Reviews of Modern Plasma Physics (2024) 8:36
https://doi.org/10.1007/s41614-024-00175-2

SPECIAL TOPICS

Formulation of a one-dimensional electrostatic plasma model for testing the validity of kinetic theory

F. Pegoraro[1] · P. J. Morrison[2] · D. Manzini[1,3] · F. Califano[1]

Received: 28 July 2024 / Accepted: 12 November 2024

Abstract

We present a one-dimensional (1-D) model composed of aligned, electrostatically interacting charged disks, conceived to address in a computable model the validity of the Bogoliubov assumption on the decay of particle correlations in the Born–Bogoliubov–Green–Kirkwood–Yvon hierarchy. This assumption is a basic premise of plasma kinetic theory. The disk model exhibits spatially 1-D features at short distances, but retains 3-D features at large distances. Here the collective dynamics of this model plasma is investigated by solving the corresponding Vlasov equation. In addition, the implementation of the model for the numerical validation of the Bogoliubov assumption is formulated.

Keywords Plasma kinetic theory · BBGKY hierarchy · Plasma correlation functions · 1-D plasma model

P. J. Morrison, D. Manzini and F. Califano have contributed equally to this work.

✉ F. Pegoraro
francesco.pegoraro@unipi.it

P. J. Morrison
morrison@physics.utexas.edu

D. Manzini
davide.manzini@lpp.polytechnique.fr

F. Califano
francesco.califano@unipi.it

[1] Physics Department, University of Pisa, Pisa, Italy

[2] Physics Department and Institute for Fusion Studies, University of Texas at Austin, Austin, TX, USA

[3] Laboratoire de Physique des Plasmas, École Polytechnique, Sorbonne Université, Paris, France

1 Introduction

The Born–Bogoliubov–Green–Kirkwood–Yvon (BBGKY) hierarchy of phase-space equations is the starting block (Cercignani et al. 1997) in the description of a wide class of multi-particle interacting systems, ranging from stellar systems (Chavanis 2008), to electromagnetic plasmas, in both the classical (Ichimaru 1973; van Kampen and Felderhoff 1967) and relativistic (Vereshchagin and Aksenov 2017) regimes, to quantum electrodynamics plasmas (Smolyansky et al. 2020), down to ultracold systems (Krönke and Schmelcher 2018; Bergeson et al. 2019), and to quantum spin dynamics (Pucci et al. 2016). A major difficulty is related to the need to construct a consistent closure of the infinite chain of equations of the BBGKY hierarchy.

The problem of dealing with this infinite chain has been addressed in the literature with different approaches and under different conditions, both from the mathematical and from the physical viewpoints. For example, in Ryabukha (2006) a regularized representation of the solution of the BBGKY hierarchy for a 1-D infinite system of hard spheres was proven for given initial data, while (Gubal 2014) addressed the Cauchy problem for an infinite 1-D particle system, where the particles interact with each other by a finite-range pair potential with a hard core. In Baalrud and Daligault (2019) a closure of the BBGKY hierarchy was developed by enforcing an exact equilibrium limit at all orders. This leads to a convergent kinetic equation that can extend plasma kinetic theory into the regime of strong Coulomb coupling. In Singleton (2020) dimensional reduction and dimensional continuation techniques allowed the adoption of a regularization scheme of the spatial divergences that occur both at short and at long distances in a Coulomb plasma.

While most of the investigations on the solution or the truncation of the BBGKY hierarchy are restricted to spatially homogeneous systems, in Chavanis (2008) the growth of correlations with time was studied in the context of the relaxation of self-gravitating systems and a general kinetic equation was derived that can be applied to spatially inhomogeneous systems, including ones that take into account memory effects that cannot be accounted for if exact equilibrium conditions are imposed. With a different perspective in Marsden et al. (1984) it was shown that the equations for the evolution of the i-point functions in the BBGKY hierarchy can be cast in Hamiltonian form with Lie-Poisson brackets that involve a Lie algebra constructed from the algebra of n-point functions. It was stated that this structure can be inherited by truncated subsystems in the hierarchy.

As shown above, most of these investigations rely on the assumption of a time scale separation, the so-called Bogoliubov assumption (Bogoliubov 1946) (see also (van Kampen and Felderhoff 1967)), where the particle correlations decay on a fast time scale and reach an asymptotic form that is uniquely determined by the instantaneous form of the uncorrelated functions.

The aim of the present article is to provide a means for testing this assumption numerically, in the case of the 2-point function of a Coulomb plasma, by retaining in the hierarchy only contributions of order $\mathcal{O}(g)$, where $g \ll 1$ is the standard plasma parameter.

It must however be remarked that, at present, such a numerical test is prohibitive in the case of a 3-D system, since the equation for the time evolution of the two-point function would be 13-dimensional (one time dimension plus the 12 dimensions of the two-particle phase space). In 1-D the equation for the two-point distribution would be 5-dimensional (time plus the 4-dimensions of the two-particle phase space). However in 1-D the electric field generated by a one-dimensional charge (a charge foil) does not decay with distance so that the two-foil interaction energy diverges at infinity. This makes it unclear how to define a finite correlation length and to define binary collisions between charge foils. A reasonable compromise is to introduce an interaction potential that behaves as a 1-D system at short distances while it decays as the inverse of the distance at large distances. It must be noted that any interaction potential of this type introduces a length scale, independent of the plasma parameters such as temperature and density, that is not present in the Coulomb interaction. This length will appear explicitly in the collective plasma dynamics.

As a specific example of such a potential, here we consider (Morrison and Pegoraro 2021) a 1-D plasma made of aligned, uniformly charged disks (see Fig. 1), where the characteristic scale length is the disk radius b. In this case the electrostatic interaction energy between two charged disks is finite both at zero and at infinite distance from the source and its absolute value is a monotonically decreasing function of the distance.

This article is organized as follows. In Sect. 2 the basic equations of the BBGKY hierarchy are briefly recalled, together with the formulation of the Bogoliubov assumption. In Sect. 3 the electrostatic properties of the disk model are described and some useful expressions are listed. In Sect. 4 the collective disk plasma dynamics is investigated by solving the corresponding Vlasov equation to clarify where and how the dynamics of such a model system can mimic the dynamics of a Coulomb plasma, at least as long as the longitudinal electric field limit is concerned. Finally in Sect. 5 the system of equations to be solved numerically is derived in explicit form and in Section 6 the numerical approach that will be applied is briefly described.

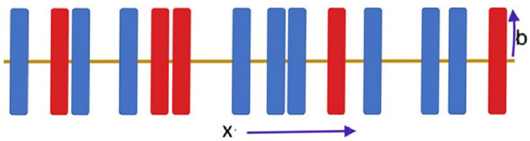

Fig. 1 Plasma of two species of oppositely charged disks: blue positively charged, red negatively charged disks. The disks are aligned and can only move in the x direction, i.e. along their axis, but can pass each other

2 The BBGKY hierarchy and Bogoliubov's assumption

The time evolution of a system of N classical particles obeys the Liouville equation for the N-point distribution function in $6N$-dimensional phase space (3 space and 3 momentum coordinates per particle)

$$F^{(N)} = F^{(N)}(\mathbf{q}_1...\mathbf{q}_N, \mathbf{p}_1...\mathbf{p_N}, t), \tag{1}$$

that can be written as

$$\frac{\partial F^{(N)}}{\partial t} + \sum_{i=1}^{N} \frac{\mathbf{p}_i}{m_i} \cdot \frac{\partial F^{(N)}}{\partial \mathbf{q}_i} + \sum_{i=1}^{N} \mathcal{F}_i \cdot \frac{\partial F^{(N)}}{\partial \mathbf{p}_i} = 0\,, \tag{2}$$

with $\mathbf{q}_i, \mathbf{p}_i$ the coordinate and momentum of the i^{th} particle with mass m_i, and $\mathcal{F}_i$ the total force acting on the i^{th} particle,

$$\mathcal{F}_i = \sum_{j=1, j\neq i}^{N} \mathcal{F}_{i,j} = - \sum_{j=1, j\neq i}^{N} \frac{\partial V_{ij}}{\partial \mathbf{q}_i}, \tag{3}$$

with V_{ij} the interaction potential, and

$$\int d\mathbf{q}_1..d\mathbf{q}_N\, d\mathbf{p}_1..d\mathbf{p}_N \;\; F^{(N)}(\mathbf{q}_1...\mathbf{q}_N\, \mathbf{p}_1...\mathbf{p_N}, t) = 1\,, \tag{4}$$

where the spatial integrations are over a finite volume.

By integration over the variables of a subset of particles, the Liouville equation can be transformed into a chain of equations for the s-point distribution function

$$F^{(s)}(\mathbf{q}_1...\mathbf{q}_s\, \mathbf{p}_1...\mathbf{p_s}, t) = \int d\mathbf{q}_{s+1}..d\mathbf{q}_N\, d\mathbf{p}_{s+1}..d\mathbf{p}_N \;\; F^{(N)}(\mathbf{q}_1...\mathbf{q}_N, \mathbf{p}_1...\mathbf{p_N}, t), \tag{5}$$

that read as

$$\begin{aligned} &\frac{\partial F^{(s)}}{\partial t} + \sum_{i=1}^{s} \frac{\mathbf{p}_i}{m_i} \cdot \frac{\partial F^{(s)}}{\partial \mathbf{q}_i} + \sum_{i=1}^{s} \sum_{j=1, j\neq i}^{s} \mathcal{F}_{i,j} \cdot \frac{\partial F^{(s)}}{\partial \mathbf{p}_i} \\ &\qquad\qquad = -(N-s) \sum_{i=1}^{s} \int \mathcal{F}_{i,s+1} \cdot \frac{\partial F^{(s+1)}}{\partial \mathbf{p}_i} d\mathbf{q}_{s+1} d\mathbf{p}_{s+1}\,. \end{aligned} \tag{6}$$

This system of equations is not closed: the first equation connects the evolution of the one-particle distribution function $F^{(1)}$ with the two-particle distribution function $F^{(2)}$, the second equation connects the two-particle distribution function $F^{(2)}$ with the three-particle distribution function $F^{(3)}$ and so on.

For the case of the one particle distribution function ($s = 1$) we obtain

$$\frac{\partial F^{(1)}}{\partial t} + \frac{\mathbf{p}_1}{m_1} \cdot \frac{\partial F^{(1)}}{\partial \mathbf{q}_1} = -(N-1) \int \mathcal{F}_{1,2} \cdot \frac{\partial F^{(2)}}{\partial \mathbf{p}_1} d\mathbf{q}_2 d\mathbf{p}_2, \tag{7}$$

where $F^{(2)}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{p}_1, \mathbf{p}_2, t)$ is linked to $F^{(3)}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, t)$ by

$$\begin{aligned} \frac{\partial F^{(2)}}{\partial t} + \frac{\mathbf{p}_1}{m_1} \cdot \frac{\partial F^{(2)}}{\partial \mathbf{q}_1} + \frac{\mathbf{p}_2}{m_2} \cdot \frac{\partial F^{(2)}}{\partial \mathbf{q}_2} + \mathcal{F}_{1,2} \cdot \frac{\partial F^{(2)}}{\partial \mathbf{p}_1} + \mathcal{F}_{2,1} \cdot \frac{\partial F^{(2)}}{\partial \mathbf{p}_2} = \\ = -(N-2) \int \left[\mathcal{F}_{1,3} \cdot \frac{\partial F^{(3)}}{\partial \mathbf{p}_1} + \mathcal{F}_{2,3} \cdot \frac{\partial F^{(3)}}{\partial \mathbf{p}_2} \right] d\mathbf{q}_3 d\mathbf{p}_3 . \end{aligned} \tag{8}$$

Integration of (8) over $\mathbf{q}_2$ and $\mathbf{p}_2$ returns (7). The space integration is over a finite volume and all $F^{(s)}$ are normalized to one, i.e.,

$$\int d\mathbf{q}_1..d\mathbf{q}_s, d\mathbf{p}_1..d\mathbf{p}_s\, F^{(s)}(\mathbf{q}_1...\mathbf{q}_s, \mathbf{p}_1...\mathbf{p}_s, t) = 1.$$

As is standard (van Kampen and Felderhoff 1967; Krall and Trivelpiece 1986), we adopt an expansion in terms of the plasma parameter $g \ll 1$ (defined as the inverse of the number of particles in a Debye sphere) and write

$$F^{(2)}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{p}_1, \mathbf{p}_2, t) = F^{(1)}(\mathbf{q}_1, \mathbf{p}_1, t)F^{(1)}(\mathbf{q}_2, \mathbf{p}_2, t) + \Delta F^{(2)}(\mathbf{q}_1, \mathbf{p}_1, \mathbf{q}_2, \mathbf{p}_2, t), \tag{9}$$

where $\Delta F^{(2)}(\mathbf{q}_1, \mathbf{p}_1, \mathbf{q}_2, \mathbf{p}_2, t)$ is the correlated part, and similarly

$$\begin{aligned} F^{(3)}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, t) &= F^{(1)}(\mathbf{q}_1, \mathbf{p}_1, t)F^{(1)}(\mathbf{q}_2, \mathbf{p}_2, t)F^{(1)}(\mathbf{q}_3, \mathbf{p}_3, t) \\ &+ F^{(1)}(\mathbf{q}_1, \mathbf{p}_1, t)\, \Delta F^{(2)}(\mathbf{q}_2, \mathbf{p}_2, \mathbf{q}_3, \mathbf{p}_3, t) + F^{(1)}(\mathbf{q}_2, \mathbf{p}_2, t)\, \Delta F^{(2)}(\mathbf{q}_3, \mathbf{p}_3, \mathbf{q}_1, \mathbf{p}_1, t) \\ &+ F^{(1)}(\mathbf{q}_3, \mathbf{p}_3, t)\, \Delta F^{(2)}(\mathbf{q}_1, \mathbf{p}_1, \mathbf{q}_2, \mathbf{p}_2, t) + \mathcal{O}(g^2) . \end{aligned} \tag{10}$$

This expansion leads to a closed system of evolution equations for the correlated part of the 2-point distribution function $\Delta F^{(2)}$.

The basic step in the Bogolyubov's method of solution of the above system of equations is the assumption that $\Delta F^{(2)}(\mathbf{q}_1, \mathbf{p}_1, \mathbf{q}_2, \mathbf{p}_2, t)$ rapidly reaches an asymptotic form $\tilde{\Delta} F^{(2)}(\mathbf{q}_1, \mathbf{p}_1, \mathbf{q}_2, \mathbf{p}_2, t)$ that is uniquely determined by the instantaneous form of the function $F^{(1)}(\mathbf{q}, \mathbf{p}, t)$ (time scale separation), i.e., at each time t we have

$$\tilde{\Delta} F^{(2)}(\mathbf{q}_1, \mathbf{p}_1, \mathbf{q}_2, \mathbf{p}_2, t) = \tilde{\Delta} F^{(2)}(\mathbf{q}_1, \mathbf{p}_1, \mathbf{q}_2, \mathbf{p}_2 | F^{(1)}(\mathbf{q}, \mathbf{p}, t)). \tag{11}$$

When (11) is substituted into (7) and (9), a closed equation for $F^{(1)}$ is obtained (van Kampen and Felderhoff 1967) that is the basis for the derivation of the collision operators of the Vlasov–Landau–Lenard–Balescu (Landau 1936; Lenard 1960; Balescu 1960) theories.

3 The disk–disk interaction

As indicated by (3) a central feature in the BBGKY hierarchy is the interparticle potential V_{ij} as it determines the particle interaction range and finally the dimensionless expansion parameter that can be used to truncate the BBGKY hierarchy and construct a closed set of equations on which the Bogoliubov assumption can be tested.

As already mentioned in the Introduction, to reduce the dimensionality of the system of equations to be solved numerically, we resort to a model one-dimensional interaction potential regularized at large distances, as is the case of the interaction potential between aligned charged disks. Therefore in this and in the next section the main properties of the disk-disk interaction and of the collective disk plasma dynamics are described with the aim of identifying which properties correspond to and which differ from those of a three-dimensional Coulomb plasma (Pegoraro and Morrison 2022). This point will be readdressed at the end of Sec. 5 in a comment on the relevance of a one-dimensional model to the validation of the Bogoliubov assumption in a three-dimensional plasma.

The interaction potential between aligned disks is known in explicit form (Ciftja 2019) but, more generally, the interaction potential could be assigned without referring to uniformly charged disks in a form that is regular at short distances and decays monotonically at large distances as the inverse of the distance.

The interaction energy $\mathcal{W}(x_1, x_2)$ between two uniformly charged infinitely thin disks of radius b located at x_1 and x_2, with charge Q_1, and Q_2, respectively, can be written as (see Appendix A),

$$\mathcal{W}(x_1, x_2) = 4\frac{Q_1\, Q_2}{b} V(\xi) = Q_1 \varphi_2 = Q_2 \varphi_1, \tag{12}$$

where

$$V(\xi) = -\frac{\xi}{2}\left\{1 - \frac{1}{3\pi}\left[(4-\xi^2)E(-4/\xi^2) + (4+\xi^2)K(-4/\xi^2)\right]\right\}, \tag{13}$$

and φ_1, φ_2 are the potentials generated by the disks 1 and 2, respectively. Here $\xi(x_1, x_2) = |x_1 - x_2|/b$, $Q_1 = \pi b^2 \sigma_1$ and $Q_2 = \pi b^2 \sigma_2$ are the (fixed) charges of the two disks, $\sigma_{1,2}$ are their surface density and $E(-4/\xi^2)$ and $K(-4/\xi^2)$ are elliptic integrals (Abramowitz and Stegun 1964) [24]. A plot of $V(\xi)$ is given in Fig. 2.

The electric force $\mathcal{F}_{Q_2}$ acting on disk 1 due to disk 2 (see Fig. 3) is given by

$$\mathcal{F}_{Q_2}(x_1) = -\mathcal{F}_{Q_1}(x_2) = -(4Q_1 Q_2/b^2)\, V'(\xi)\, \mathrm{sign}\,(x_1 - x_2), \tag{14}$$

with

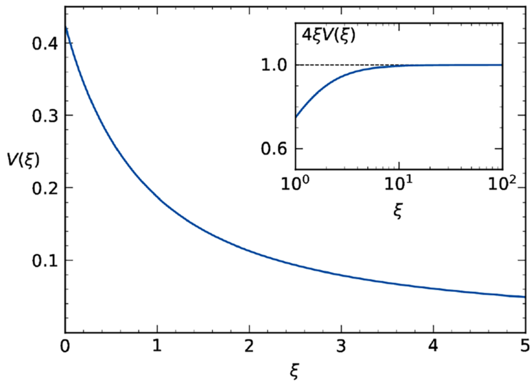

Fig. 2 Plot of the potential $V(\xi)$ versus the normalized distance ξ. Note the $1/(4\xi)$-behavior for large ξ shown in the insert and the finite value $4/(3\pi)$ at $\xi = 0$

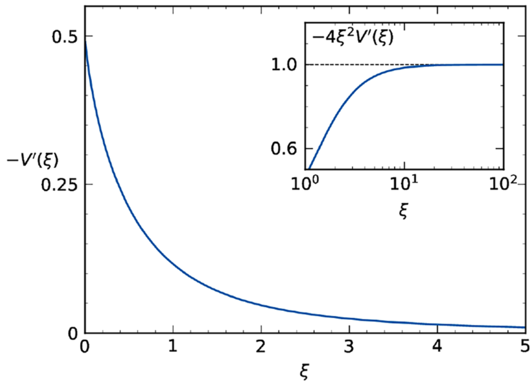

Fig. 3 Plot of $-V'(\xi)$ versus ξ . Note the $1/(4\xi^2)$ behavior for large ξ shown in the insert, and the value $-V'(0) = 1/2$

$$V'(\xi) = \frac{dV(\xi)}{d\xi} = \frac{1}{2} + \frac{1}{2\pi}\left[\xi^2 E(-4/\xi^2) - (4+\xi^2)K(-4/\xi^2)\right]. \tag{15}$$

The Fourier transform of $V(\xi)$ (see Appendix B) is given by

$$\begin{aligned}\hat{V}(kb) &= \int_{-\infty}^{+\infty} \frac{dx}{b}\, V(|x|/b) \exp -[i(kb)\,(x/b)] \\ &= 2\int_0^{+\infty} ds\, [J_1(s)/s]^2 \,\frac{s}{(kb)^2 + s^2} = \frac{1}{\pi^{1/2}(kb)^2} G^{2,2}_{2,4}\left((kb)^2 \middle|^{1/2,1}_{1,1,-1,0}\right),\end{aligned} \tag{16}$$

where $G^{2,2}_{2,4}\left((kb)^2 \middle|^{1/2,1}_{1,1,-1,0}\right)$ is the Meijer G function (Abramowitz and Stegun 1964) [24].

In the limits of long wavelengths with respect to the disk radius b, i.e., $(kb)^2 \ll 1$, we find from (16)

$$\begin{aligned}\hat{V}(kb) &= (1/8)\left\{5 - 6\gamma - 2\ln[(kb)^2] - 2\psi(3/2)\right\} \\ &\quad + (1/48)\left\{13 - 9\gamma - 3\ln[(kb)^2] - 3\psi(5/2)\right\}(kb)^2 + \dots,\end{aligned} \tag{17}$$

with $\gamma = 0.577216$, $\psi(z) = d \ln \Gamma(z)/dz$ and $\Gamma(z)$ the Gamma function (Abramowitz and Stegun 1964) [24]. For short wavelengths, i.e., $(kb)^2 \gg 1$, we find

$$\hat{V}(kb) = \frac{1}{(kb)^2} + \dots . \tag{18}$$

For short wavelengths we recover the Coulomb spectrum $\hat{V} \propto 1/k^2$ while for small kb the disk and Coulomb spectra differ. A plot of $\hat{V}(kb)$ is given in Fig. 4. The logarithmic behavior at small wave-numbers does not depend on the specific model adopted and arises from the imposed $1/\xi$ dependence of the interaction potential at large distances in a 1-D configuration.

4 Collective disk plasma dynamics

In this section, the interaction potential described in Sec. 3 is used to define the characteristic dynamical time scales and lengths of a one-dimensional disk plasma (Sec. 4.1,4.2) and to compare them with the corresponding quantities in a Coulomb three-dimensional plasma. In Secs.4.4, 4.5 the expressions of the screening length and of the dimensionless expansion parameter that will be used for truncating the BBGKY hierarchy in a disk plasma are derived. In Sec 4.3 we consider the disk Vlasov equation which in the present context corresponds to assuming that the disks in the plasma are and remain uncorrelated. Solving the disk Vlasov equation serves a dual purpose. First it allows to confront the wave spectra in a disk plasma and in a three-dimensional Coulomb plasma and, through the wave group velocity, the speed of the transmission of information through the plasma. Second it allows us to compare the results obtained for the disk plasma excitations in the uncorrelated Vlasov case with those that will be found in the case when the correlations are retained and explicitly solved for, i.e. with the solutions that will be obtained by integrating numerically equations (58) and (59) of Sec. 5.1.

We consider a globally neutral disk plasma and, in analogy with a Coulomb plasma, take the continuous density limit. We introduce positive and negative charged disk densities and a collective (mean) disk electric field. The mean electric

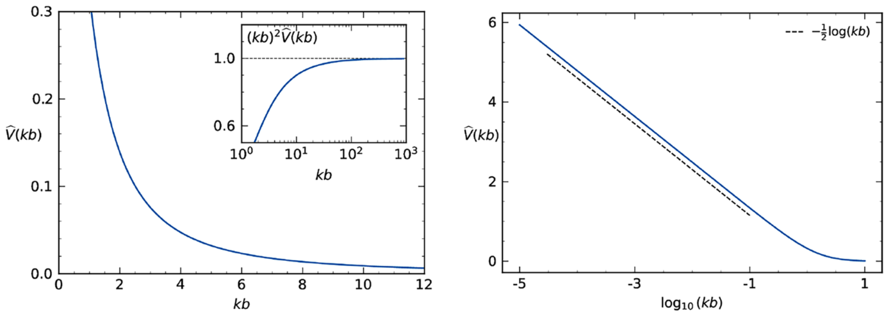

Fig. 4 Plot of $\hat{V}(kb)$ versus kb. The insert in the left frame shows the $1/(kb)^2$ for large kb, while the right frame displays the logarithmic behaviour as $kb \to 0$

field is calculated from the total disk charge density ρ and a Green's function with a kernel that involves the interaction electrostatic potential $V(\xi)$ of (13), which plays the role of the Coulomb potential in a 3-D plasma.

4.1 Cold fluid equations

Let $n_e(x,t)$ be the electron disk number density (number of negatively charged disks per unit length) and $n_i(x,t)$ be the ion disk number density (number of positively charged disks per unit length). For simplicity, in this section, we assume the charges on the disks satisfy $Q = Q_i = -Q_e$, so that the total disk-charge density is given by

$$\rho(x) = Q\,[n_i - n_e(x,t)], \tag{19}$$

where n_i is constant. The electron disk density satisfies the continuity equation

$$\partial_t n_e(x,t) + \partial_x [n_e(x,t)\,u_e(x,t)] = 0, \tag{20}$$

where $u_e(x,t)$ is the electron disk fluid velocity.

The electrostatic potential created by the disk charge density can be written in convolution integral form as

$$\varphi(x) = \frac{4}{b}\int_{-\infty}^{+\infty} dx'\ \rho(x')V(|x-x'|/b), \tag{21}$$

which for $\rho(x') = Q\,\delta(x'-x_1)$ returns the potential generated by a single disk. The corresponding electric force acting on a disk of charge $Q_j \quad (j = i, e)$ can be written as

$$\mathcal{F}_j(x) = -\frac{4Q_j}{b}\frac{d}{dx}\int_{-\infty}^{+\infty} dx'\ \rho(x')V(|x-x'|/b). \tag{22}$$

To close the system we may supplement the continuity equation with the momentum equation of a "cold" disk plasma

$$M_e n_e \left(\partial_t + u_e \partial_x\right) u_e = n_e \mathcal{F}_e, \tag{23}$$

where here and in the following the electron and ion disk masses are denoted by $M_{e.i}$
.

4.2 Cold plasma disk waves

We consider the case where the disk thermal motion can be neglected with respect to the wave phase velocity and derive the linear dispersion relation of the cold disk plasma. Linearization of (20) and (23) for a monochromatic wave,

$$\tilde{n}_e(x,t) = \tilde{n}_0 \exp[-i(\omega t - kx)] \quad \text{and} \quad \tilde{u}_e(x,t) = \tilde{u}_0 \exp[-i(\omega t - kx)], \tag{24}$$

in a stationary homogeneous equilibrium with density n_0 and immobile ion disks yields

$$\tilde{u}_e = \frac{\tilde{n}_e}{n_0}\frac{\omega}{k} \quad \text{and} \quad \tilde{u}_e = \frac{4k\tilde{n}_e}{M_e\omega} Q^2 \hat{V}(kb), \tag{25}$$

which give

$$\omega^2 = \omega_{Dpe}^2 (kb)^2 \hat{V}(kb), \quad \text{where} \quad \omega_{Dpe}^2 = \frac{4Q^2 n_0}{M_e b^2}. \tag{26}$$

Note that the ratio $n_0/(\pi b^2)$ has the dimension of a volume density.

In the short wavelength limit $kb \gg 1$ these disk waves become Langmuir waves with frequency ω_{Dpe}, but with a small group velocity ($\mathcal{O}(1/(kb)^2)$ that does not depend on the disk temperature. In the long wavelength limit we find (logarithmically corrected) "cold electron disk sound waves" with dispersion relation of the form

$$\omega^2 = k^2 v_D^2 \left[C_1 + C_2 \ln (kb)^2\right], \tag{27}$$

where the velocity v_D is defined by

$$v_D^2 = b^2 \omega_{Dpe}^2 = \frac{4Q^2 n_o}{M_e}, \tag{28}$$

and C_1 and C_2 are the constants defined in (17). A plot of the normalized frequency ω/ω_{Dpe}, and of the group velocity is shown in Fig. 5. We recall that in a Coulomb plasma there is no velocity directly corresponding to v_D and thus no cold electron-disk sound waves.

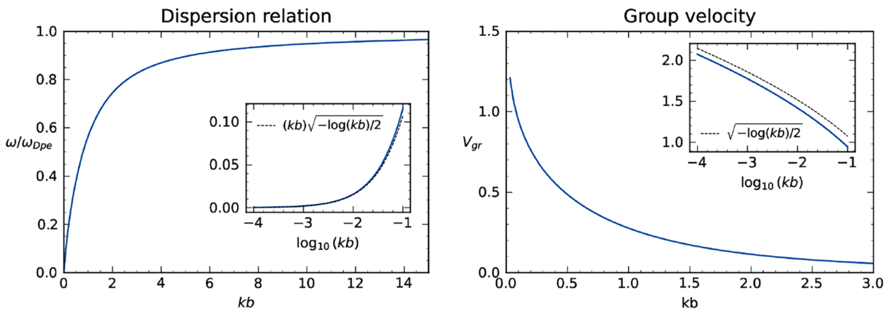

Fig. 5 Plots of the normalized frequency ω/ω_{Dpe} (left frame) and of the group velocity (right frame) versus kb. Note their square-root logarithmic behaviour for $kb \to 0$

4.3 Vlasov equation and the contribution of resonant disks

The treatment of the Vlasov kinetic equation for a disk plasma follows standard lines with two minor differences: the expression of the interaction potential $V(\xi)$ is different from the Coulomb potential and it may be more convenient to use the disk charge density as the independent variable instead of the electrostatic potential.

Thus we have the integro-differential equation

$$\frac{\partial f_j(x,v,t)}{\partial t} + v\frac{\partial f_j(x,v,t)}{\partial x} + \frac{\mathcal{F}_j(x,t)}{M_j}\frac{\partial f_j(x,v,t)}{\partial v} = 0, \tag{29}$$

where $f_j(x,v,t)$ is the phase space distribution function of the j species ($j = e, i$), v is the disk velocity coordinate in phase space and $\mathcal{F}_j(x,t)$ is the electric force, see (22), due to the disk charge density

$$\rho(x,t) = Q\left[\int_{-\infty}^{+\infty} dv f_i(x,v,t) - \int_{-\infty}^{+\infty} dv f_e(x,v,t)\right], \tag{30}$$

where the electron and ion disk number densities are given by

$$n_{i,e}(x,t) = \int_{-\infty}^{+\infty} dv f_{i,e}(x,v,t)\,. \tag{31}$$

In a homogeneous disk plasma the linearized Vlasov equation takes the form

$$\frac{\partial \tilde{f}_j(x,v,t)}{\partial t} + v\frac{\partial \tilde{f}_j(x,v,t)}{\partial x} + \frac{\tilde{\mathcal{F}}_j(x,t)}{M_j}\frac{\partial f_{j0}(v)}{\partial v} = 0, \tag{32}$$

where $f_{j0}(v)$ and $\tilde{f}_j(x,v,t)$ are the equilibrium and the perturbed distribution functions and $\tilde{\mathcal{F}}_j(x,t)$ is the perturbed force.

Assuming immobile ion disks and wave perturbations of the form

$$\tilde{f}_e(x,v,t) = \tilde{f}_0(v)\exp[-i(\omega t - kx)], \quad \tilde{\rho}(x,t) = \tilde{\rho}_0 \exp[-i(\omega t - kx)], \tag{33}$$

with

$$\tilde{\rho}_0 = -Q\tilde{n}_{e0} = -Q\int_{-\infty}^{+\infty} dv \tilde{f}_{e0}(v), \tag{34}$$

we obtain the dispersion relation

$$\begin{aligned} 1 &= -4kQ^2\frac{\hat{V}(kb)}{M_e}\int_{-\infty}^{+\infty} dv\,\frac{\partial f_{e0}(v)/\partial v}{\omega - kv} \\ &= -\frac{\omega_{Dpe}^2}{k}[(kb)^2\hat{V}(kb)]\int_{-\infty}^{+\infty} dv\,\frac{\partial[f_{e0}(v)/n_0]/\partial v}{\omega - kv}\,. \end{aligned} \tag{35}$$

The velocity integral in (35) has to be taken according to the Landau prescription and, as for charged particles in a Coulomb plasma, resonant disks satisfy the condition $\omega = kv$.

In the cold limit when $\omega/k \gg v_{\text{th}e}$, with $v_{\text{th}e}$ the thermal velocity defined as the halfwidth of the distribution function f_{e0}, (35) becomes

$$1 = 4Q^2 n_0 \frac{k^2}{\omega^2} \frac{\hat{V}(kb)}{M_e} = \frac{\omega_{Dpe}^2}{\omega^2} \left[(kb)^2 \hat{V}(kb)\right], \tag{36}$$

which coincides with (26). The dispersion relation (35) depends on the normalized wave number kb and in addition on the equilibrium dimensionless parameter

$$\Theta = (v_{\text{th}e}/v_D)^2\,, \tag{37}$$

that can be interpreted as the dimensionless electron disk "temperature" $\Theta = T_e/(4Q^2 n_0)$ or, equivalently, as the square of the ratio between the disk Debye length λ_{Dpe} defined as,

$$\lambda_{Dpe} = v_{\text{th}e}/\omega_{Dpe}, \tag{38}$$

and the disk radius b, (see Sect. 4.4). For disk Langmuir waves the cold limit $\omega/k \gg v_{\text{th}e}$ implies $v_{\text{th}e}/v_D \ll 1/(kb) \ll 1$, while for cold sound waves it implies the condition $v_{\text{th}e}/v_D \ll |\ln{(kb)^2}|$ where $|\ln{(kb)^2}| \gg 1$.

Normalizing velocities over v_D the dispersion relation can be rewritten in terms of $\bar{\omega} = \omega/\omega_{Dpe}$, $\bar{v} = v/v_d$ and $\kappa = kb$ as

$$1 = -\kappa \hat{V}(\kappa) \int_{-\infty}^{+\infty} dv\, \frac{\partial \bar{f}_{e0}(\bar{v})/\partial \bar{v}}{\bar{\omega} - \kappa \bar{v}}, \tag{39}$$

where $\bar{f}_{e0}(\bar{v}) = v_D f_{e0}(vt)/n_0$ is the dimensionless disk distribution function.

The linear dispersion relation of electron disk waves, as obtained by integrating (39) numerically for a Maxwellian electron disk distribution and immobile ion disks, is shown in Fig. 6 while the phase space trapping of electron disks in a finite amplitude wave at $t\,\omega_{Dpe} = 190$ is shown in Fig. 7.

4.4 Disk charge screening at thermodynamic equilibrium

To determine the screening of a stationary test disk of charge Q_t at $x = 0$ produced by the spatial rearranging of the other disks, we write the electron disk density in terms of the screened potential φ_s as

$$n_e(x) = n_e(0) \exp{[(Q\varphi_s)/T_e]} \sim n_e(0)[1 + Q\varphi_s/T_e], \tag{40}$$

and neglect, for the sake of simplicity the perturbation of the density of the ion disks. Note that in this disk model, contrary to a 3-D Coulomb plasma, the approximation $Q\varphi_s/T_e \ll 1$ remains valid even at close distance. Setting $n_i = n_e(0) = n_0$,

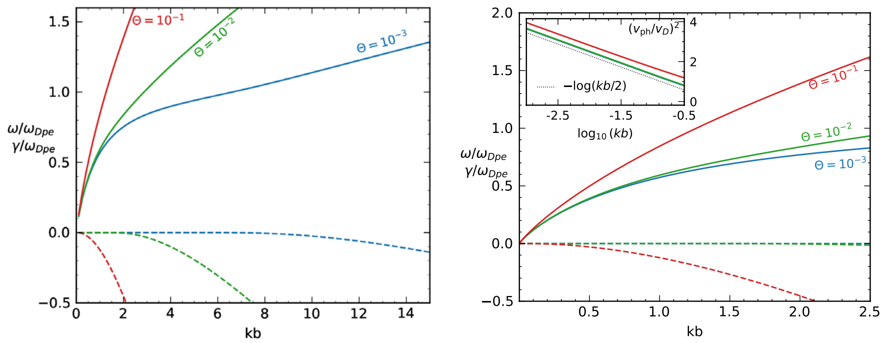

Fig. 6 Normalized frequency, solid lines, and negative growth, dashed lines (Landau damping due to resonant disks) of electron disk waves versus kb for different values of the dimensionless temperature Θ. The right frame presents an enlarged plot that covers small values of kb and returns, for positive values approaching zero, the square-root logarithmic behaviour of cold waves, as shown in the insert in the right frame. For warm Langmuir waves the adiabatic index = 3 as in a 1-D Coulomb plasma

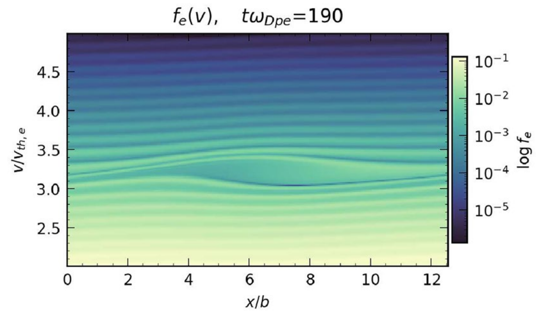

Fig. 7 Phase space of electron disks at the resonant velocity in a finite amplitude electron disk plasma wave for: $\Theta = 0.2$, mobile ion disks with $M_i/M_e = 1836$. The amplitude of the initial perturbation of the normalized density of the electron disks is $\tilde{n}_e/n_0 = 0.03$ and $kb = 0.5$

we can write the charge density of the test disk, plus the screening density given by the second term in (40), as

$$\rho(x) = Q_t\delta(x) - Q^2 n_0 \varphi_s/T_e, \tag{41}$$

which when inserted into (21) gives the integral equation

$$\varphi_s(x) = Q_t(4/b)V(|x|/b) - 4(Q^2n_0/T_e)\int_{-\infty}^{+\infty}\frac{dx'}{b}\,\varphi_s(x')V(|x-x'|/b). \tag{42}$$

Since the integral term in (42) is a convolution product, (42) can be solved by performing a Fourier transform with respect to the x variable (See (16) and Appendix A for analytical details of the functions involved). Note that the factor in front of the integral is dimensionless and can be used to define the disk Debye length λ_{Dpe} setting $4(Q^2n_0/T_e) = 1/\Theta = b^2/\lambda_{Dpe}^2$ (see (38)).

The Fourier transform of (42) reads

$$\hat{\varphi}_s(kb) = Q_t(4/b)\hat{V}(kb) - 4(Q^2 n_0/T_e)\hat{\varphi}_s(kb)\,\hat{V}(kb), \tag{43}$$

i.e.,

$$\hat{\varphi}_s(kb) = \frac{Q_t(4/b)\hat{V}(kb)}{1 + 4(Q^2 n_0/T_e)\hat{V}(kb)]} \leq Q_t(4/b)\hat{V}(kb), \tag{44}$$

where we define $\hat{\varphi}_s$ as

$$\hat{\varphi}_s(kb) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \frac{dx}{b}\, \varphi_s(x)\, \exp\{-i[(kb)\,(x/b)]\}\,. \tag{45}$$

Inverting the Fourier transform and defining $\Phi_s(X) = (b/4)\,\varphi_s(x)$, from (44) we obtain in dimensionless variables with $X = x/b$

$$\Phi_s(X) = \frac{2Q_t}{(2\pi)^{1/2}} \int_0^{+\infty} d\kappa\, \frac{\hat{V}(\kappa)}{1 + \hat{V}(\kappa)/\Theta}\, \cos(\kappa X)\,. \tag{46}$$

It is not direct to perform the integration in (46) analytically because of the logarithmic dependence of $\hat{V}(kb)$ for $kb \ll 1$ (see (17)), which leads to a non-exponential decay at large distances of the screened potential that is not present in a 3-D Coulomb plasma. This non-exponential decay is confirmed by integrating (46) numerically for different values of the dimensionless temperature Θ.

In Fig. 8 we show the screened electric field E_{scr}, as defined by differentiating (46) with respect to X, for $\Theta = 2$. We see that for large values of X, the screened electric field has a power law behaviour $\propto X^{-2.2}$ and decays faster than the unscreened field.

4.5 Plasma Parameter

Disregarding corrections due to the presence of the non-exponential decay of the screened potential, we define the plasma parameter g for a disk plasma by referring to the length λ_{Dpe} in (38) as

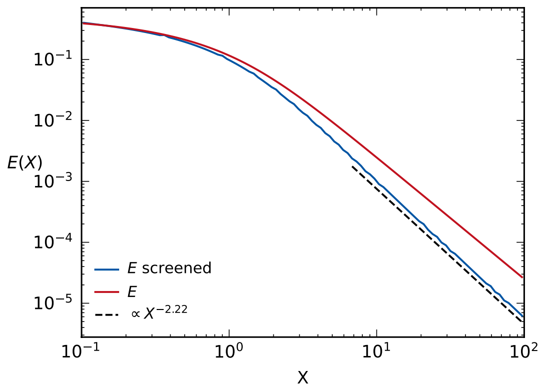

Fig. 8 Log-Log plot of the (normalized) screened electric field (blue) drawn against the unscreened field (red) for $\theta = 2$. The power law behaviour of the screened electric field is noted explicitly (black dashed line)

$$g^{-1} = n\lambda_{Dpe} = n\,\frac{v_{\rm the}}{\omega_{Dpe}} = (nb/2)^{1/2}\left[\frac{(M_e v_{\rm the}^2/2)}{(Q^2/b)}\right]^{1/2}, \tag{47}$$

where it is assumed that $g \ll 1$. Here n^{-1} plays the role of the average distance between disks while $(M_e v_{\rm the}^2/2)/(Q^2/b)$ gives the ratio between the disk kinetic energy and a reference value of the disk-disk interaction energy. The plasma parameter g can be expressed in terms of the disk radius b and the dimensionless temperature $\Theta = T_e/[4(Q^2 n_o)]$ as

$$g^{-1} = n\lambda_{Dpe} = nb\,\Theta^{1/2}\,. \tag{48}$$

5 BBGKY hierarchy for a disk plasma

In this section we apply the results obtained for the disk electrostatics to the system of equations of the BBGKY hierarchy in Sect. 2.

We insert the disk interaction potential defined by (13) into the time evolution equation (see (7)), for the one point distribution function $F^{(1)}(x_1, v_1, t)$ involving the two point distribution function $F^{(2)}(x_1, v_1, x_2, v_2, t)$ and write for $N \gg 1$, where N is the total number of disks,

$$\frac{\partial F^{(1)}(x_1, v_1, t)}{\partial t} + v_1\frac{\partial F^{(1)}(x_1, v_1, t)}{\partial x_1} - \frac{N}{M_1}\frac{4Q_1Q_s}{b}\frac{\partial}{\partial v_1}\int dx_s \frac{\partial V(|x_1 - x_s|/b)}{\partial x_1}\int dv_s\, F^{(2)}(x_1, v_1, x_s, v_s, t) = 0\,, \tag{49}$$

where

$$F^{(2)}(x_1, v_1, x_s, v_s, t) = F^{(1)}(x_1, v_1, t)F^{(1)}(x_s, v_s, t) + \Delta F^{(2)}(x_1, v_1, x_s, v_s, s). \tag{50}$$

Here, the electron and the ion disk distribution functions have not been separated for the sake of notational compactness, but indices have been added to the charge Q and to the mass M depending on the charge and on the mass of the disks involved. In addition x - v coordinates with disk indices are used in the 2-D disk phase space of each disk.

Similarly, the evolution equation for $F^{(2)}(x_1, v_1, x_2, v_2, t)$ in terms of the three point distribution function $F^{(3)}(x_1, v_1, x_2, v_2, x_3, v_3, t)$ (see (8)), becomes

$$
\begin{aligned}
&\frac{\partial F^{(2)}(x_1,v_1,x_2,v_2,t)}{\partial t} + v_1 \frac{\partial F^{(2)}(x_1,v_1,x_2,v_2,t)}{\partial x_1} + v_2 \frac{\partial F^{(2)}(x_1,v_1,x_2,v_2,t)}{\partial x_2} \\
&- \frac{1}{M_1}\frac{4Q_1Q_2}{b}\left[\frac{\partial V(|x_1-x_2|/b)}{\partial x_1}\frac{\partial F^{(2)}(x_1,v_1,x_2,v_2,t)}{\partial v_1}\right] \\
&- \frac{1}{M_2}\frac{4Q_1Q_2}{b}\left[\frac{\partial V(|x_1-x_2|/b)}{\partial x_2}\frac{\partial F^{(2)}(x_1,v_1,x_2,v_2,t)}{\partial v_2}\right] \\
&-\frac{N}{M_1}\frac{4Q_1Q_s}{b}\left[\frac{\partial}{\partial v_1}\int dx_s \frac{\partial V(|x_1-x_s|/b)}{\partial x_1}\int dv_s\, F^{(3)}(x_1,v_1,x_2,v_2,x_s,v_s,t)\right] \\
&-\frac{N}{M_2}\frac{4Q_2Q_s}{b}\left[\frac{\partial}{\partial v_2}\int dx_s \frac{\partial V(|x_2-x_s|/b)}{\partial x_2}\int dv_s\, F^{(3)}(x_1,v_1,x_2,v_2,x_s,v_s,t)\right] = 0\,.
\end{aligned}
\tag{51}
$$

Clearly (51) implies (49) as can be seen by setting $N-2 \sim N-1 \sim N$ and integrating over x_2 and v_2. Recalling the expansion in (10) we rewrite

$$
\begin{aligned}
F^{(3)}(x_1,x_2,x_s,v_1,v_2,v_s,t) &= F^{(1)}(x_1,v_1,t)F^{(1)}(x_2,v_2,t)F^{(1)}(x_s,v_s,t) \\
&+ F^{(1)}(x_s,v_s,t)\,\Delta F^{(2)}(x_1,v_1,x_2,v_2,t) + F^{(1)}(x_1,v_1,t)\,\Delta F^{(2)}(x_2,v_2,x_s,v_s,t) \\
&+ F^{(1)}(x_2,v_2,t)\,\Delta F^{(2)}(x_1,v_1,x_s,v_s,t) + \mathcal{O}(g^2)
\end{aligned}
\tag{52}
$$

in the form

$$
\begin{aligned}
F^{(3)}(x_1,x_2,x_s,v_1,v_2,v_s,t) &= F^{(2)}(x_1,x_2,v_1,v_2,t)F^{(1)}(x_s,v_s,t) \\
&+ F^{(1)}(x_1,v_1,t)\,\Delta F^{(2)}(x_2,v_2,x_s,v_s,t) \\
&+ F^{(1)}(x_2,v_2,t)\,\Delta F^{(2)}(x_1,v_1,x_s,v_s,t) + \mathcal{O}(g^2)\,,
\end{aligned}
\tag{53}
$$

where the first and the second terms on the r.h.s. of (52) have been recombined into the first term on the r.h.s. of (53). This reorganization of the terms is convenient since the disk marked by s plays a different role than the disks marked as 1 and 2 in (51).

5.1 Closed system of equations

We refer now to a disk plasma in a finite spatial linear domain of length L, corresponding to the finite volume in Sect. 2 for a 3-D configuration, and, to connect to the notation used in Sect. 4.3 for the Vlasov equation, we introduce an average linear disk density

$$
\bar{n} = N/L\,, \tag{54}
$$

and define

$$
f^{(1)}(x_1,v_1,t) = \bar{n}\,F^{(1)}(x_1,v_1,t)\,, \tag{55}
$$

$$f^{(2)}(x_1, v_1, x_2, v_2, t) = \bar{n}^2 F^{(2)}(x_1, v_1, x_2, v_2, t)\,. \tag{56}$$

Then (50), with $s \to 2$, becomes

$$f^{(2)}(x_1, v_1, x_2, v_2, t) = f^{(1)}(x_1, v_1, t) f^{(1)}(x_2, v_2, t) + \Delta f^{(2)}(x_1, v_1, x_2, v_2, t)\,. \tag{57}$$

In terms of the functions $f^{(1)}(x_1, v_1, t)$ and $f^{(2)}(x_1, v_1, x_2, v_2, t)$, (49) and (51) can be rewritten as

$$\left[\frac{\partial}{\partial t} + v_1 \frac{\partial}{\partial x_1}\right] f^{(1)}(x_1, v_1, t) = \mathcal{S}^{(1)}(f^{(2)}), \tag{58}$$

and

$$\left[\frac{\partial}{\partial t} + v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2}\right] f^{(2)}(x_1, v_1, x_2, v_2, t) = \mathcal{S}^{(2)}(f^{(1)}, f^{(2)}, \Delta f^{(2)})\,. \tag{59}$$

Here

$$\mathcal{S}^{(1)}(f^{(2)}) = \frac{L}{M_1} \frac{4Q_1 Q_s}{b} \frac{\partial}{\partial v_1} \int_{-L/2}^{+L/2} dx_s \frac{\partial V(|x_1 - x_s|/b)}{\partial x_1} \int_{-\infty}^{+\infty} dv_s f^{(2)}(x_1, v_1, x_s, v_s, t)\,, \tag{60}$$

and

$$\begin{aligned}
\mathcal{S}^{(2)}(f^{(1)}, f^{(2)}, \Delta f^{(2)}) = {} & \frac{1}{M_1} \frac{4Q_1 Q_2}{b} \left[\frac{\partial V(|x_1 - x_2|/b)}{\partial x_1} \frac{\partial f^{(2)}(x_1, v_1, x_2, v_2, t)}{\partial v_1}\right] \\
& + \frac{1}{M_2} \frac{4Q_1 Q_2}{b} \left[\frac{\partial V(|x_1 - x_2|/b)}{\partial x_2} \frac{\partial f^{(2)}(x_1, v_1, x_2, v_2, t)}{\partial v_2}\right] \\
& + \frac{L}{M_1} \frac{4Q_1 Q_s}{b} \left[\frac{\partial}{\partial v_1} f^{(2)}(x_1, v_1, x_2, v_2, t) \int_{-L/2}^{+L/2} dx_s \frac{\partial V(|x_1 - x_s|/b)}{\partial x_1} \int_{-\infty}^{+\infty} dv_s f^{(1)}(x_s, v_s, t)\right] \\
& + \frac{L}{M_2} \frac{4Q_2 Q_s}{b} \left[\frac{\partial}{\partial v_2} f^{(2)}(x_1, v_1, x_2, v_2, t) \int_{-L/2}^{+L/2} dx_s \frac{\partial V(|x_2 - x_s|/b)}{\partial x_2} \int_{-\infty}^{+\infty} dv_s f^{(1)}(x_s, v_s, t)\right] \\
& + \frac{L}{M_1} \frac{4Q_1 Q_s}{b} \left[\frac{\partial}{\partial v_1} \int_{-L/2}^{+L/2} dx_s \frac{\partial V(|x_1 - x_s|/b)}{\partial x_1} f^{(1)}(x_1, v_1, t) \int_{-\infty}^{+\infty} dv_s\, \Delta f^{(2)}(x_2, v_2, x_s, v_s, t)\right] \\
& + \frac{L}{M_2} \frac{4Q_2 Q_s}{b} f^{(1)}(x_1, v_1, t) \left[\frac{\partial}{\partial v_2} \int_{-L/2}^{+L/2} dx_s \frac{\partial V(|x_2 - x_s|/b)}{\partial x_2} \int_{-\infty}^{+\infty} dv_s\, \Delta f^{(2)}(x_2, v_2, x_s, v_s, t)\right] \\
& + \frac{L}{M_1} \frac{4Q_1 Q_s}{b} f^{(1)}(x_2, v_2, t) \left[\frac{\partial}{\partial v_1} \int_{-L/2}^{+L/2} dx_s \frac{\partial V(|x_1 - x_s|/b)}{\partial x_1} \int_{-\infty}^{+\infty} dv_s\, \Delta f^{(2)}(x_1, v_1, x_s, v_s, t)\right] \\
& + \frac{L}{M_2} \frac{4Q_2 Q_s}{b} \left[\frac{\partial}{\partial v_2} \int_{-L/2}^{+L/2} dx_s \frac{\partial V(|x_2 - x_s|/b)}{\partial x_2} f^{(1)}(x_2, v_2, t) \int_{-\infty}^{+\infty} dv_s\, \Delta f^{(2)}(x_1, v_1, x_s, v_s, t)\right].
\end{aligned} \tag{61}$$

Equations (57) and (61) are a closed system of equations for the functions $f^{(1)}$, $f^{(2)}$ and $\Delta f^{(2)}$.

This system can be solved by assigning at $t = 0$ the initial values of $f^{(1)}$ and $\Delta f^{(2)}$. This makes it possible to compute

$$f^{(2)}(x_1, v_1, x_2, v_2, 0) = f^{(1)}(x_1, v_1, 0) f^{(1)}(x_2, v_2, t=0) + \Delta f^{(2)}(x_1, v_1, x_2, v_2, 0) \quad (62)$$

algebraically. The condition

$$\int_{L/2}^{L/2} dx_2 \int_{-\infty}^{\infty} dv_2 f^{(2)}(x_1, x_2, v_1, v_2, t) = \bar{n} f^{(1)}(x_1, v_1, t), \quad (63)$$

i.e.,

$$\int_{L/2}^{L/2} dx_2 \int_{-\infty}^{\infty} dv_2 \; \Delta f^{(2)}(x_1, x_2, v_1, v_2, t) = 0 \quad (64)$$

is valid by definition at all times and thus the initial choice of $\Delta f^{(2)}(x_1, v_1, x_2, v_2, 0)$ must satisfy it.

For $t > 0$, the functions $f^{(1)}(x_1, v_2, t)$ and $f^{(2)}(x_1, x_2, v_1, v_2, t)$ can be advanced in time using (58) and (59) respectively and calculating at every step $\Delta f^{(2)}(x_1, x_2, v_1, v_2, t)$ algebraically. Since (58) and (59) are integrated independently, one must verify that they satisfy the consistency condition (63) for all times. This check provides a tool that will be used to test the accuracy of their numerical integration.

Conversely, one can avoid to advance in time both (58) and (59) and use (63) to transform equation (59) for $f^{(2)}(x_1, x_2, v_1, v_2, t)$ into an integro-differential equation by expressing $f^{(1)}$ in terms of $f^{(2)}$ by direct space and velocity integration. In this approach a check of the accuracy of the numerical integration will be the conservation of the total number of particles.

By integrating (58) and (59) it will be possible to ascertain directly whether, and under which conditions, $\Delta f^{(2)}(x_1, v_1, x_2, v_2, t)$ does reach an asymptotic form that is uniquely determined by the instantaneous form of $f^{(1)}(x, v, t)$ i.e. whether we can write (see (11))

$$\Delta f^{(2)}(x_1, v_1, x_2, v_2, t) = \Delta f^{(2)}[f^{(1)}(x_1, v_1, t), f^{(1)}(x_2, v_2, t)]\,, \quad (65)$$

and determine how the characteristic time that may be needed to reach this asymptotic form compares with the dynamical time scale of the system.

6 Generalizations and future developments

The model plasma introduced in the present article can also provide interesting information when applied to problems different from the closure of the BBGKY hierarchy for an electrostatic system.

In a direct generalization one can conceive of a plasma of gravitationally interacting disks with different masses (light and heavy disks) by suitably modifying the coefficients and the sign in the expression of the interaction energy in (12). Such a modification would be of interest for dealing with the problem of the mass segregation in 1-D stellar system (Bertin and Pegoraro 2022) representing, e.g., a spatially flat and uniform distribution of stars. Mass segregation in star systems cannot be described within the (mean field) Vlasov framework, where the star trajectories do not depend on their mass, but are allowed if a collision operator, arising from star correlations, is included. For such a system the model presented in this article would make it possible to address the problem of the mass segregation by deriving a collision operator in a 1-D configuration while still maintaining that the potential vanishes at infinity. Simulations with a large but finite number of gravitationally interacting disks can be easily performed and show disk evaporation (i.e., disks becoming unbound), mass segregation, and the disk phase space behavior for the different disk masses (Pegoraro and Morrison 2023).

Returning to the problem of the validation of the Bogoliubov assumption, the system of (58) and (59) will be integrated using an Eulerian Vlasov code (Mangeney et al. 2002) that solves the Vlasov equation in a multidimensional phase space. The advantage of the Eulerian approach (Califano and Cerri 2022) is that it provides an almost zero-noise integration procedure, even at the smaller scales in a fully non-linear, turbulent regime; however, at the expense of a huge computational cost. The original version of the code must be adapted to a larger phase space but the splitting approach of a multi-advection equation will be preserved. Periodic space boundary conditions will be imposed. With respect to the usual electrostatic approach in plasma physics where the Vlasov equation is coupled to the Poisson equation, here the electric potential is obtained from the disk charge density by means of a Green's function where the kernel is the interaction potential between individual disks. This procedure has been already implemented for the numerical integration of the disk Vlasov equation that led to the plots of the warm plasma dispersion relation in Sect. 4.3.

The numerical solution of the system of (58) remains a formidable challenge even on the latest generation of GPU supercomputers. We plan to investigate a selected set of physically relevant conditions with the aim of verifying within the model introduced in this article under which conditions the Bogoliubov approach holds and whether a "collision" operator can be defined under general plasma conditions.

However dimensionality plays a major role in plasma dynamics, as is well known for example in the case of plasma turbulence. In a one-dimensional disk plasma the charged disk dynamics is more constrained than that of charged particles in a real three-dimensional plasma and it is not possible to define and impact parameter. In this sense one cannot distinguish for example between soft and hard "collisions". Thus the model introduced in this article cannot be expected to provide a conclusive validation or disproval of the Bogoliubov assumption in a three-dimensional plasma but can provide information on how the time separation that is at the basis of this assumption depends on the initial particle correlation, on the presence of distribution functions with highly energetic particle tails, of large electric field fluctuations or strong plasma spatial inhomogeneities. These features can also occur in a three-dimensional plasma so that the information that can be gained from the model described in the present manuscript can provide useful indications to identify relevant processes that can be at work in a three dimensional plasma and possibly not be covered by the Bogoliubov assumption.

Disks electrostatics

A set of relevant formulae derived in Ciftja (2019) are listed below with some notational adaptations together with the derivation of (16).

The axially symmetric electrostatic potential $\varphi(r,x)$ created by a uniformly charged disk with its center at the coordinate origin, with total charge Q and radius b can be written as

$$\varphi(r,x) = 2Q\int_0^{+\infty} ds\, J_0(sr)\frac{J_1(sb)}{sb}\exp(-s|x|), \tag{A1}$$

where J_0 and J_1 are Bessel functions, while the x-component of the electric field can be written as

$$\epsilon_x(r,x) = \mathrm{sign(x)}\,\frac{2Q}{b}\int_0^{+\infty} ds\, J_0(sr)J_1(sb)\,\exp(-s|x|). \tag{A2}$$

At $r = 0$, it reduces to

$$\begin{aligned}\epsilon_x(0,x) &= \mathrm{sign(x)}\,\frac{2Q}{b}\int_0^{+\infty} ds\, J_1(sb)\,\exp(-s|x|)\\ &= \mathrm{sign(x)}\,\frac{2Q}{b^2}\left[1-\frac{|x|/b}{(1+x^2/b^2)^{1/2}}\right].\end{aligned} \tag{A3}$$

The expression for the interaction energy in (12) is given by

$$\mathcal{W}(|x|) = 4\frac{Q_1Q_2}{b}V(|x|/b) = \frac{4Q_1Q_2}{b}\int_0^{+\infty} ds\,[J_1(s)/s]^2\,\exp(-s|x|/b)\,, \tag{A4}$$

which follows by integrating (A1) over the disk surface (see (15) of Ciftja (2019)).

Fourier transform of $V(\xi)$

Using (A4), the Fourier transform of $V(\xi)$ in (13) can be obtained as follows

$$\begin{aligned}\hat{V}(kb) &= \int_{-\infty}^{+\infty}\frac{dx}{b}\,V(|x|/b)\exp\{-[i(kb)\,(x/b)]\}\\ &= \int_0^{+\infty} ds\,[J_1(s)/s]^2\int_{-\infty}^{+\infty}\frac{dx}{b}\,\exp(-s|x|/b)\exp\{-[i(kb)\,(x/b)]\}\\ &= 2\int_0^{+\infty} ds\,[J_1(s)/s]^2\int_0^{+\infty}\frac{dx}{b}\,\exp(-sx/b)\cos[(kb)\,(x/b)]\\ &= 2\int_0^{+\infty} ds\,[J_1(s)/s]^2\,\frac{s}{(kb)^2+s^2} = \frac{1}{\pi^{1/2}(kb)^2}\,G^{2,2}_{2,4}\left((kb)^2\middle|\begin{matrix}1/2,1\\1,1,-1,0\end{matrix}\right),\end{aligned} \tag{B5}$$

where Meijer G is the Meijer G function [24]. In the notation used by "Mathematica" as (see https://reference.wolfram.com/language/ref/MeijerG.html) the Meijer G function is denoted as

$$G^{2,2}_{2,4}\left(z\middle|{a_1,a_2 \atop b_1,b_2,b_3,b_4}\right) = \mathrm{Meijer}G[\{\{a_1,a_2\},\{...\}\},\{\{b_1,b_2\},\{b_3,b_4\}\},z]. \quad \text{(B6)}$$

Acknowledgements It is a pleasure to thank Prof. G. Moruzzi for his help in the study of the discrete multi-disk dynamics for both the cases of electrostatic and of gravitational interaction. PJM acknowledges support from the US Department of Energy DE-FG02-04ER54742.

Data availability There is no data availability statement to make.

Declarations

Conflict of interest The authors declare that there are no Conflict of interest.

References

M. Abramowitz, I.A. Stegun I.A, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, (Dover, New York, 1964) https://personal.math.ubc.ca/~cbm/aands/

S.D. Baalrud, J. Daligault, Mean force kinetic theory: a convergent kinetic theory for weakly and strongly coupled plasmas. Phys. Plasmas **26**, 082106 (2019) https://pubs.aip.org/aip/pop/article/26/8/082106/631827/Mean-force-kinetic-theory-A-convergent-kinetic

R. Balescu, Irreversible processes in ionized gases, Phys. Fluids, **3**, 52-63 (1960) https://pubs.aip.org/aip/pfl/article-abstract/3/1/52/943789/Irreversible-Processes-in-Ionized-Gases?redirectedFrom=fulltext

S.D. Bergeson, S.D. Baalrud, C.L. Ellison, E. Grant, F.R. Graziani, T.C. Killian, M.S. Murillo, J.L. Roberts, L.G. Stanton, Exploring the crossover between high-energy-density plasma and ultracold neutral plasma physics. Phys. Plasmas **26**, 100501 (2019). https://doi.org/10.1063/1.5119144

G. Bertin, F. Pegoraro, Two-component self-gravitating isothermal slab models, Eur. Phys. J. Plus, **137**, 538 (2022) https://link.springer.com/article/10.1140/epjp/s13360-022-02727-2

N.N. Bogoliubov, Kinetic Equations. Journal of Physics USSR **10**, 265–274 (1946)

F. Califano, S.S. Cerri, Eulerian approach to the Vlasov equation: codes and simulations, Space and Astrophysical plasma simulations, Ed. J. Buchner, Lecture Notes in Physics, Springer Cham (2022) https://link.springer.com/chapter/10.1007/978-3-031-11870-8_5

C. Cercignani, V.I. Gerasimenko, D.Ya. Petrinam, Many-Particle Dynamics and Kinetic Equations, Mathematics and Its Applications (MAIA), **420** (1997) https://link.springer.com/book/10.1007/978-94-011-5558-8

O. Ciftja, Electrostatic interaction energy between two coaxial parallel uniformly charged disks, Results in Physics, **15**, 102684 (2019) https://doi.org/10.1016/j.rinp.2019.102684

G.N. Gubal, On the existence of a weak local-in-time solution in the form of a cumulant expansion for a chain of Bogolyubov equations for a one-dimensional symmetric particle system. J. Math. Sci. **199**, 654–666 (2014) https://link.springer.com/article/10.1007/s10958-014-1892-1

S. Ichimaru, Basic principles of plasma physics, Benjamin, Reading, MA, (1973) https://doi.org/10.1201/9780429502118

N.A. Krall, A.W. Trivelpiece, Principles of plasma physics, (San Francisco Press, San Francisco, CA) (1986) https://books.google.it/books/about/Principles_of_Plasma_Physics.html?id=b0BRAAAAMAAJ&redir_esc=y

S. Krönke, P. Schmelcher, Born-Bogoliubov-Green-Kirkwood-Yvon hierarchy for ultracold bosonic systems. Phys. Rev. A **98**, 013629 (2018) https://journals.aps.org/pra/abstract/10.1103/PhysRevA.98.013629

L.D. Landau, Kinetic equation for the case of Coulomb interaction. J. Phys. USSR **10**, 154–164 (1936)

A. Lenard, On Bogoliubov's kinetic equation for a spatially homogeneous plasma. Ann. Phys. **10**, 390–400 (1960) https://www.sciencedirect.com/science/article/pii/0003491660900038

A. Mangeney, F. Califano, C. Cavazzoni, P. Travnicek, A numerical scheme for the integration of the Vlasov - Maxwell system of equations, J. Comput. Phys., 179, 495-530 (2002) https://www.sciencedirect.com/science/article/pii/S0021999102970713

J.E. Marsden, P.J. Morrison, A. Weinstein, The Hamiltonian Structure of the BBGKY Hierarchy Equations. Contemporary Mathematics **28**, 115–124 (1984) https://w3fusion.ph.utexas.edu/ifs/ifsreports/122_morrison.pdf

P.J. Morrison, F. Pegoraro, On the Hamiltonian structure of the BBGKY hierarchy, 63rd Annual Meeting of the APS Division of Plasma Physics, UP11.00040 (2021) https://meetings.aps.org/Meeting/DPP21/Session/UP11.40

NIST : Digital Library of Mathematical Functions, https://dlmf.nist.gov/

P-H. Chavanis, Hamiltonian and Brownian systems with long-range interactions: III. The BBGKY hierarchy for spatially inhomogeneous systems, Physica A, **387**, 787-805 (2008) https://www.sciencedirect.com/science/article/pii/S0378437107010291

F. Pegoraro, P.J. Morrison, A 1-dimensional plasma model for testing the validity of kinetic theory, presentation at the workshop "Mathematical Justification for the Kinetic and Fluid Equations of Plasmas and Self-Gravitating Systems" held at CIRM (Centre International de Rencontres Mathématiques), France (2023) https://conferences.cirm-math.fr/wp-content/uploads/2023/07/Booklet2962.pdf

F. Pegoraro, P.J. Morrison, Notes on a 1-dimensional electrostatic plasma model, arXiv/2210.04254, (2022) https://arxiv.org/pdf/2210.04254

L. Pucci, A. Roy, M. Kastner, Simulation of quantum spin dynamics by phase space sampling of Bogoliubov-Born-Green-Kirkwood-Yvon trajectories. Phys. Rev. B **93**, 174302 (2016) https://journals.aps.org/prb/abstract/10.1103/PhysRevB.93.174302

T.V. Ryabukha, On regularized solution for BBGKY Hierarchy of One-Dimensional Infinite System, Symmetry, Integrability and Geometry: Methods and Applications (SIGMA), **2**. 053 (2006) https://www.emis.de/journals/SIGMA/2006/Paper053/sigma06-053.pdf

R. Singleton BPS Explained III: The Leading Order Behavior of the BBGKY Hierarchy in a Plasma, arXiv:2007.10103v2, (2020) https://arxiv.org/pdf/2007.10103v2

S.A. Smolyansky, A.M. Fedotov, V.V. Dmitriev BBGKY Method in Strong Field QED, Physics of Particles and Nuclei **51**, 595–598, Pleiades Publishing, Ltd. (2020) https://link.springer.com/article/10.1134/S106377962004067X

N. G. van Kampen, B. U. Felderhoff, Theoretical Methods in Plasma Physics, Wiley, New York, (1967) https://archive.org/search?query=external-identifier%3A%22urn%3Aoclc%3Arecord%3A1151442984%22

G.V. Vereshchagin, A.G. Aksenov, Relativistic Kinetic Theory: With Applications in Astrophysics and Cosmology, Cambridge University Press; 299-325 (2017) https://www.cambridge.org/core/books/relativistic-kinetic-theory/55F3257D8A0B6595C5265D15C93445B8