ALGEBRAIC STRUCTURE OF THE PLASMA QUASILINEAR EQUATIONS ☆

Allan N. KAUFMAN
Lawrence Berkeley Laboratory
and Physics Department, University of California, Berkeley, CA 94720, USA

and

Philip J. MORRISON [1]
Plasma Physics Laboratory, Princeton University, Princeton, NJ 08544, USA

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The standard quasilinear equations of plasma physics are shown to possess an algebraic structure, although the system is dissipative. The energy functional yields the evolution equations and the conservation laws, in analogy to hamiltonian systems.

Our recent discovery [1] of a hamiltonian structure for the Vlasov equation with Coulomb interaction (discovered independently by Gibbons [2]) has led us to search for an algebraic structure for the corresponding dissipative system, the quasilinear diffusion equations for an unstable plasma. By analogy to the hamiltonian structure, we desire a bracket and an energy functional that yield the evolution equations and conservation laws. However, this bracket cannot be a Lie algebra, implying a hamiltonian structure, since the quasilinear system possesses a Liapunov functional, the entropy, expressing irreversibility.

In the interests of simplicity and clarity, we deal here with the simplest case, a uniform unmagnetized plasma, with one species of resonant particles and one wave branch. The particle distribution in momentum space is $f(\boldsymbol{p})$, and the total particle energy functional is

$$\mathcal{H}(f) = \int \mathrm{d}^3p\, H(\boldsymbol{p}) f(\boldsymbol{p}) , \tag{1}$$

where $H(\boldsymbol{p})$ is the single-particle energy. The wave action distribution in wave vector space is $J(\boldsymbol{k})$, and the total wave energy functional is

$$\mathcal{H}(J) = \int \mathrm{d}^3k\, \omega(\boldsymbol{k}) J(\boldsymbol{k}) , \tag{2}$$

where $\omega(\boldsymbol{k})$ is the wave dispersion relation. The total energy,

$$\mathcal{H}(f, J) = \mathcal{H}(f) + \mathcal{H}(J) , \tag{3}$$

contains no interaction term. The resonant wave–particle interaction appears in the bracket, eq. (5).

Consider now two observables, $Q_1(f, J)$ and $Q_2(f, J)$. We search for a bracket algebra: $\{Q_1, Q_2\} = Q_3$, which is bilinear, antisymmetric, and operates on Q_1 and Q_2 with first functional derivatives. In addition, we demand that observables evolve in time as

$$\dot{Q} = \{Q, \mathcal{H}(f, J)\} . \tag{4}$$

Since the quasilinear evolution equations are known, a short search yields the desired result:

$$\{Q_1, Q_2\} = \int \mathrm{d}^3p \int \mathrm{d}^3k \left(\frac{\delta Q_1}{\delta f} \frac{\delta Q_2}{\delta J} - \frac{\delta Q_1}{\delta J} \frac{\delta Q_2}{\delta f} \right) \times J(\boldsymbol{k}) R f(\boldsymbol{p}) , \tag{5}$$

with

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[1] Present address: Physics Department and Institute for Fusion Studies, University of Texas, Austin, TX, USA.

$$R = \alpha(k) k \cdot \nabla \delta[\omega(k) - k \cdot \nabla H(p)] k \cdot \nabla , \quad \nabla = \partial/\partial p , \tag{6}$$

and $\alpha(k)$ is a coupling constant. The resonant wave–particle interaction resides in R. This bracket does not satisfy the Jacobi identity, and hence is not a Lie algebra.

Applying eqs. (4) and (3) to $f(p)$, we obtain the diffusion equation:

$$\dot{f}(p) = \nabla \cdot \mathsf{D}(p) \cdot \nabla f(p) , \tag{7}$$

$$\mathsf{D}(p) = \int d^3k \, kk \alpha(k) \delta[\omega(k) - k \cdot \nabla H(p)] J(k) .$$

Applying eq. (4) to $J(k)$, we obtain the linear growth equation:

$$\dot{J}(k) = 2\gamma(k) J(k) , \tag{8}$$

$$2\gamma(k) = \int d^3p \, \alpha(k) \delta[\omega(k) - k \cdot \nabla H(p)] k \cdot \nabla f(p) .$$

These are the standard equations of quasilinear theory, with resonant interactions, and no refinements (such as resonance broadening).

The conservation laws should now follow directly from (4). Energy conservation,

$$\dot{\mathcal{H}} = \{\mathcal{H}, \mathcal{H}\} = 0 , \tag{9}$$

is a trivial consequence of the antisymmetry of (5). For conservation of momentum

$$P(f, J) = \int d^3p \, p f(p) + \int d^3k \, kJ(k) , \tag{10}$$

we have

$$\dot{P} = \int d^3p \int d^3k [p\omega(k) - kH(p)] J(k) R f(p) ,$$

which vanishes, upon integration by parts. The Liapunov functional,

$$S(f) = -\int d^3p \, f(p) \ln f(p) , \tag{11}$$

evolves monotonically, as found from eq. (14):

$$\dot{S} = \int d^3p \int d^3k \, \alpha\omega J f^{-1} (k \cdot \nabla f)^2 \delta(\omega - k \cdot \nabla H)$$

$$\geqslant 0 . \tag{12}$$

These results raise a number of questions for future investigation:

(i) How is the algebraic structure discussed here related to the underlying Lie structure of the Vlasov system, in particular to the fundamental work of Marsden and Weinstein [3]?

(ii) How can this structure be modified to take into account resonance broadening and more recent improvements to quasilinear theory [4]?

(iii) How can this structure be generalized to deal with nonuniform magnetized plasma, and with nonlinearities, such as the ponderomotive hamiltonian [5]?

(iv) Do similar algebraic structures exist for other dissipative systems, such as the Boltzmann equation?

References

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