BRACKET FORMULATION FOR IRREVERSIBLE CLASSICAL FIELDS

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A bracket formulation for irreversible fields analogous to that for hamiltonian fields is presented. The formulation contains a bracket with symmetric and antisymmetric components and a generator of time translation. Plasma examples are given when the generator of time translation is the energy, entropy and Helmholtz free energy.

In recent times many fundamental nondissipative equations describing fluids and plasmas have been shown to be hamiltonian field theories in terms of generalized Poisson brackets (GPB). (For review see refs. [1–4] [‡1]). Here we report on a formalism for irreversible yet conservative systems. As an example we consider the plasma kinetic equation that is composed of the Vlasov–Poisson (VP) equation with a collision term, which includes the Landau as well as the Lenard–Balescu forms (see e.g. ref. [5]). Additional examples including fluids and nonconservative systems were given in refs. [6,7].

Recall that a GPB is a bilinear, antisymmetric operator that is a derivation on functionals and satisfies the Jacobi identity. The GPB need not be the usual Poisson bracket; hence fields that do not possess standard or canonical form can sometimes still be expressed as follows:

$$\partial\psi^i/\partial t = \{\psi^i, H\}_{\mathrm{GPB}}\,, \quad i = 1, 2, \dots N\,, \tag{1}$$

where the hamiltonian functional H is the generator of time translation and the quantities ψ^i are the field components. This formulation can capsulate the Lie symmetries of the field, and has been instrumental in obtaining nonlinear criteria for fluid and plasma equilibria [8,9].

Systems that are dissipative would not a priori be expected to fit into the form of eq. (1). Indeed it is not clear what functional should be the generator of time translation, and which algebraic properties of the binary bracket operator will lead to a rich structure.

We address the first point above by recalling that in classical thermodynamics the equilibrium state can be obtained by either the energy or entropy extremum principles. In this sense we view the energy, a function of the extensive variables, as the "generator" of equilibria, or alternatively the entropy can generate equilibria. Moreover, additional extremum principles exist in terms of the thermodynamic potentials. For dynamical systems a natural extension of this is to choose these quantities as the generators of time translation. That is, we desire a binary operator such that the dynamical field equations can be represented in the form

$$\partial\psi^i/\partial t = \{\psi^i, M\}_M\,, \quad i = 1, 2, \dots N\,. \tag{2}$$

where the quantity M can be the energy, entropy, etc. For the examples addressed here we present brackets where M is the energy E, entropy S and Helmholtz free energy F. At the end of this note we show how these representations can be unified.

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[‡1] Ch. 9 of ref. [1] deals with GPBs for ordinary differential equations.

If eq. (2) is to govern the evolution of all functionals of the dynamical variables then the binary operator must be bilinear. Also the brackets presented are derivations in each argument. In each of the representations given here, as in ref. [6], there is a symmetric as well as antisymmetric component. This is analogous to splitting an operator into self-adjoint and skew-adjoint parts. Additional properties will be subsequently noted.

The dynamical system we consider is

$$\frac{\partial f}{\partial t}(z,t) = -\boldsymbol{v}\cdot\frac{\partial f}{\partial \boldsymbol{x}} + \frac{\partial \phi}{\partial \boldsymbol{x}}(x;f)\cdot\frac{\partial f}{\partial \boldsymbol{v}} + \frac{\partial}{\partial v_i}\int \omega_{ij}(z,z')\left(\frac{\partial f(z)}{\partial v_j}f(z') - \frac{\partial f(z')}{\partial v_j'}f(z)\right)\mathrm{d}z', \tag{3}$$

where $f(z,t)$ is the phase space density for a species of particles and $z = (\boldsymbol{x}, \boldsymbol{v})$ denotes a point in phase space. For simplicity only one species is treated. The quantity $\phi(x;f) = \int V(\boldsymbol{x},\boldsymbol{x}')f(z')\,\mathrm{d}z'$, where V is the single particle potential (assumed spatially invariant). The tensor ω_{ij} is also a function of $z - z'$ and need not be further specified except for the following symmetries: (i) $\omega_{ij}(z,z') = \omega_{ji}(z,z')$, (ii) $\omega_{ij}(z,z') = \omega_{ij}(z',z)$, and importantly (iii) $(v_i - v_i')\omega_{ij} = 0$. These properties are satisfied by both the Landau form, where

$$\omega_{ij}^{(L)} = (L/g)\,(\delta_{ij} - g_i g_j/g^2)\,\delta(\boldsymbol{x}-\boldsymbol{x}') \tag{4}$$

(here L is a constant, δ_{ij} is the Kronecker delta, $\delta(\boldsymbol{x}-\boldsymbol{x}')$ is the Dirac delta and $g_i = v_i - v_i'$) and the Lenard–Balescu form (see ref. [5]). These properties are not fortuitous for they guarantee momentum and energy conservation.

Vlasov–Poisson bracket. If the tensor ω_{ij} is set to zero then eq. (3) becomes the VP equation. The GPB for this system was introduced in ref. [10]. In this case the generator of time translation, the hamiltonian, is the total functional

$$D[f] = \int T(z)f(z)\,\mathrm{d}z + \tfrac{1}{2}\iint V(z,z')f(z)f(z')\,\mathrm{d}z\,\mathrm{d}z', \tag{5}$$

where $T(z) = \frac{1}{2}v^2$ is the particle kinetic energy. The GPB is the following:

$$\{A,B\}_{\mathrm{VP}} = \int f(z')\,[\delta A/\delta f(z'), \delta B/\delta f(z')]\,\mathrm{d}z'. \tag{6}$$

Here A and B are functionals and $\delta A/\delta f(z)$, the functional derivative, is defined by

$$\delta A/\delta f(z) = (\mathrm{d}/\mathrm{d}\epsilon)A[f(z') + \epsilon\delta(z-z')]\,|_{\epsilon=0}$$

and $[f,g] = \partial f/\partial \boldsymbol{x}\cdot\partial g/\partial \boldsymbol{v} - \partial f/\partial \boldsymbol{v}\cdot\partial g/\partial \boldsymbol{x}$. Observe $\delta f(z)/\delta f(z') = \delta(z-z')$ and $\delta E/\delta f(z') = T + V \equiv h$, the particle energy. Evidently the following is equivalent to the VP equation:

$$\partial f/\partial t = \{f,E\}_{\mathrm{VP}} = -[f,h]. \tag{7}$$

The VP bracket is an essential ingredient in the following.

Energy representation. Eq. (2) for $\psi^i = f$ in this case is

$$\partial f/\partial t = \{f,E\}_E. \tag{8}$$

It is reasonable that a portion of $\{A,B\}_E$ should be $\{A,B\}_{\mathrm{VP}}$; the remaining portion should conserve momentum and energy. We introduce

$$\{A,B\}_E = \{A,B\}_{\mathrm{VP}} + (A,B)_E, \tag{9}$$

where

$$(A,B)_E = \int \left(\frac{\partial}{\partial v_j} \frac{\delta A}{\delta f(z)} - \frac{\partial}{\partial v_j'} \frac{\delta A}{\delta f(z')} \right) \left(\frac{\partial}{\partial v_i} \frac{\delta B}{\delta f(z)} - \frac{\partial}{\partial v_i'} \frac{\delta B}{\delta f(z')} \right) T_{ij}^{(E)}(z,z')\, \mathrm{d}z\, \mathrm{d}z' \tag{10}$$

and

$$T_{ij}^{(E)}(z,z') = \tfrac{1}{2} \left(f(z') \frac{\partial f(z)}{\partial v_k} \frac{\partial \omega_{ij}}{\partial v_k}(z,z') + f(z) \frac{\partial f(z')}{\partial v_k'} \frac{\partial \omega_{ij}}{\partial v_k'}(z,z') \right) . \tag{11}$$

Noting that $T_{ij}^{(E)}(z,z') = T_{ji}^{(E)}(z,z')$, $T_{ij}^{(E)}(z,z') = T_{ij}^{(E)}(z',z)$, and $\partial\omega_{ij}/\partial v_k = -\partial\omega_{ij}/\partial v_k'$, it is easy to show that eq. (8) using eqs. (5), (6), (9)–(11) is equivalent to eq. (3). We note that in spite of the fact that $(A,B)_E$ is symmetric it conserves energy and $(P_i, B)_E = 0$ for all B where $P_i = \int v_i f(z)\, \mathrm{d}z$. Because of bilinearity it must yield entropy production upon insertion of $S = -\int f(z) \ln f(z)\, \mathrm{d}z$ with E. Recall S has the property $\{S, B\}_{\mathrm{VP}} = 0$ for all B [‡2].

Entropy representation. In this case the quantity M is S as defined above. We introduce antisymmetric and symmetric parts as follows:

$$\{A,B\}_S = \{A,B\}_{\mathrm{VPS}} + (A,B)_S , \tag{12}$$

where

$$\{A,B\}_{\mathrm{VPS}} = \int f(z') h(z';f) [\delta A/\delta f(z'),\ \delta B/\delta f(z')]\, \mathrm{d}z' , \tag{13}$$

and h is the particle energy as previously defined. Observe that while $\{A,B\}_{\mathrm{VP}}$ is the expected value of the ordinary phase space Poisson bracket, $\{A,B\}_{\mathrm{VPS}}$ is the moment of the energy times this quantity. [We note that there is room for generalization where the entropy, and hence eq. (13) can be defined in terms of any convex function of f]. The symmetric part is given by

$$(A,B)_S = \int \left(\frac{\partial}{\partial v_j} \frac{\delta A}{\delta f(z)} - \frac{\partial}{\partial v_j'} \frac{\delta A}{\delta f(z')} \right) \left(\frac{\partial}{\partial v_i} \frac{\delta B}{\delta f(z)} - \frac{\partial}{\partial v_i'} \frac{\delta B}{\delta f(z')} \right) T_{ij}^{(S)}(z,z')\, \mathrm{d}z\, \mathrm{d}z' , \tag{14}$$

where $T_{ij}^{(S)} = \frac{1}{2}\omega_{ij} f(z) f(z')$. Eqs. (13) and (14) with S produce eq. (3). Observe that $(E,B)_S = (P_i, B)_S = 0$ for all B, and that any functional C such that $\delta \mathrm{C}/\delta f$ = function of f has the property $\{\mathrm{C}, B\}_{\mathrm{VPS}} = 0$ for all B.

Helmholtz free energy representation. The generator of time translation in this representation is the free energy $F[f] = E[f] - TS[f]$, where T is constant [‡3]. The bracket in this case is composed of eq. (6) and (14), i.e.

$$\{A,B\}_F = \{A,B\}_{\mathrm{VP}} + (A,B)_F , \tag{15}$$

where $(A,B)_F$ is identical to eq. (14) except $T_{ij}^{(S)}$ is replaced by

$$T_{ij}^{(F)} = -(\omega_{ij}/2T) f(z) f(z') . \tag{16}$$

It is evident from the preceeding that eq. (15) with F will produce eq. (3).

Unifying three-forms. All of the brackets presented are contained within the following trilinear operators:

$$\{A; B, C\} = \int f(z) [\delta A/\delta f(z')] [\delta B/\delta f(z'), \delta C/\delta f(z')]\, \mathrm{d}z' , \tag{17}$$

‡2 Kaufman reports results similar to these for a hybrid formulation where the symmetric bracket uses entropy to generate evolution [11].

‡3 Grmela introduced a bracket for the Boltzmann collision term for which F is the generator. His bracket is neither symmetric nor antisymmetric, but does possess similar degeneracy properties [12].

and

$$(A, B, C) = \tfrac{1}{2}\int \left(\frac{\partial}{\partial v_i}\frac{\delta A}{\delta f(z)} - \frac{\partial}{\partial v_i'}\frac{\delta A}{\delta f(z')}\right)\left(\frac{\partial}{\partial v_j}\frac{\delta B}{\delta f(z)} - \frac{\partial}{\partial v_j'}\frac{\delta B}{\delta f(z')}\right)\left(\frac{\partial}{\partial v_k}\frac{\delta C}{\delta f(z)} - \frac{\partial}{\partial v_k'}\frac{\delta C}{\delta f(z')}\right)$$
$$\times \frac{\partial \omega_{ij}}{\partial v_k'}(z, z')\, f(z) f(z')\, \mathrm{d}z\, \mathrm{d}z' \,. \tag{18}$$

We note

$$\{E; A, B\} = \{A, B\}_{\mathrm{VPS}} \,, \qquad \{A; S, B\} = \{A, B\}_{\mathrm{VP}} \,, \tag{19, 20}$$

$$(A, B, E) = (A, B)_S \,, \qquad (A, B, S) = (A, B)_E \,. \tag{21, 22}$$

Clearly eq. (17) is antisymmetric in B and C. (In the case where the entropy is proportional to f^2 it further becomes permutation symmetric.) From eq. (5) we see that $\partial\omega_{ij}/\partial v_k$ is symmetric under interchange of indices; hence eq. (18) is symmetric. (Note also $g_k \partial\omega_{ij}/\partial v_k = -\omega_{ij}$).

Remarks. 1. Collision operators like those presented here are derived by truncating the BBGKY hierarchy. Some truncations are hamiltonian (e.g. the VP equation), while others that involve assumptions like Bogoliubov's hypothesis result in diffusion. Since the hamiltonian formulation for the hierarchy is now available [13], we hope to understand the structure presented here in this context.

2. Generalized Poisson brackets can be used as a means of classifying equations. Many different equations, when represented in their natural physical variables, possess the same GPB (e.g. the VP, two-dimensional Euler and plasma guiding center drift equations [2]) but possess different hamiltonians. Although the equations are different, since the GPBs are the same these systems automatically have a common infinite set of conservation laws (Casimirs). These constants manifest degeneracy in the bracket. New conservation laws have been discovered in this manner [6]. The symmetric brackets also possess constants {e.g. $(E[f], B)_S = 0$ for all B} and hence can similarly serve as a means of classification. Results presented in ref. [7] indicate that constitutive relations can be couched in the form of these brackets. In the case of the Navier–Stokes equation the relevant brackets contain terms that correspond to entropy production via heat flow and viscous dissipation. For systems with coupled fluxes the Onsager relations are contained within the brackets. The formalism may provide a useful framework for the covariant description of media.

3. The notion of splitting an operator into two parts in order to isolate behavior has precedence (e.g. response functions are split into hermitian and anti-hermitian components). The degree to which our symmetric forms can describe behavior of the solution is currently under investigation. Since these forms describe the "non-hamiltonian" part of a system, it is evident that they embody the breaking of Liouville's theorem. Hence, they should describe "non-hamiltonian" behavior such as the existence of attracting or strange attracting sets. An example of this occurs here where the H-theorem is embodied in the definitness of $(E, S)_E$ and $(S, S)_S$. Since attracting sets are related to stability we speculate that a generalization of the technique used in ref. [8] may be possible.

4. Just as GPBs possess underlying geometrical interpretation one would expect the same for the brackets presented here, or perhaps similar structures. As noted, present efforts are concerned with understanding this formalism in light of the structure underlying the hierarchy. The hierarchy bracket [13] is related to a filtered Lie algebra, a structure that also appears in the context of pseudo-differential operators (see ref. [14] for proceedings of a conference dealing with Kac–Moody algebras). Moreover, since we have brackets with symmetric as well as antisymmetric components a connection with graded Lie algebras and supersymmetry is sought [15].

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