COMMENTS ON: THE MAXWELL–VLASOV EQUATIONS AS A CONTINUOUS HAMILTONIAN SYSTEM

Alan WEINSTEIN
Department of Mathematics, University of California, Berkeley, CA 94720, USA

and

Philip J. MORRISON [1]
Plasma Physics Laboratory, Princeton University, Princeton, NJ 08544, USA

Received 6 July 1981
Revised manuscript received 17 September 1981

The Poisson structure previously introduced by Morrison for the Maxwell–Vlasov equations does not satisfy the Jacobi identity. The corrected Poisson structure has been found by Marsden and Weinstein. It is derived by standard constructions in symplectic geometry and therefore satisfies the Jacobi identity.

In ref. [1] it was shown that the Maxwell–Vlasov equations for a collisionless plasma in an electromagnetic field could be written in hamiltonian form in terms of a Poisson bracket structure on the space of functionals $F(f, \boldsymbol{E}, \boldsymbol{B})$. In this reference it was claimed that the Jacobi identity had been proven for the terms of the Poisson structure which produce Maxwell's equations in vacuum and the Vlasov equation without acceleration. These terms are correct. It was also indicated that the Jacobi identity had not been proven in complete generality for the terms which provide coupling between the Maxwell and Vlasov equations. Although the electric field coupling term is correct, the magnetic field coupling term is in error.

This note contains an example which points out how this magnetic field coupling term causes the Jacobi identity to fail. This means, for example, that the Poisson bracket of two constants of the motion may not be a constant of the motion. Following the example, we correct this error along with several additional minor errors contained in ref. [1].

[1] Present address: Department of Physics and Institute for Fusion Studies, University of Texas at Austin, Austin, TX 78712, USA.

The example depends upon the following simple consequence of the Jacobi identity $\{F, \{G,H\}\} + \{G, \{H,F\}\} + \{H,\{F, G\}\} = 0$: if $\{F, G\} = 0$, then $\{F, \{G,H\}\} = \{G, \{F,H\}\}$ for all H. If $F(f, \boldsymbol{E}, \boldsymbol{B})$ and $G(f, \boldsymbol{E}, \boldsymbol{B})$ depend upon $\boldsymbol{B}$ alone, they satisfy $\{F, G\} = 0$ because the lower right hand box of matrix (8) of ref. [1] is zero.

Now let $F(f, \boldsymbol{E}, \boldsymbol{B}) = \int B_1(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}$, $G(f, \boldsymbol{E}, \boldsymbol{B}) = \int B_2(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}$, and $H(f, \boldsymbol{E}, \boldsymbol{B}) = \int \phi(\boldsymbol{x}, \boldsymbol{v}) f(\boldsymbol{x}, \boldsymbol{v})\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}\boldsymbol{v}$, where ϕ is a function of $\boldsymbol{x}$ and $\boldsymbol{v}$ to be chosen later. Evaluation of the Poisson brackets (only the upper right and the lower left boxes of (8) have nonzero contributions) gives:

$$\{F, H\}(f, \boldsymbol{E}, \boldsymbol{B}) = \int \left(v_3 \frac{\partial \phi}{\partial v_1} - v_1 \frac{\partial \phi}{\partial v_3} \right) f\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}\boldsymbol{v}\,,$$

$$\{G, H\}(f, \boldsymbol{E}, \boldsymbol{B}) = \int \left(v_1 \frac{\partial \phi}{\partial v_2} - v_2 \frac{\partial \phi}{\partial v_1} \right) f\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}\boldsymbol{v}\,.$$

But the operators $v_3 \partial/\partial v_1 - v_1 \partial/\partial v_3$ and $v_1 \partial/\partial v_2 - v_2 \partial/\partial v_1$ do not commute. Specifically,

$$\{F, \{G,H\}\} - \{G, \{F,H\}\}(f, \boldsymbol{E}, \boldsymbol{B}) = \int \left(v_3 \frac{\partial \phi}{\partial v_2} - v_2 \frac{\partial \phi}{\partial v_3} \right) f\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}\boldsymbol{v}\,.$$

Choosing $\phi = v_2$, for example, gives $\int v_3 f \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}\boldsymbol{v}$, which is a non-zero functional of f.

The corrected Poisson bracket was found by Marsden and Weinstein [2] by using standard constructions in symplectic geometry. It follows from general theory that the corrected bracket necessarily satisfies the Jacobi identity. This is of great utility since direct proofs can be quite lengthy and difficult.

The corrected bracket is obtained by setting $\boldsymbol{P}_B^{1\to 2}$ and $\boldsymbol{P}_B^{2\to 1}$ to zero in eqs. (8), (10) and (11) of ref. [1], and adding

$$-\frac{e_\alpha}{m_\alpha^2}\boldsymbol{B}\cdot\left[\frac{\partial f_\alpha}{\partial \boldsymbol{v}}\times\frac{\partial}{\partial \boldsymbol{v}}(\cdot)\right]$$

to the upper left-hand diagonal entry of eq. (8). Equivalently, one must add to eq. (10) the term

$$\sum_\alpha \frac{e_\alpha}{m_\alpha^2}\int_{\mathrm{R}_1} f_\alpha \boldsymbol{B}\cdot\left[\frac{\partial}{\partial \boldsymbol{v}}\frac{\delta F}{\delta f_\alpha}\times\frac{\partial}{\partial \boldsymbol{v}}\frac{\delta G}{\delta f_\alpha}\right]\mathrm{d}z\,.$$

Additional errata for ref. [1] are:

Page 384, column 2, paragraph 2, line 3 should read: "...invariant and a bilinear bracket...".

The usual Poisson bracket in the upper left-hand diagonal entry of eq. (8) should be divided by m_α.

The first term of eq. (10) and eq. (12) should contain an m_α in their denominators.

The hamiltonian $\hat{H}_E$ following eq. (12) should read

$$\hat{H}_E = \sum_\alpha \int_{\mathrm{R}_1} H_1^\alpha(z) f_\alpha(z)\,\mathrm{d}z + \sum_{\alpha,\beta}\frac{1}{2}\int_{\mathrm{R}_1}\int_{\mathrm{R}_1} f_\alpha(z) f_\beta(z') H_2^{\alpha\beta}(z|z')\,\mathrm{d}z\,\mathrm{d}z'.$$

In the next to last line of the paper $H_2^\alpha = \ldots$ should be replaced by $H_2^{\alpha\beta} = -e_\alpha e_\beta/|\boldsymbol{r}-\boldsymbol{r}'|$.

We would like to thank Allan Kaufman for his continuing encouragement. He has played a great role in the dissemination of ideas related to this work. One of us (A.W.) would like to acknowledge work done in collaboration with Jerry Marsden on the Maxwell–Vlasov equations and to thank Boris Kupershmidt for planting seeds of doubt concerning the Jacobi identity. The research was partially supported by NSF MCS 77-23579 and DOE Contract No. DE-AC02-76-CH03073.

References

[1] P.J. Morrison, Phys. Lett. 80A (1980) 383.
[2] J. Marsden and A. Weinstein, The hamiltonian structure of the Maxwell–Vlasov equations, Physica D, to be published (1982).