to $+\infty$, parts of it may be retraced.

In the case of a point monopole, the cone in this sense is the same as the usual cone, and the curve on the unit sphere is a small circle or an arc thereof; for a free particle ($\mathbf{B} = 0$), the curve is an arc of a great circle, and for the uniform distribution of monopolium, an example of the curve on the unit sphere is given in Fig. 1.

The cone in this sense is locally isometric to the Euclidean plane. Let s be the arc length of the curve on the unit sphere. Then, an increment of arc length on the unit sphere $ds = d\theta$ is also an increment of angle, and the sector of the cone defined by $d\theta$ is isometric to

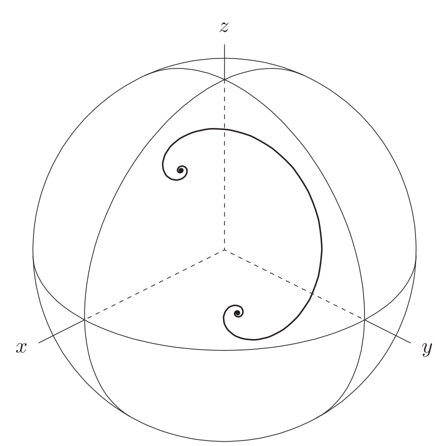

FIG. 1. An orbit projected onto the unit sphere, with $p = 0.553$, corresponding to a scattering angle of $\theta = 135^\circ$. The orbit has been rotated to appear in the principal octant of the sphere. The initial point at $t = 0$ is the symmetry point. The asymptotic spirals as $t \longrightarrow \pm\infty$ are clearly seen.

the corresponding sector of the Euclidean plane, where θ is the usual polar coordinate. More formally, if θ is a length or accumulated angle coordinate along the curve on the unit sphere, then (r, θ) are coordinates of points on the cone, and the metric on the cone, as inherited from the Euclidean metric in $\mathbb{R}^3$, is $dr^2 + r^2 d\theta^2$, the same as the metric in Euclidean $\mathbb{R}^2$ in polar coordinates. Poincaré's argument applies to the cone as we have defined it, and the orbits are geodesics, the images of straight lines on $\mathbb{R}^2$ under the mapping in which points are identified by their (r, θ) coordinates.

III. CONCLUSIONS

In this article, we have analyzed the motion of a charged particle in the magnetic field produced by a spherically symmetric distribution of magnetic monopoles. We have found some general features of such solutions, that is, for any spherically symmetric distribution of monopoles, including the fact that the equations of motion can be converted into a linear system and the orbit is a geodesic on a generalized cone. In the special case of a uniform, spherically symmetric distribution of monopoles, we have given a complete solution for the orbits, including an S-matrix that connects asymptotic states and an explicit formula for the scattering angle. These results enlarge the repertoire of systems of distributions of magnetic monopoles for which the orbits of charged particles are known.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Robert Littlejohn: Conceptualization (lead); Writing – original draft (lead). **Philip Morrison**: Writing – original draft (supporting). **Jeffrey Heninger**: Software (supporting).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

APPENDIX: PARABOLIC CYLINDER FUNCTIONS

We have found that the best reference for parabolic cylinder functions for our purposes is Magnus and Oberhettinger.[18] The edition cited has a small and obvious error in the asymptotic forms, which is corrected in (A6) and (A7); earlier editions have more serious errors.

Parabolic cylinder functions, denoted as $D_\nu(z)$, are entire analytic functions of z. In this appendix, z and ν are arbitrary complex numbers, while in the main body of this paper, z and ν have the specific values (41) and (44). The parabolic cylinder functions satisfy the differential equation,

$$\frac{d^2 f(z)}{dz^2} + \left(\nu + \frac{1}{2} - \frac{z^2}{4}\right) f(z) = 0, \tag{A1}$$

of which the functions $D_\nu(\pm z)$ and $D_{-\nu-1}(\pm iz)$ are solutions. These have the linear dependencies,

$$\begin{aligned} D_\nu(z) &= e^{-i\nu\pi} D_\nu(-z) + \frac{\sqrt{2\pi}}{\Gamma(-\nu)} e^{-i(\nu+1)\pi/2} D_{-\nu-1}(iz) \\ &= e^{i\nu\pi} D_\nu(-z) + \frac{\sqrt{2\pi}}{\Gamma(-\nu)} e^{i(\nu+1)\pi/2} D_{-\nu-1}(-iz). \end{aligned} \tag{A2}$$

These functions satisfy the recursion relations,

$$D'_\nu(z) + \frac{z}{2} D_\nu(z) - \nu D_{\nu-1}(z) = 0, \tag{A3a}$$

$$D'_\nu(z) - \frac{z}{2} D_\nu(z) + D_{\nu+1}(z) = 0, \tag{A3b}$$

and the initial conditions,

$$D_\nu(0) = \frac{\sqrt{\pi}\,2^{\nu/2}}{\Gamma((1-\nu)/2)}, \qquad D'_\nu(0) = -\frac{\sqrt{\pi}\,2^{(\nu+1)/2}}{\Gamma(-\nu/2)}. \tag{A4}$$

The parabolic cylinder functions have the integral representation,

$$D_\nu(z) = \frac{e^{-i\nu\pi/2}}{\sqrt{2\pi}} \int_C dt\, t^\nu\, e^{-t^2/2+itz+z^2/4}, \tag{A5}$$

where the contour C runs just above the real axis and where t^ν has a branch cut just below the positive real axis. This is not the same as the integral representations quoted in Ref. 18, but is convenient because it applies to all $\nu \in \mathbb{C}$. Equation (A5) can be proved by showing that $D_\nu(z)$, so defined, satisfies the differential equation [Eq. (A1)] and the initial conditions (A4).

When $|z| \gg 1, |\nu|$, the integral representation (A5) leads via the method of steepest descent to the asymptotic forms,

$$D_\nu(z) \approx e^{-z^2/4}\, z^\nu \left[1 - \frac{\nu(\nu-1)}{1!\cdot 2z^2} + \frac{\nu(\nu-1)(\nu-2)(\nu-3)}{2!\cdot 2^2 z^4} - \cdots\right] \tag{A6}$$

for $-\pi/2 < \arg z < \pi/2$ and

$$D_\nu(z) \approx \text{Series(A6)} - \frac{\sqrt{2\pi}}{\Gamma(-\nu)}\, e^{i\nu\pi}\, e^{z^2/4}\, z^{-\nu-1}\left[1 + \frac{(\nu+1)(\nu+2)}{1!\cdot 2z^2} + \frac{(\nu+1)(\nu+2)(\nu+3)(\nu+4)}{2!\cdot 2^2 z^4} + \cdots\right] \tag{A7}$$

for $\pi/2 < \arg z < \pi$. The asymptotic form changes at $\arg z = \pi/2$ because of a bifurcation in the steepest descent contour; see the work of Dingle[20] for details. For the purposes of this paper, we need only $\arg z = -\pi/4$ and $3\pi/4$.

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