Hamiltonian formulation of reduced magnetohydrodynamics

P. J. Morrison[a] and R. D. Hazeltine
Institute for Fusion Studies, The University of Texas at Austin, Austin, Texas 78712

(Received 28 July 1983; accepted 5 January 1984)

Reduced magnetohydrodynamics (RMHD) is a principal tool for understanding nonlinear processes, including disruptions, in tokamak plasmas. Although analytical studies of RMHD turbulence are useful, the model's impressive ability to simulate tokamak fluid behavior has been revealed primarily by numerical solution. A new analytical approach, not restricted to turbulent regimes, based on Hamiltonian field theory is described. It is shown that the nonlinear (ideal) RMHD system, in both its high-beta and low-beta versions, can be expressed in Hamiltonian form. Thus a Poisson bracket, $\{\ ,\ \}$, is constructed such that each RMHD field quantity ξ_i evolves according to $\dot{\xi}_i = \{\xi_i, H\}$, where H is the total field energy. The new formulation makes RMHD accessible to the methodology of Hamiltonian mechanics; it has lead, in particular, to the recognition of new RMHD invariants and even exact, nonlinear RMHD solutions. A canonical version of the Poisson bracket, which requires the introduction of additional fields, leads to a nonlinear variational principle for time-dependent RMHD.

I. INTRODUCTION

A. Reduced magnetohydrodynamics

The term "reduced magnetohydrodynamics" (RMHD) refers to a number of simplified approximations to ordinary magnetohydrodynamics (MHD). The original versions of RMHD, with which this work is concerned, were constructed to describe nonlinear plasma dynamics in large aspect-ratio tokamak geometry.[1–3] Thus the ordering parameter ϵ is the inverse aspect ratio; one assumes the following ratios, in particular, to be of order ϵ: (1) scale length transverse to the magnetic field **B**: scale length along **B**; (2) poloidal component of **B**: toroidal component of **B**; and (3) time for compressional equilibration (compressional Alfvén time): time scale of interest (shear Alfvén time). In addition, the plasma pressure p is assumed small, either $p \sim \epsilon^2 B^2$ ("low-beta RMHD") or $p \sim \epsilon B^2$ ("high-beta RMHD"). The RMHD set is presented in Sec. II; for a detailed derivation we refer the reader to the original work by Strauss.[2,3]

As a model for high-temperature tokamak plasma behavior, RMHD is crude in several respects. Of course its MHD origin precludes any treatment of potentially important, nonideal or kinetic effects, a circumstance which is inadequately remedied by resistive versions of RMHD. Even within the ideal context, RMHD omits, for example, density gradient terms and ion acoustic propagation. Perhaps most seriously, the RMHD simplification of tokamak geometry can yield misleading results in certain linear contexts (e.g., interchange stability); it provides an inaccurate version of tokamak magnetic field curvature.[4]

To be weighed against such drawbacks are the four main advantages of RMHD.

(1) It is numerically tractable. The ideal version, being parameter-free, involves only a single temporal scale. Furthermore, only two or three scalar fields need to be advanced in time.

(2) It is conceptually simple. The significance of the field quantities (magnetic flux, electrostatic potential, pressure) is transparent and the physical content of the equations is clear.

(3) Its derivation is internally consistent. The equations result from a systematic neglect of $O(\epsilon^3)$ terms, with few additional simplifications.

(4) Most importantly, the RMHD system simulates the actual nonlinear behavior of tokamak discharges.[5] Its predictions—concerning nonlinear kink deformations, flux surface destruction, and plasma disruption, for example—have a qualitative reliability which few tokamak theoretic constructs can equal.

For these reasons (especially the last), RMHD has become a principal tool in the interpretation of tokamak experiments. Most major tokamak facilities routinely use computer solutions to some version of RMHD, and several research teams are devoted to uncovering its implications. It is significant, if unsurprising, that the great bulk of this theoretical effort has been strictly numerical. The relatively few analytical investigations of RMHD have been devoted either to improving the system itself (for example, by the inclusion of various nonideal effects) or to examining its consequences in certain turbulent regimes.

The present work is motivated by the belief that RMHD deserves more extensive analytical study. Our central theme is the Hamiltonian description of RMHD, in both its low-beta and high-beta versions.

B. Hamiltonian dynamics

In this subsection we briefly review what is meant by a Hamiltonian system of equations. Contrary to conventional textbook treatments, we emphasize the algebraic properties of the Poisson bracket. This emphasis frees one from the requirement of canonical variables and thus is a more general setting.[6–8] In recent times there has been a wealth of work for both finite and infinite degrees of freedom systems that is related to this point of view.[9–14] For simplicity of

[a] Also the Department of Physics, University of Texas, Austin, Texas 78712.

 0031-9171/84/040886-12$01.90

exposition we describe finite systems prior to the field formulation that is our concern.

The standard route to a Hamiltonian description is to Legendre-transform the Lagrangian, which is constructed on physical bases. This yields the Hamiltonian and the following $2N$ first-order ordinary differential equations:

$$\dot{q}_k = [q_k, H], \quad \dot{p}_k = [p_k, H], \quad k = 1,2, \ldots, N.$$

Here the Poisson bracket has the form

$$[f,g] = \sum_{k=1}^{N} \left(\frac{\partial f}{\partial q_k} \frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k} \frac{\partial g}{\partial q_k} \right) = \frac{\partial f}{\partial z^i} J^{ij} \frac{\partial g}{\partial z^j}. \tag{1}$$

The last equality of Eq. (1) follows from the substitutions,

$$z^i = \begin{cases} q_k, & i = k = 1,2, \ldots, N, \\ p_k, & i = N + k = N+1, N+2, \ldots, 2N, \end{cases}$$

and

$$(J^{ij}) = \begin{pmatrix} 0 & I_N \\ -I_n & 0 \end{pmatrix}, \tag{2}$$

where I_n is the $N \times N$ unit matrix. (Repeated index convention is used.) The matrix J^{ij} is called the cosymplectic form and it can be shown to transform as a contravariant tensor under a change of coordinates. Recall those transformations that preserve its form are canonical.

The approach taken here is that there is no concern that the J^{ij} take the form given by Eq. (2). Rather, we require only that the J^{ij} endow the Poisson bracket, as given by Eq. (1), with the following properties:

(i) $[f,g] = -[g,f]$,

(ii) $[f,[g,h]] + [g,[h,f]] + [h,[f,g]] = 0$.

There must hold for all functions f, g, and h defined on phase space. Property (i) requires that J^{ij} be antisymmetric and property (ii), the Jacobi identity, requires the following:

$$S^{ijk} = J^{il} \frac{\partial J^{jk}}{\partial z^l} + J^{jl} \frac{\partial J^{ki}}{\partial z^l} + J^{kl} \frac{\partial J^{ij}}{\partial z^l} = 0. \tag{3}$$

Equation (3) is trivially satisfied for the form of J^{ij} given by Eq. (2), though in general it is a severe restriction. It can be shown that S^{ijk} transforms contravariantly; hence if the Jacobi identity is satisfied in one frame it is satisfied in all frames. Similarly, antisymmetry is coordinate independent. This suggests the following outlook: if a system of equations possesses the form

$$\dot{z}^i = \tilde{J}^{ij} \frac{\partial H}{\partial z^j}, \quad i,j = 1,2, \ldots, N,$$

where $\tilde{J}^{ij}$ is antisymmetric and satisfies Eq. (3), then it is Hamiltonian. This outlook is justified by a theorem due to Darboux which states that assuming $\det(\tilde{J}^{ij}) \neq 0$ (locally) a canonical coordinate system exists.

Turning now to systems of infinite dimensions we note that the generalization of Eq. (1) for a system of field equations is

$$\{F,G\} = \sum_{k=1}^{M} \int \left(\frac{\delta F}{\delta \eta_k} \frac{\delta G}{\delta \pi_k} - \frac{\delta G}{\delta \eta_k} \frac{\delta F}{\delta \pi_k} \right) d\tau \equiv \left\langle \frac{\delta F}{\delta u^i} \middle| O^{ij} \frac{\delta G}{\delta u^j} \right\rangle. \tag{4}$$

Here the Poisson bracket acts on functionals F, G of the field variables η_k and π_k and partial derivatives are replaced by functional derivatives that are defined in the usual way by

$$\frac{dF}{d\epsilon} [\eta_k + \epsilon w] \bigg|_{\epsilon = 0} = \left\langle \frac{\delta F}{\delta \eta_k} \middle| w \right\rangle. \tag{5}$$

The bracket stands for the usual inner product

$$\langle f | g \rangle = \int fg \, d\tau.$$

We now carry over the ideas for finite degree of freedom systems. We define a system to be Hamiltonian if it can be written, for some Hamiltonian functional H, in the form

$$\frac{\partial u^i}{\partial t} = O^{ij} \frac{\delta H}{\delta u^j},$$

where O^{ij} is a matrix (in general nonlinear) operator that endows a Poisson bracket defined by the second equality of Eq. (4) with the properties (i) and (ii). Antisymmetry requires that O^{ij} be anti-self-adjoint. The Jacobi requirement for a specific case is taken up in the text. For the general case we direct the reader to Ref. 14. A major goal of this paper is to present the operator O^{ij} with these desired properties such that the u^i's are the usual field variables for RMHD.

C. Overview of results

Section II is composed of two subsections. In Sec. II A we briefly review how RMHD is asymptotically obtained from MHD. Here we define our notation and our coordinate system. In Sec. II B we discuss integral invariants. A comparison is made between the invariants of ideal MHD and those of RMHD that survive the asymptotics. In the course of investigating these invariants, a class of exact, nonlinear, uniformly propagating solutions to RMHD were discovered.[15] A novel result of this subsection is the presentation of a new class of invariants for single-helicity RMHD. These invariants are a natural by-product of the generalized Poisson structure obtained in Sec. III. Quantities that commute with all Hamiltonians are known as Casimir invariants[16]—the new invariants are of this type. Casimir invariants are important because together with the Poisson bracket they enable the construction of global nonlinear stability criteria for nonlinear solutions. Arnold[16] used the Hamiltonian structure for two-dimensional inviscid, incompressible fluids to prove nonlinear stability. Arnold's result has been invoked by Meiss and Horton[17] in order to ascertain the stability of solitary drift waves. Recently the technique was utilized by Holm *et al.*[18] to prove stability for three-dimensional compressible fluid flow. Applications involving RMHD nonlinear solutions will be the subject of a future publication.[19]

The main portion of this paper is presented in Secs. III and IV. The Poisson brackets are described and the Jacobi identity is proven for both the low- and high-beta theories. In Sec. IV B we present the Hamiltonian description in terms of the usual discretization employed for tokamak numerics, i.e., we use Fourier transforms in the poloidal and toroidal angles. We leave the radial variable alone, but finite differ-

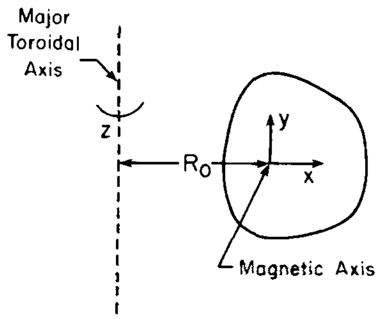

FIG. 1. Tokamak coordinate system. R_0 is the distance to the minor toroidal axis. The closed curve is used to schematically indicate a poloidal plane, which has a characteristic size a.

ence schemes can be worked out within the generalized Poisson bracket context. Discretization in this manner automatically insures energy conservation.

Section V is concerned with the transformation of our generalized Poisson brackets to canonical form. The equations of motion in these variables are presented; analogous equations for ordinary fluids have been numerically integrated. Having obtained a canonical Hamiltonian description we take the short step to produce a variational principle that yields Hamilton's equations of motion. Nonlinear variational principles are useful in that one can employ Rayleigh–Ritz or trial function approximations. A variational principle for the regularized-long-wave equation, which was obtained by the same route as that described in Sec. V, has been used to successfully predict the phase shift of solitary wave scattering.[20]

Section VI summarizes our conclusions and this application. Comments concerning incorporation of dissipation into the formalism are also included.

II. REDUCED MHD: EQUATIONS AND CONSTANTS

A. Equations of motion

The reduced MHD equations are obtained by asymptotically ordering the equations of ideal MHD. The fundamental small parameter is the inverse aspect ratio, $\epsilon = a/R_0$, where a and R_0 are the minor and major radii, respectively (see Fig. 1). The fluid velocity $\mathbf{v} = v_\parallel \hat{z} + \mathbf{u}_\perp$ is scaled with the poloidal Alfvén speed $v_p = B_{p0}/\sqrt{4\pi\rho}$, where ρ is the mass density and $B_{p0} = aB_0/R$ (B_0 is the scale for the toroidal field). The ordering causes parallel dynamics to become decoupled, so that $v_\parallel$ does not enter the final closed system. The ordering also renders $\nabla\cdot\mathbf{v} = O(\epsilon)$. Time is scaled with $\tau_p = a/v_p$ while distances in the toroidal (z) direction are scaled with R_0 and poloidal distances are scaled with a. The dimensionless gradient operator is $\nabla = \epsilon\hat{z}(\partial/\partial z) + \nabla_\perp$. The scaled magnetic field is represented in the form

$$\mathbf{B} = \hat{z}/(1+\epsilon x) + \epsilon\nabla_\perp\psi\times\hat{z} + \epsilon\hat{z}h + O(\epsilon^2). \tag{6}$$

The first term is the vacuum toroidal field, the second is the poloidal field represented in terms of the scaled poloidal flux ψ, and the third term represents the deviation from the $1/R$ toroidal field due to the presence of the plasma. Note that $R = R_0(1+\epsilon x)$. The function h is determined from the ideal MHD momentum balance equation. One obtains to order ϵ

$$\nabla_\perp(\beta/2 + h) = 0, \quad \beta = 8\pi p/B_0^2\epsilon.$$

Here p is the plasma pressure. In the high-beta version of RMHD, p is chosen to scale as ϵ. The previous low-beta version avoided pressure effects by scaling $p\sim\epsilon^2$; this version may be obtained from that of Eqs. (2)–(4) by setting $\beta\equiv 0$.

The dynamical equations obtained from the ordering described in the previous paragraph are

$$\frac{\partial\psi}{\partial t} + \frac{\partial\phi}{\partial z} = \hat{z}\cdot\nabla_\perp\psi\times\nabla_\perp\phi, \tag{7}$$

$$\frac{\partial U}{\partial t} + \frac{\partial J}{\partial z} = \hat{z}\cdot\nabla_\perp\psi\times\nabla_\perp J + \hat{z}\cdot\nabla_\perp U\times\nabla_\perp\phi - \frac{\partial\beta}{\partial y}, \tag{8}$$

and

$$\frac{\partial\beta}{\partial t} = \hat{z}\cdot\nabla_\perp\beta\times\nabla_\perp\phi. \tag{9}$$

Here we have introduced the stream function ϕ defined by $\mathbf{v}_\perp = \hat{z}\times\nabla\phi$, the vorticity $U\equiv\nabla_\perp^2\phi$, and the toroidal current $J\equiv\nabla_\perp^2\psi$. The Poisson brackets for Eqs. (2)–(4) and important subsets thereof are obtained in Secs. III and IV.

Equation (7) simply expresses $\mathbf{E}\cdot\mathbf{B} = 0$, where $\mathbf{E} = -\nabla\Phi - c^{-1}\,\partial\mathbf{A}/\partial t$ is the electric field. The variable Ψ is a normalized measure of $\mathbf{B}\cdot\mathbf{A}$ ($\mathbf{A}$ is the vector potential) while ϕ measures the electrostatic potential Φ. The nonlinear term corresponds to $\mathbf{B}\cdot\nabla\Phi$. Equation (8) is the reduced vorticity equation, usually obtained from the curl of the equation of motion; its two nonlinearities correspond to advection and the parallel current gradient. Finally, Eq. (9) describes pressure advection, corresponding to the approximate incompressibility of RMHD dynamics.

The version of RMHD given above evidently does not contain resistive effects; however, consistent with the RMHD ordering, one can add $\eta\nabla_\perp^2\psi$ to the right-hand side of Eq. (18), where η is a normalized resistivity. One would not expect such a dissipative system to be Hamiltonian. We comment further upon this issue in Sec. VI.

B. Constants of motion

A dynamical system such as RMHD possesses a conserved density if there exists a quantity R that satisfies an equation of the form

$$\frac{\partial R}{\partial t} + \nabla\cdot\mathbf{C} = 0, \tag{10}$$

where R and $\mathbf{C}$ are composed of the dynamical variables of the system. Clearly for each such quantity R there corresponds an integral constant of motion, since

$$\frac{d}{dt}\int R\,d\tau = \int\nabla\cdot\mathbf{C}\,d\tau = 0. \tag{11}$$

In Eq. (6) the integral extends over the fixed domain of interest and the second equality arises if the surface term vanishes.

The equations of ideal MHD are known to possess many conserved densities. These are shown in Table I along with the RMHD remnant obtained under the ordering of the previous subsection. For a discussion of the ideal MHD constants and the symmetries they generate we refer the reader to Ref. 14. In the table, cases where the remnant appears to be trivial are left blank. Of the nontrivial remnants the natu-

TABLE I. Invariants of MHD and reduced MHD.

MHD invariant	Comments	RMHD remnant	Comments
$M=\int \rho\, d\tau$	Casimir invariant		
$\mathbf{P}=\int \rho\mathbf{v} d\tau$			
$\mathbf{L}=\int \mathbf{x}\times\rho\mathbf{v}\, d\tau$		$\mathbf{L}=\hat{z}\int \phi\, d\tau$	
$\mathbf{H}=\int\left(\frac{1}{2}\rho v^2+\rho U+\frac{B^2}{2}\right)d\tau$	$U(\rho, s)$ is the internal energy per unit mass. s is the entropy per unit mass.	$H=\int \frac{1}{2}(\lvert\nabla_\perp\phi\rvert^2+\lvert\nabla_\perp\psi\rvert^2-2\beta x)\, d\tau$	For low-β version $\beta\to 0$. x is Cartesian coordinate in poloidal plane.
$\mathbf{G}=\int(\rho\mathbf{x}-\rho\mathbf{v}t)\, d\tau$	Center of mass constant. Appeared in Ref. 14.		
$S=\int \rho S(s)\, d\tau$	S arbitrary. Casimir invariant.	$P_r=\int g(\beta)\, d\tau$	g arbitrary. This should be opposite $\int \rho g(p\rho^{-\gamma})\, d\tau$. We write the equation of state more generally using S.
$\mathbf{B}=\int \mathbf{B}\, d\tau$			
$V=\int \mathbf{v}\cdot\mathbf{B}\, d\tau$	Barotropic flow or $\mathbf{B}\cdot\nabla s=0$. Casmir invariant.	$V=\int \nabla_\perp\phi\cdot\nabla_\perp\psi\, d\tau$	
$A=\int \mathbf{A}\cdot\mathbf{B}\, d\tau$	Casimir invariant.	$A=\int \psi\, d\tau$	
$Q=\int \rho f\left(\frac{\mathbf{B}}{\rho}\cdot\nabla\Phi\right)d\tau$	f arbitrary. Φ any advected quantity. Appeared in Ref. 21.	$Q=\int f\left(\frac{\partial\Phi}{\partial z}-[\psi,\Phi]\right)d\tau$	The special case where Φ is pressure and f is the identity function was given in Ref. 2.
		$C=\int Uh(\psi)\, d\tau$	h arbitrary. Single helicity. Casimir invariant. Low beta.

ral choice for the Hamiltonian is, of course, H. This is used in the upcoming sections.

The quantity C, which appears to have no MHD antecedent, is the Casimir invariant mentioned in the Introduction. It is conserved for the two-dimensional and hence single-helicity models of Sec. III. It was obtained by recognizing that the Poisson bracket in this case is identical to that for the incompressible Euler equations in two dimensions. This structure is well understood.[22,23]

It is easy to see that V, the cross helicity, is a special case of C where $h(\psi)=\psi$. The invariants C are the cross-helicity analog to the class of invariants associated with the magnetic helicity. These invariants have been proposed as constraints on turbulent relaxation.[24,25]

III. LOW-BETA THEORY

A. Two dimensions

In this section we construct a Poisson bracket for the simplest version of reduced MHD, in which the interchange term on the right-hand side of Eq. (8) is neglected. The resulting system describes the nonlinear behavior of current-driven modes, such as the kink mode, and is consistent with the ordering $8\pi p/B_0^2 \lesssim \epsilon^2$. We further simplify, initially, by neglecting z derivatives, thus considering a two-dimensional system. Axisymmetric disturbances are of limited interest in themselves. However (as becomes explicit in the following subsection) the axisymmetric system is equivalent to one possessing helical symmetry, and the helically symmetric case has considerable intrinsic importance.[1]

Hence we consider the system

$$\dot{U}=[\psi,J]+[U,\phi], \tag{12}$$

$$\dot{\psi}=[\psi,\phi], \tag{13}$$

$$U=\nabla_\perp^2\phi, \quad J=\nabla_\perp^2\psi. \tag{14}$$

Here we use a bracket notation which has become conventional[3]:

$$[f,g]\equiv\hat{z}\cdot\nabla_\perp f\times\nabla_\perp g. \tag{15}$$

Because this bracket presently will be embedded in the field Poisson bracket, we refer to it as the "inner" bracket. The inner bracket is a divergence,

$$[f,g]=\nabla_\perp\cdot(g\hat{z}\times\nabla_\perp f), \tag{16}$$

which satisfies the crucial identity

$$\int d\mathbf{x}_\perp f[g,h]=\int d\mathbf{x}_\perp g[h,f]=\int d\mathbf{x}_\perp h[f,g], \tag{17}$$

for any functions f, g, and h. Equation (17), in which $d\mathbf{x}_\perp \equiv dx\, dy$ and the integrals extend over the entire plasma volume, depends upon the neglect of surface terms. Such neglect is not usually serious; however, the present formalism must be applied with care to situations in which the plasma boundary significantly affects the dynamics. We note in passing that several of the conservation laws presented in Sec. II are immediate consequences of Eq. (17).

Our objective is to write Eqs. (12)–(14) in Hamiltonian form. That is, we seek a suitably defined "outer" bracket $\{F,G\}$, which acts on functionals of U and ψ. The outer bracket must be antisymmetric,

$$\{F,G\} = -\{G,F\}; \tag{18}$$

must satisfy Jacobi's identity,

$$\{E,\{F,G\}\} + \{F,\{G,E\}\} + \{G,\{E,F\}\} = 0; \tag{19}$$

and must yield Hamilton's equations, in the (generally noncanonical) form

$$\dot{\psi} = \{\psi,H\}, \tag{20}$$

$$\dot{U} = \{U,H\}. \tag{21}$$

Here H is the energy introduced in Sec. II, appropriately simplified for low beta and axisymmetry:

$$H = \frac{1}{2}\int d\mathbf{x}_\perp\, [(\nabla_\perp\phi)^2 + (\nabla_\perp\psi)^2]. \tag{22}$$

We simplify notation by using the same symbol to denote the general energy integral and its various simplified versions.

The quantity H is manifestly a functional of the reduced MHD fields. Note that the fields ψ and U can themselves also be interpreted as functionals; for example,

$$\psi(\mathbf{x}_\perp) = \int d\mathbf{x}'_\perp\, \delta(\mathbf{x}_\perp - \mathbf{x}'_\perp)\psi(\mathbf{x}'_\perp). \tag{23}$$

Such interpretation is called for in Eqs. (20) and (21).

A generic form for the Poisson bracket can be inferred from previous work:

$$\{F,G\} = \int d\mathbf{x}_\perp\, W_{ij}\left[\frac{\delta F}{\delta\xi_i}, \frac{\delta G}{\delta\xi_j}\right], \tag{24}$$

where $(\xi_1,\xi_2) = (\psi, U)$, a sum over repeated indices is implied, and the functional derivative as noted in the Introduction is defined by

$$\frac{d}{d\epsilon}F[\xi + \epsilon w] = \int d\mathbf{x}_\perp\, w\frac{\delta F}{\delta\xi}. \tag{25}$$

The quantities W_{ij} are to be chosen to satisfy Eqs. (18)–(21). From Eqs. (20) and (21) it can be seen that W_{ij} must depend linearly upon the ξ_i.

Before proceeding further with Eq. (24), we turn our attention to H, which must now be considered as a functional of ψ and U. After partial integration, Eq. (22) becomes

$$\begin{aligned} H &= -\frac{1}{2}\int d\mathbf{x}_\perp\, [U\phi + \psi J] \\ &= -\frac{1}{2}\int d\mathbf{x}_\perp\, [UK(U) + \psi\nabla_\perp^2\psi], \end{aligned}$$

where K represents the operator inverse to $\nabla_\perp^2$: $K(\nabla_\perp^2 f) = f$. Because $\nabla_\perp^2$, and therefore K, are self-adjoint operators, we see that

$$\frac{\delta H}{\delta U} = -\phi, \quad \frac{\delta H}{\delta\psi} = -J. \tag{26}$$

It follows in particular that the equations of motion can be expressed as

$$\dot{U} = \left[\frac{\delta H}{\delta\psi},\psi\right] + \left[\frac{\delta H}{\delta U},U\right], \tag{27}$$

and

$$\dot{\psi} = \left[\frac{\delta H}{\delta U},\psi\right]. \tag{28}$$

We now return to Eq. (24), and consider the quantity $\{\psi,H\}$. In view of the identity (17), this can be written as

$$\{\psi,H\} = \int d\mathbf{x}'_\perp\, \frac{\delta\psi}{\delta\xi_i}\left[\frac{\delta H}{\delta\xi_j}, W_{ij}\right].$$

But $\delta\psi/\delta\xi_i$ vanishes unless $i = 1$, in which case it is the identity operator [cf. Eq. (23)]. Thus

$$\{\psi,H\} = \left[\frac{\delta H}{\delta\xi_j}, W_{1j}\right]. \tag{29}$$

Comparing the right-hand sides of Eqs. (28) and (29), we see that Eq. (20) will hold only if

$$W_{11} = 0, \quad W_{12} = \psi.$$

Analogous consideration of $\{U,H\}$ readily shows that we must choose

$$W_{12} = \psi, \quad W_{22} = U,$$

in order to reproduce the right-hand side of Eq. (27).

We conclude that the bracket defined by

$$\begin{aligned} \{F,G\}_2 \equiv \int d\mathbf{x}_\perp \biggl\{\psi\biggl(\left[\frac{\delta F}{\delta\psi}, \frac{\delta G}{\delta U}\right] + \left[\frac{\delta F}{\delta U}, \frac{\delta G}{\delta\psi}\right]\biggr) \\ + U\left[\frac{\delta F}{\delta U}, \frac{\delta G}{\delta U}\right]\biggr\} \end{aligned} \tag{30}$$

yields the correct two-dimensional equations of motion (the subscript refers to the dimensionality). It is a proper Poisson bracket if, in addition, $\{\ \ ,\ \ \}_2$ is antisymmetric and satisfies Jacobi's identity. Since antisymmetry is a trivial consequence of the antisymmetry of the *inner* bracket and the symmetry of $W_{ij} = W_{ji}$, the remainder of this subsection is devoted to verifying the Jacobi identity.

Our demonstration is grossly simplified because of two symmetries: antisymmetry in the bracket and symmetry in the second variation. Consider the general bracket of Eq. (24). We can evidently write

$$\frac{\delta}{\delta\xi_k}\{F,G\} = \int d\mathbf{x}_\perp\, \frac{\delta W_{ij}}{\delta\xi_k}\left[\frac{\delta F}{\delta\xi_i}, \frac{\delta G}{\delta\xi_j}\right] + A_k, \tag{31}$$

where A_k involves higher-order functional derivatives of F and G. For the purpose of verifying Jacobi's identity, one can always neglect A_k. The point is that the terms in A_k are consistent with Eq. (19) for any symmetric W_{ij}, essentially because $\delta^2/\delta\xi_i\,\delta\xi_j$ is effectively self-adjoint. The reader interested in seeing a proof of this is directed to Ref. 14.

In our case, Eqs. (30) and (31) yield

$$\frac{\delta}{\delta\psi}\{F,G\}_2 = \left[\frac{\delta F}{\delta\psi}, \frac{\delta G}{\delta U}\right] + \left[\frac{\delta F}{\delta U}, \frac{\delta G}{\delta\psi}\right] + A_1,$$

$$\frac{\delta}{\delta U}\{F,G\}_2 = \left[\frac{\delta F}{\delta U}, \frac{\delta G}{\delta U}\right] + A_2,$$

and therefore,

$$\begin{aligned} \{E,\{F,G\}_2\}_2 = \int d\mathbf{x}_\perp \biggl\{\psi\biggl(\left[\frac{\delta E}{\delta\psi}, \left[\frac{\delta F}{\delta U}, \frac{\delta G}{\delta U}\right]\right] \\ + \left[\frac{\delta E}{\delta U}, \left[\frac{\delta F}{\delta\psi}, \frac{\delta G}{\delta U}\right] + \left[\frac{\delta F}{\delta U}, \frac{\delta G}{\delta\psi}\right]\right]\biggr) \\ + U\left[\frac{\delta E}{\delta U}, \left[\frac{\delta F}{\delta U}, \frac{\delta G}{\delta U}\right]\right]\biggr\}. \end{aligned} \tag{32}$$

Here irrelevant terms, involving the A_k, have been omitted. It can be seen from Eq. (32) that the outer bracket will satisfy Jacobi's identity provided only that the inner one does. This is obvious with regard to the term which is weighted by U,

$$\left[\frac{\delta E}{\delta U},\left[\frac{\delta F}{\delta U},\frac{\delta G}{\delta U}\right]\right].$$

It is also true for the ψ-weighted terms, because functional derivatives with respect to ψ occur uniformly in these terms: once on E, once on F, and once on G. Hence it suffices to verify that

$$[f,[g,h]]+[g,[h,f]]+[h,[f,g]]=0. \tag{33}$$

But Eq. (33) can be established by elementary means (for example, by noting the resemblance of the inner bracket to the classical Poisson bracket).

We conclude that the equations

$$\dot\psi=\{\psi,H\}_2,\quad \dot U=\{U,H\}_2 \tag{34}$$

indeed yield a Hamiltonian representation of two-dimensional, low-beta, reduced MHD.

We remark in closing this subsection that the bracket given by Eq. (30) has a mathematical interpretation as the dual of the Lie algebra of a semidirect product.[11] This will be discussed in a forthcoming publication.[26]

B. Three dimensions

Here we generalize the low-beta bracket to allow for arbitrary asymmetry. It is convenient to use cylindrical coordinates,

$$(x,y,z)\rightarrow(r,\theta,\zeta),$$

where

$$x=r\cos\theta,\quad y=r\sin\theta,\quad z=\zeta.$$

The coordinates θ and ζ are conventional poloidal and toroidal coordinates, respectively, while r is a dimensionless minor radius. Evidently, the operator $\nabla_\perp=\hat x\partial/\partial x+\hat y\partial/\partial y$ becomes

$$\nabla_\perp=\hat r\frac{\partial}{\partial r}+\frac{\hat\theta}{r}\frac{\partial}{\partial\theta}.$$

To treat the ζ (or z) derivatives in Eqs. (7) and (8), it is helpful to introduce a three-dimensional gradient operator, defined by

$$\hat\nabla\equiv\nabla_\perp+\hat\zeta\frac{\partial}{\partial\zeta}. \tag{35}$$

Note that $\hat\nabla$ differs from the true, normalized gradient, which contains a factor a/R in the toroidal derivative term. The present definition implies $\hat\zeta=\hat\nabla\zeta$ and therefore

$$[f,g]=\hat\nabla\zeta\cdot\hat\nabla f\times\hat\nabla g=\hat\nabla\cdot(g\hat\nabla\zeta\times\hat\nabla f), \tag{36}$$

$$[f,g]=\frac{1}{r}\left(\frac{\partial f}{\partial r}\frac{\partial g}{\partial\theta}-\frac{\partial g}{\partial r}\frac{\partial f}{\partial\theta}\right). \tag{37}$$

We next introduce a new inner bracket, the "poloidal" inner bracket. It is defined analogously to Eq. (36):

$$[f,g]_p\equiv\hat\nabla\theta\cdot\hat\nabla f\times\hat\nabla g=\hat\nabla\cdot[g\hat\nabla\theta\times\hat\nabla f]. \tag{38}$$

The word "poloidal" refers to the $\hat\nabla\theta$ factor; in this sense, Eq. (36) provides the "toroidal" inner bracket. Both brackets can be seen to satisfy

$$\int d\mathbf{x}\,f[g,h]_{(p)}=\int d\mathbf{x}\,g[h,f]_{(p)}=\int d\mathbf{x}\,h\,[f,g]_{(p)}, \tag{39}$$

as in Eq. (17). Here and below

$$\int d\mathbf{x}=\int d\mathbf{x}_\perp\,dz=\int_0^1 r\,dr\oint d\theta\oint d\zeta,$$

and surface terms are presumed to vanish as usual.

The essential property of the poloidal inner bracket is that it allows us to write, for any function f,

$$\frac{\partial f}{\partial\zeta}=-\left[\frac{r^2}{2},f\right]_p. \tag{40}$$

Hence the three-dimensional, low-beta equations of motion [Eqs. (7) and (8)] can be written as

$$\dot\psi=[\psi,\phi]+\left[\frac{r^2}{2},\phi\right]_p,$$

$$\dot U=[\psi,J]+\left[\frac{r^2}{2},J\right]_p-[\phi,U].$$

Alternatively, we may use the three-dimensional Hamiltonian,

$$H=\frac{1}{2}\int d\mathbf{x}[(\nabla_\perp\phi)^2+(\nabla_\perp\psi)^2],$$

which also satisfies Eqs. (26), to write

$$\dot\psi=\left[\frac{\delta H}{\delta U},\psi\right]+\left[\frac{\delta H}{\delta U},\frac{r^2}{2}\right]_p, \tag{41}$$

$$\dot U=\left[\frac{\delta H}{\delta\psi},\psi\right]+\left[\frac{\delta H}{\delta\psi},\frac{r^2}{2}\right]_p+\left[\frac{\delta H}{\delta U},U\right]. \tag{42}$$

Observe that Eqs. (41) and (42) differ from the two-dimensional system only in that $[\psi,f]$ is replaced by $[\psi,f]+[r^2/2,f]_p$. We therefore obtain the three-dimensional outer bracket by making an analogous replacement in Eq. (32):

$$\begin{aligned}\{F,G\}_3\equiv&\int d\mathbf{x}\Bigg\{\psi\left(\left[\frac{\delta F}{\delta\psi},\frac{\delta G}{\delta U}\right]+\left[\frac{\delta F}{\delta U},\frac{\delta G}{\delta\psi}\right]\right)\\&+\frac{r^2}{2}\left(\left[\frac{\delta F}{\delta\psi},\frac{\delta G}{\delta U}\right]_p+\left[\frac{\delta F}{\delta U},\frac{\delta G}{\delta\psi}\right]_p\right)\\&+U\left[\frac{\delta F}{\delta U},\frac{\delta G}{\delta U}\right]\Bigg\}\\=&\{F,G\}_2+\int d\mathbf{x}\,\frac{r^2}{2}\left(\left[\frac{\delta F}{\delta\psi},\frac{\delta G}{\delta U}\right]_p\right.\\&\left.+\left[\frac{\delta F}{\delta U},\frac{\delta G}{\delta\psi}\right]_p\right).\end{aligned} \tag{43}$$

Alternatively, Eq. (43) can be written in the form

$$\begin{aligned}\{F,G\}_3=\{F,G\}_2+\int d\mathbf{x}\Bigg(&\frac{\delta F}{\delta\psi}\frac{\partial}{\partial\zeta}\frac{\delta G}{\delta U}\\&-\frac{\delta G}{\delta\psi}\frac{\partial}{\partial\zeta}\frac{\delta F}{\delta U}\Bigg).\end{aligned} \tag{44}$$

A bracket of the form of the second term of Eq. (44) has previously appeared in Refs. 13, 14, and 27. For its geometrical interpretation, see Ref. 13. The argument of the previous subsection quickly shows that this bracket yields the correct equations of motion,

$$\dot\psi=\{\psi,H\}_3,\quad \dot U=\{U,H\}_3, \tag{45}$$

and it is obviously antisymmetric. Hence we turn our attention to Jacobi's identity.

The nested three-dimensional bracket $\{E,\{F,G\}_3\}_3$, will contain: nested two-dimensional brackets, coming from the first term of Eq. (44); nested brackets involving only poloidal inner brackets, corresponding to the second term; and cross terms involving both poloidal and toroidal inner brackets. The first two of these contributions are easily seen to satisfy Eq. (19), so we may restrict our attention to the cross terms. These can be simplified by means of Eq. (31), and there remains only

$$\{E,\{F,G\}_3\}_3 = \int d\mathbf{x}\, \frac{r^2}{2} \Big\{ \Big[\frac{\delta E}{\delta \psi}, \Big[\frac{\delta F}{\delta U}, \frac{\delta G}{\delta U} \Big] \Big]_p + \Big[\frac{\delta E}{\delta U}, \Big[\frac{\delta F}{\delta \psi}, \frac{\delta G}{\delta U} \Big] + \Big[\frac{\delta F}{\delta U}, \frac{\delta G}{\delta \psi} \Big] \Big]_p \Big\} + A. \tag{46}$$

Here, as usual, A represents the terms which are already known to satisfy Jacobi's identity. Because functional ψ derivatives are symmetrically distributed in Eq. (46), it can be seen that $\{\ \ ,\ \ \}_3$ will satisfy Jacobi's identify provided that the quantity

$$Z \equiv \int d\mathbf{x}\, \frac{r^2}{2} \{ [e,[f,g]]_p + [f,[g,e]]_p + [g,[e,f]]_p \}$$

vanishes, for any functions e, f, and g. We use Eqs. (39) and (40) to find

$$Z = \int d\mathbf{x} \Big\{ e \frac{\partial}{\partial \zeta} [f,g] + f \frac{\partial}{\partial \zeta} [g,e] + g \frac{\partial}{\partial \zeta} [e,f] \Big\},$$

and then combine Eqs. (36) and (38) to obtain

$$Z = -\int d\mathbf{x}\, \hat{\nabla} e \cdot \hat{\nabla} f \times \hat{\nabla} g.$$

The integrand in this last expression is a divergence. Hence, with our usual neglect of surface contributions,

$$Z = 0,$$

and the Jacobi identity is satisfied.

We close this section by considering the specialization of the three-dimensional bracket to the single-helicity, or helically symmetric, case. The helical symmetry constraint

$$\frac{\partial f}{\partial \zeta} = -\frac{1}{q_0} \frac{\partial f}{\partial \theta}, \tag{47}$$

where q_0 is the helicity (or rational safety factor), can be seen to imply

$$[f,g]_p = (1/q_0)[f,g]. \tag{48}$$

Hence Eqs. (41) and (42) can be written as

$$\dot{\psi}_h = [\psi_h, \phi], \tag{49}$$

$$\dot{U} = [\psi_h, J_h] - [\phi, U], \tag{50}$$

where

$$\psi_h \equiv \psi + r^2/2q_0$$

is the helical flux and $J_h = \nabla_\perp^2 \psi_h$. We have noted that $\dot{\psi} = \dot{\psi}_h$ and that

$$[\psi_h, J] = [\psi_h, J_h],$$

since J_h differs from $J = \nabla_\perp^2 \psi$ only by a constant. It follows that Eqs. (49) and (50) coincide with the two-dimensional system studied in the previous subsection; one needs merely to interpret ψ, in the two-dimensional formalism, as the helical flux. Similarly, in terms of the helically symmetric Hamiltonian,

$$H = \frac{1}{2} \int d\mathbf{x} [(\nabla_\perp \phi)^2 + (\nabla_\perp \psi_h)^2], \tag{51}$$

the Poisson bracket of Eq. (30) can be obtained as the helically symmetric version of Eq. (43).

IV. HIGH-BETA THEORY

The results of the previous section applied to the equations obtained in the ordering $8\pi p/B_0^2 \lesssim \epsilon^2$. Here we consider the case where $8\pi p/B_0^2 = O(\epsilon)$. This results in the inclusion of the interchange term, $-\partial\beta/\partial y$, in Eq. (3) and the pressure is seen to advect as in Eq. (4). The equations are thus generalized to include pressure-gradient driven instability.

A. High-beta Poisson bracket

In Sec. II it was noted that the conserved energy for high-beta RMHD is

$$H = \int \frac{1}{2} (|\nabla_\perp \phi|^2 + |\nabla_\perp \psi|^2 - 2x\beta) d\mathbf{x}. \tag{52}$$

Note that this form differs from that used for the Hamiltonian in Sec. III by the addition of the pressure term, $-2x\beta$, which comes from the internal energy term of the ideal MHD Hamiltonian. The Poisson bracket of the previous section with this Hamiltonian will still produce the low-beta equations. In order to produce the high-beta equations, additional terms must be added to the bracket, Eq. (32). These terms will naturally involve functional derivatives with respect to β. Furthermore, since the equation for β is coupled to the equation for U, the Poisson bracket must involve functional derivatives with respect to U. These remarks suggest that the following should be added to Eq. (43):

$$\{F,G\}_4 = \int \beta \Big\{ \Big[\frac{\delta F}{\delta \beta}, \frac{\delta G}{\delta U} \Big] + \Big[\frac{\delta F}{\delta U}, \frac{\delta G}{\delta \beta} \Big] \Big\} d\mathbf{x}. \tag{53}$$

This form is clearly antisymmetric, but let us investigate its effect upon the equations of motion. Inserting U with the Hamiltonian, Eq. (52), yields

$$\{U,H\}_4 = -\frac{\partial \beta}{\partial y}.$$

This is the interchange term that is desired for the right-hand side of Eq. (8). Inserting β and the Hamiltonian in Eq. (53) yields

$$\{\beta,H\}_4 = -[\beta,\phi].$$

This is clearly seen to be the right-hand side of Eq. (9) written in "inner" bracket form. In summary, Eq. (43) plus Eq. (53) produces the high-beta RMHD equations with the Hamiltonian Eq. (52). It remains to show that this large bracket satisfies the Jacobi identity.

As in Sec. II B, we observe that in order for $\{\ \ ,\ \ \}_3 + \{\ \ ,\ \ \}_4$ to satisfy the Jacobi identity, the following must vanish:

$$\{F,\{G,F\}\} + \uparrow = \{F,\{G,H\}_3\}_3 + \{F,\{G,H\}_3\}_4 + \{F,\{G,H\}_4\}_3 + \{F,\{G,H\}_4\}_4 + \uparrow,$$

where the arrow indicates cyclic permutation. We have already shown that the first term makes no contribution. Likewise the third term vanishes since $\{F,G\}_4$ has no explicit dependence on ψ or U. Hence it remains to show that

$$\{F,\{G,H\}\}_4 + \uparrow = 0, \tag{54}$$

where we only need to worry about functional derivatives acting on explicit dynamical variable dependence. Equation (54) thus becomes

$$\{F,\{G,H\}\}_4 + \uparrow = \int d\mathbf{x}\,\beta\left\{\left[\frac{\delta F}{\delta\beta},\left[\frac{\delta G}{\delta U},\frac{\delta H}{\delta U}\right]\right] + \left[\frac{\delta F}{\delta U},\left[\frac{\delta G}{\delta\beta},\frac{\delta H}{\delta U}\right]\right] + \left[\frac{\delta F}{\delta U},\left[\frac{\delta G}{\delta\beta},\frac{\delta H}{\delta U}\right]\right]\right\} + \uparrow.$$

Clearly this vanishes, as is always the case for brackets that depend linearly on the dynamical variables, by virtue of the Jacobi identity for the inner bracket.

To summarize, we denote the three-dimensional, high-beta Poisson bracket by

$$\{\quad\} = \{\quad\}_3 + \{\quad\}_4,$$

or

$$\{F,G\} = \int d\mathbf{x}\left\{\psi\left(\left[\frac{\delta F}{\delta\psi},\frac{\delta G}{\delta U}\right] + \left[\frac{\delta F}{\delta U},\frac{\delta G}{\delta\psi}\right]\right) + \frac{r^2}{2}\left(\left[\frac{\delta F}{\delta\psi},\frac{\delta G}{\delta U}\right]_p + \left[\frac{\delta F}{\delta U},\frac{\delta G}{\delta\psi}\right]_p\right) + U\left[\frac{\delta F}{\delta U},\frac{\delta G}{\delta U}\right] + \beta\left(\left[\frac{\delta F}{\delta\beta},\frac{\delta G}{\delta U}\right] + \left[\frac{\delta F}{\delta U},\frac{\delta G}{\delta\beta}\right]\right)\right\}. \tag{55}$$

Then we have shown that the high-beta reduced MHD equations can be expressed as

$$\dot\psi = \{\psi,H\},\quad \dot U = \{U,H\},\quad \dot\beta = \{\beta,H\}, \tag{56}$$

where H is the general Hamiltonian given by Eq. (52).

B. Fourier decomposition

In applications of reduced MHD, it is often convenient to represent the $\boldsymbol\theta$ variation of the fields in terms of Fourier components. We use the convention

$$f(r,\boldsymbol\theta) = \sum_{\mathbf{m}} \exp(i\mathbf{m}\cdot\boldsymbol\theta) f_{\mathbf{m}}(r),$$

so that

$$f_{\mathbf{m}}(r) = (2\pi)^{-2}\oint d\,\boldsymbol\theta\,\exp(-i\mathbf{m}\cdot\boldsymbol\theta) f(r,\boldsymbol\theta),$$

with $f_{-\mathbf{m}} = f^*_{\mathbf{m}}$. Here we have introduced the convenient abbreviations $\boldsymbol\theta \equiv (\theta,\zeta)$,

and

$$\mathbf{m} \equiv (m,-n).$$

The asterisk denotes complex conjugation and

$$\oint d\,\boldsymbol\theta \equiv \int_{-\pi}^{\pi} d\theta \int_{-\pi}^{\pi} d\zeta.$$

In order to express the Hamiltonian theory in terms of Fourier amplitudes, we consider first the decomposition of the inner brackets. For the toroidal bracket of Eq. (37), we compute

$$\begin{aligned}[][f,g]_{\mathbf{m}} &= (2\pi)^{-2}\oint d\theta\,\exp(-i\mathbf{m}\cdot\boldsymbol\theta)[f,g] \\ &= \frac{i}{r}\sum_{\mathbf{m}'}\left((m-m')g_{\mathbf{m}-\mathbf{m}'}\frac{\partial f_{\mathbf{m}'}}{\partial r} - m' f_{\mathbf{m}'}\frac{\partial g_{\mathbf{m}-\mathbf{m}'}}{\partial r}\right) \\ &= \frac{i}{r}\sum_{\mathbf{m}'}\left(m g_{\mathbf{m}-\mathbf{m}'}\frac{\partial f_{\mathbf{m}'}}{\partial r} - m'\frac{\partial}{\partial r}g_{\mathbf{m}-\mathbf{m}'}f_{\mathbf{m}'}\right).\end{aligned} \tag{57}$$

Notice that the radial derivative in the last term of Eq. (57) acts on both functions to its right. The poloidal bracket yields a similar form:

$$[f,g]_{p\mathbf{m}} = \frac{i}{r}\sum_{m'}\left(n g_{\mathbf{m}-\mathbf{m}'}\frac{\partial f_{\mathbf{m}'}}{\partial r} - n'\frac{\partial}{\partial r}g_{\mathbf{m}-\mathbf{m}'}f_{\mathbf{m}'}\right). \tag{58}$$

Consider next some functional F of a field f,

$$F[f] = \int d\mathbf{x}\,\widetilde F(f,\nabla f,\ldots),$$

where $\widetilde F$ is the corresponding density and the omitted arguments are higher-order derivatives on f. It is clear that Fourier decomposition of f will induce a functional of the Fourier coefficients,

$$\overline F[f_1,f_2,\ldots] = \int r\,dr\oint d\,\boldsymbol\theta\,\widetilde F\left(\sum_m f_{\mathbf{m}}\exp(i\mathbf{m}\cdot\boldsymbol\theta)\right). \tag{59}$$

What is needed is a relation between the functional derivatives $\delta F/\delta f$ and $\delta\overline F/\delta f_{\mathbf{m}}$. A convenient expression for $\delta F/\delta f$ is obtained from Eqs. (25):

$$\frac{\delta F}{\delta f} = \frac{\partial\widetilde F}{\partial f} - \nabla\cdot\frac{\partial\widetilde F}{\partial(\nabla f)} + \cdots, \tag{60}$$

while $\delta\overline F/\delta f_{\mathbf{m}}$ is defined by

$$\frac{d}{d\epsilon}\overline F[\ldots,f_{\mathbf{m}}+\epsilon\eta_{\mathbf{m}},\ldots] = \int r\,dr\,\eta_{\mathbf{m}}\frac{\delta\overline F}{\delta f_{\mathbf{m}}}. \tag{61}$$

From Eqs. (59) and (60) we compute

$$\frac{d\overline F}{d\epsilon} = \int r\,dr\,\eta_{\mathbf{m}}(r)\oint d\,\boldsymbol\theta \times\exp(i\mathbf{m}\cdot\boldsymbol\theta)\left(\frac{\partial\widetilde F}{\partial f} - \nabla\cdot\frac{\partial\widetilde F}{\partial(\nabla f)} + \cdots\right). \tag{62}$$

After comparing the integrand in Eq. (62) to the definition of Eq. (61), we see that

$$\frac{\delta\overline F}{\delta f_{\mathbf{m}}} = (2\pi)^2\left(\frac{\delta F}{\delta f}\right)_{-\mathbf{m}}. \tag{63}$$

Let us apply this formula to the general Hamiltonian functional

$$H = \frac{1}{2}\int dr\,r\oint d\,\boldsymbol\theta\left[(\nabla_\perp\phi)^2 + (\nabla_\perp\psi)^2 - 2r\cos\theta\beta\right].$$

The induced functional, $\overline H$, is readily computed:

$$\overline H = \frac{1}{2}\sum_{\mathbf{m}} dr\,r\left\{\left|\frac{\partial\phi_{\mathbf{m}}}{\partial r}\right|^2 + \frac{m^2}{r^2}|\phi_{\mathbf{m}}|^2 + \left|\frac{\partial\psi_{\mathbf{m}}}{\partial r}\right|^2 + \frac{m^2}{r^2}|\psi_{\mathbf{m}}|^2 - 2r(\cos\theta)_{\mathbf{m}}\beta_{\mathbf{m}}\right\}.$$

Notice that

$$2(\cos\theta)_{\mathbf{m}} = \delta_{n,0}(\delta_{m,1} + \delta_{m,-1})$$

in terms of Kronecker delta functions. Hence only $\beta_{1,0}$ and $\beta_{-1,0}$ contribute to $\overline{H}$. Recalling that

$$\frac{\delta H}{\delta\psi} = -J, \quad \frac{\delta H}{\delta U} = -\phi, \quad \frac{\delta H}{\delta\beta} = -x,$$

we can use Eq. (63) to obtain the formulas

$$\frac{\delta\overline{H}}{\delta\psi_{\mathbf{m}}} = -(2\pi)^2 J_{-\mathbf{m}}, \tag{64}$$

$$\frac{\delta\overline{H}}{\delta U_{\mathbf{m}}} = -(2\pi)^2 \phi_{-\mathbf{m}}, \tag{65}$$

and

$$\frac{\delta\overline{H}}{\delta\beta_{\mathbf{m}}} = -(2\pi)^2 \frac{r}{2}\delta_{n0}(\delta_{m,1} + \delta_{m,-1}). \tag{66}$$

Equations (55), (57), and (58) can be combined to write the general Poisson bracket in terms of the Fourier coefficients $\psi_{\mathbf{m}}$, $U_{\mathbf{m}}$, and $\beta_{\mathbf{m}}$. We omit the result, which is straightforward to obtain, but consider explicitly the most important special case: that in which the functional F is a Fourier component of one of the three basic fields. It is evident that any Fourier coefficient, $f_{\mathbf{m}}(r)$, can be considered as a functional of $f(r,\boldsymbol{\theta})$; Eq. (25) provides the functional derivative

$$\frac{\delta f_{\mathbf{m}}(r)}{\delta f(r_0,\boldsymbol{\theta}_0)} = (2\pi)^{-2} r^{-1}\delta(r-r_0)\exp(-i\mathbf{m}\cdot\boldsymbol{\theta}_0). \tag{67}$$

Hence $\{f_{\mathbf{m}}, G\}$ is well-defined for any functional G. Suppose, for example, that $f_{\mathbf{m}} = \psi_{\mathbf{m}}$. Then Eq. (55) provides

$$\{\psi_{\mathbf{m}}, G\} = \int r\,dr \oint d\,\boldsymbol{\theta}\left\{\psi\left[\frac{\delta\psi_{\mathbf{m}}}{\delta\psi}, \frac{\delta G}{\delta U}\right] + \frac{r^2}{2}\left[\frac{\delta\psi_{\mathbf{m}}}{\delta\psi}, \frac{\delta G}{\delta U}\right]_p\right\}.$$

We use Eq. (39) to rearrange the integral

$$\{\psi_{\mathbf{m}}, G\} = \int r\,dr \oint d\,\boldsymbol{\theta}\,\frac{\delta\psi_{\mathbf{m}}}{\delta\psi}\left(\left[\frac{\delta G}{\delta U}, \psi\right] + \left[\frac{\delta G}{\delta U}, \frac{r^2}{2}\right]_p\right),$$

which then can be evaluated by means of Eq. (67):

$$\{\psi_{\mathbf{m}}, G\} = \oint \frac{d\,\boldsymbol{\theta}}{(2\pi)2}\exp(-i\mathbf{m}\cdot\boldsymbol{\theta}_0)\left(\left[\frac{\delta G}{\delta U}, \psi\right] + \left[\frac{\delta G}{\delta U}, \frac{r^2}{2}\right]_p\right) = \left(\left[\frac{\delta G}{\delta U}, \psi\right] + \left[\frac{\delta G}{\delta U}, \frac{r^2}{2}\right]_p\right)_{\mathbf{m}}.$$

Since a similar argument, using Eq. (23), shows that

$$\{\psi, G\} = \left[\frac{\delta G}{\delta U}, \psi\right] + \left[\frac{\delta G}{\delta U}, \frac{r^2}{2}\right]_p,$$

we have obtained the important result

$$\{\psi, G\}_{\mathbf{m}} = \{\psi_{\mathbf{m}}, G\}, \tag{68}$$

which equates the Fourier component of a Poisson bracket with ψ to the *same* bracket with the Fourier component of ψ. It can be seen that Eq. (68) also holds when ψ is replaced by U or β.

The main point of Eq. (68) is that it permits immediate Fourier decomposition of Hamilton's equations, Eqs. (56):

$$\begin{aligned} \dot{\psi}_{\mathbf{m}} &= \{\psi_{\mathbf{m}}, H\}, \\ \dot{U}_{\mathbf{m}} &= \{U_{\mathbf{m}}, H\}, \\ \dot{\beta}_{\mathbf{m}} &= \{\beta_{\mathbf{m}}, H\}, \end{aligned} \tag{69}$$

where, as in Eq. (56), the Poisson bracket is that defined by Eq. (55). Thus the Fourier coefficients obey precisely the same equations of motion as the corresponding fields, when (and only when) these equations are written in Hamiltonian form. In this sense, Hamilton's equations are invariant under Fourier decomposition. However, we note that the Jacobi identity does not survive truncation: all Fourier harmonics must be retained. It is also worth noting that energy conservation does survive truncation, because it depends only on antisymmetry of the bracket.

Of course ψ_m (for example) is coupled to $\psi_{\mathbf{m}'}$, $\mathbf{m}' \neq \mathbf{m}$, as well as to $U_{\mathbf{m}'}$ and $\beta_{\mathbf{m}'}$. Such couplings are explicit in Eqs. (57) and (58), and are implicitly included in Eqs. (69), by the definition of the outer bracket. This bracket similarly includes the effects of the Fourier components $\beta_{\mathbf{m}}$, $\mathbf{m} \neq (\pm 1,0)$, which are absent from the Hamiltonian.

The main conclusion of this subsection is that the transformation from the space $(r,\boldsymbol{\theta})$ to the space $(r,\mathbf{m})$ (Fourier discretization) is easily effected without modifying the definition of the outer bracket.

V. INTRODUCTION OF POTENTIALS—CANONICAL FORM

It is well-known that in order to represent Maxwell's equations in vacuum in canonical Hamiltonian form it is necessary to introduce the vector potential. In a similar manner the generalized Poisson brackets presented here can be transformed to canonical form via the decomposition of our fields into "potentials." Decomposition of physical fields into subsidiary fields has an extensive precedence that includes work of Euler[28] (1769) and Clebsch[29] (1859). The reader interested in this history is referred to Ref. 14. Recent work concerned with the interconnection between noncanonical Poisson brackets, canonical variables, gauge groups,and variation principles can be found in Refs. 12, 14, 21, 23, 30, and 31.

In this section, we restrict our attention to the low-beta, single-helicity case of Sec. III A. The transformation to canonical variables, $(\mathbf{Q},\mathbf{P})$, is effected; hence, the equations of motion are expressed in the form

$$\dot{\mathbf{Q}} = \frac{\delta H}{\delta\mathbf{P}}, \quad \dot{\mathbf{P}} = -\frac{\delta H}{\delta\mathbf{Q}}. \tag{70}$$

The canonical formulation involves four fields rather than the initial two (ψ, U)—a fact which weighs against the apparent simplicity of Eqs. (70). Nonetheless, the analogous potential decomposition for the ideal fluid has been ascerted by Buneman[32] to be of numerical advantage. Next, in this section we present the variational principle for which solutions to Eqs. (70) and hence RMHD are extremal functions. Variational principles are natural starting points for trial function or Rayleigh–Ritz approximations. The analogous variational principle for two-dimensional scalar vortex advection has been used by Salmon for numerical integration.[33]

The canonical variables are related to ψ and U through the following:

$$\psi = \hat{\zeta}\cdot\nabla_\perp Q_1 \times \nabla_\perp Q_2 = [Q_1, Q_2], \tag{71}$$

and

$$U = [Q_2, P_2] + [Q_1, P_1]. \tag{72}$$

With these definitions, we can compute the relevant functional derivatives. For example, a functional F of ψ and U yields a corresponding functional $\hat{F}$ of $(Q_i, P_i) = 1,2$ with

$$\frac{\delta \hat{F}}{\delta P_1} = \left[\frac{\delta F}{\delta U}, Q_1\right],$$

$$\frac{\delta \hat{F}}{\delta Q_1} = -\left[\frac{\delta F}{\delta \psi}, Q_2\right] - \left[\frac{\delta F}{\delta U}, P_1\right],$$

and so on. (Such formulas are derived from a functional derivative version of the chain rule.) One readily finds that the outer bracket of Eq. (30) becomes

$$\{F, G\}_2 = \sum_i \int d\mathbf{x}_\perp \left(\frac{\delta F}{\delta P_i}\frac{\delta G}{\delta Q_i} - \frac{\delta F}{\delta Q_i}\frac{\delta G}{\delta P_i}\right), \tag{73}$$

which is manifestly canonical. For simplicity, the caret notation is suppressed on the right-hand side.

The canonical bracket leads directly to Eqs. (70). Consider, for example, the equation of motion $\dot{\psi} = \{\psi, H\}$. In view of the definition, Eq. (71), we have

$$[\dot{Q}_1, Q_2] + [Q_1, \dot{Q}_2] = \{[Q_1, Q_2], H\}.$$

Then Eq. (73) provides

$$[\dot{Q}_1, Q_2] + [Q_1, \dot{Q}_2] = \left[\frac{\delta H}{\delta P_1}, Q_2\right] + \left[Q_1, \frac{\delta H}{\delta P_2}\right].$$

Therefore we can choose

$$\dot{Q}_1 = \frac{\delta H}{\delta P_i}, \quad i = 1,2. \tag{74}$$

A similar calculation shows that

$$\dot{P}_i = \frac{-\delta H}{\delta Q_i}, \quad i = 1,2, \tag{75}$$

will produce $\dot{U} = \{U, H\}$. Hence if P_i and Q_i satisfy Eqs. (74) and (75), where the right-hand sides are obtained by treating H as a functional of $\mathbf{P}$ and $\mathbf{Q}$, then the ψ and U obtained through Eqs. (71) and (72) necessarily satisfy Eqs. (12) and (13).

Let us write Eq. (74) more explicitly. In view of Eqs. (26) and (72),

$$\frac{\delta H}{\delta P_i} = \left[\frac{\delta H}{\delta U}, Q_i\right] = -[\phi, Q_i].$$

Hence we have

$$\dot{Q}_i + [\phi, Q_i] = 0,$$

or, in terms of the reduced MHD fluid velocity,

$$\mathbf{V} = \hat{\zeta} \times \nabla_\perp \phi,$$

$$\frac{dQ_i}{dt} \equiv \frac{\partial Q_i}{\partial t} + \mathbf{V}\cdot\nabla_\perp Q_i = 0. \tag{76}$$

Thus Eq. (74) simply implies that the Q_i are constant in the rest frame of the fluid,

A similar explication of Eq. (75) reveals that

$$\frac{dP_1}{dt} = \nabla_\perp\cdot(\hat{\zeta} J \times \nabla_\perp Q_2), \tag{77}$$

$$\frac{dP_2}{dt} = \nabla_\perp\cdot(\nabla_\perp Q_1 \times \hat{\zeta} J), \tag{78}$$

where, as usual, $J \equiv \nabla_\perp^2 \psi$ measures the toroidal current. The interpretation of Eqs. (77) and (78) is considered next.

We first observe that the flow velocity, defined by

$$\mathbf{V}_* \equiv -\sum_i P_i \nabla_\perp Q_i, \tag{79}$$

has the same vorticity as $\mathbf{V}$. That is,

$$\hat{\zeta}\cdot\nabla_\perp \times \mathbf{V}_* = -\hat{\zeta}\cdot\nabla_\perp \times \left(\sum_i P_i \nabla_\perp Q_i\right) = \hat{\zeta}\cdot\nabla_\perp \times \mathbf{V} \equiv U,$$

in view of Eq. (72). Thus $\mathbf{V}_*$ and $\mathbf{V}$ differ by a two-dimensional gradient,

$$\mathbf{V}_* = \mathbf{V} + \nabla_\perp \chi, \tag{80}$$

and we can ensure that $\nabla_\perp\cdot\mathbf{V}_* = \nabla_\perp\cdot\mathbf{V} = 0$ by requiring

$$\nabla_\perp^2 \chi = 0.$$

In terms of canonical variables, the same requirement yields the constraint

$$\sum_i (P_i \nabla_\perp^2 Q_i + \nabla_\perp P_i \cdot \nabla_\perp Q_i) = 0. \tag{81}$$

Equation (79) suggests choosing the Q_i as spatial coordinates: $(x, y, \zeta) \mapsto (Q_1, Q_2, \zeta)$. The P_i are then seen to be the covariant components of $-\mathbf{V}_*$, which evolve according to Eqs. (77) and (78). We write the latter in terms of the new coordinates, noting that the volume element $\sqrt{g}$ in (Q_1, Q_2, ζ) space is given by

$$1/\sqrt{g} = \hat{\zeta}\cdot\nabla_\perp Q_1 \times \nabla_\perp Q_2 = \psi. \tag{82}$$

Thus, for any vector $\mathbf{A}$,

$$\nabla_\perp\cdot\mathbf{A} = \psi\left(\frac{\partial}{\partial Q_1}\frac{\mathbf{A}\cdot\nabla_\perp Q_1}{\psi} + \frac{\partial}{\partial Q_2}\frac{\mathbf{A}\cdot\nabla_\perp Q_2}{\psi}\right),$$

and we find that Eqs. (71) and (72) can be written as

$$\frac{dP_1}{dt} = -\psi\frac{\partial J}{\partial Q_1}, \quad \frac{dP_2}{dt} = -\psi\frac{\partial J}{\partial Q_2}. \tag{83}$$

To understand Eqs. (83), we return to the single helicity equation of motion,

$$\dot{U} + [\phi, U] = [\psi, J],$$

which can be written as

$$\frac{d}{dt}\hat{\zeta}\cdot\nabla_\perp \times \mathbf{V} = \hat{\zeta}\cdot\nabla_\perp\psi \times \nabla_\perp J = \hat{\zeta}\cdot\nabla_\perp \times (\psi\nabla_\perp J).$$

Thus

$$\hat{\zeta}\cdot\nabla_\perp \times \left(\frac{d\mathbf{V}}{dt} - \psi\nabla_\perp J\right) = 0. \tag{84}$$

Here we used the identity

$$\hat{\zeta}\cdot\nabla_\perp \times [(\mathbf{V}\cdot\nabla_\perp)\mathbf{V}] = \mathbf{V}\cdot\nabla_\perp(\hat{\zeta}\cdot\nabla_\perp \times \mathbf{V}),$$

which also enters the derivation of reduced MHD, and which can be verified directly. Equation (84) implies that

$$\frac{d\mathbf{V}}{dt} = \psi\nabla_\perp J + \nabla_\perp F, \tag{85}$$

where the arbitrary function F is evidently related to the "gauge" function χ of Eq. (80). Equation (85) can be seen to be equivalent to Eqs. (83). The latter therefore compactly express the covariant fluid acceleration, with a gauge choice which eliminates the $\nabla_\perp F$ term. Since the vorticity, and thus also the dynamics of reduced MHD, are gauge independent, this gauge choice is appropriate.

Equations (83) have the nice property of emphasizing the essential free-energy source for the class of motions pertinent to low-beta reduced MHD: current gradients.

Now we construct the action principle that produces Eqs. (74) and (75) upon variation. Consider

$$A[\mathbf{Q},\mathbf{P}] = \int dt\left(\int d\mathbf{x}\,\mathbf{P}\cdot\dot{\mathbf{Q}} - H(P,Q)\right). \tag{86}$$

If we treat $\mathbf{Q}$ and $\mathbf{P}$ as independent variables, then the class of variations of A that allow the neglect of surface terms yields, for $\delta A/\delta\mathbf{Q}(\mathbf{x},t) = 0$,

$$\dot{\mathbf{P}} = -\frac{\delta H}{\delta\mathbf{Q}(\mathbf{x})}, \tag{87a}$$

and similarly for $\delta A/\delta\mathbf{P}(\mathbf{x},t) = 0$ we obtain

$$\dot{\mathbf{Q}} = \frac{\delta H}{\delta\mathbf{P}(x)}. \tag{87b}$$

If either the variational principle Eq. (86) or the symmetry manifest in Eqs. (87) is to be utilized, then initial conditions on ψ and U must be transformed into initial conditions on $\mathbf{Q}$ and $\mathbf{P}$. This transformation is not unique—the choice must be tailored to the application at hand.

VI. CONCLUSIONS

To summarize, we have presented the Hamiltonian description of RMHD in both its high-beta and low-beta versions. Our main objective has been to make the extensive machinery of Hamiltonian theory applicable to an important model of tokamak behavior. In particular, the formalism is useful for understanding symmetries of a system, as evidenced by the presentation here of new conservation laws. These conservation laws enabled the discovery of a large class of exact nonlinear solutions, some of which have been found to possess Liapunov functionals.[19] Understanding the symmetries of a system is important when one does perturbation theory. For example, it seems reasonable that one should approximate a Hamiltonian system by another Hamiltonian system. Perturbation within the noncanonical Hamiltonian context for field theories is a subject of current research.

We also feel that the formalism could be useful for constructing numerical procedures.[32,33] There are two possibilities. First, one could start with the noncanonical bracket and discretize the spatial dependence. The resulting set of ordinary differential equations in time automatically conserves energy, even upon truncation. Unfortunately, in terms of noncanonical variables, truncation usually destroys the Jacobi identity. This situation can be remedied by a second approach, which is to use the canonical form presented in Sec. V. Discretization and truncation here result in a finite degree-of-freedom Hamiltonian system; it is guaranteed not to possess attracting sets, as it would possess in the dissipative case.

A glaring inadequacy for plasma applications of Hamiltonian formalisms is that they do not allow, in general, for dissipation. A possible way of including such phenomena is to generalize the formulation presented here by adding a symmetric bracket to the antisymmetric Poisson bracket. This notion is motivated by the fact that any operator can be split into self-adjoint and anti-self-adjoint parts. Symmetric brackets are not difficult to find for a variety of dissipative systems; e.g., Burgers-type viscous dissipation, kinetic equations with various collision operators, or resistive dissipation in a magnetic fluid. Resistive RMHD possesses the simple symmetric form

$$\{F,G\}_s = -\eta\int d\mathbf{x}\,\frac{\delta F}{\delta\psi}\frac{\delta G}{\delta\psi}.$$

Clearly, $\{\psi,H\}_s = \eta\nabla_\perp^2\psi$; hence, addition of this symmetric bracket to our previous Poisson [Eq. (55)] yields resistive RMHD. An interesting consequence is that evolution of the Casimir and other invariants is now governed entirely by the symmetric bracket. Further investigation of this generalization, relating to stability and the underlying mathematical structure, is in progress.

ACKNOWLEDGMENTS

One of the authors (PJM) would like to acknowledge helpful conversations with J. Marsden.

This paper was supported by the U. S. Department of Energy Grant No. DE-FG05-80ET-53088.

[1] M. N. Rosenbluth, D. A. Monticello, H. R. Strauss, and R. B. White, Phys. Fluids **19**, 198 (1976).
[2] H. R. Strauss, Phys. Fluids **19**, 134 (1976).
[3] H. R. Strauss, Phys. Fluids **20**, 1354 (1977).
[4] See, for example, H. R. Strauss, Nucl. Fusion **23**, 649 (1983).
[5] See, for example, B. Carreras, B. V. Waddell, and H. R. Hicks, Nucl. Fusion **19**, 1423 (1979).
[6] V. I. Arnold, *Mathematical Methods of Classical Mechanics* (Springer-Verlag, New York, 1978).
[7] R. G. Littlejohn, to appear in Phys. Rep.
[8] R. Abraham and J. E. Marsden, *Foundations of Mechanics* (Benjamin, Reading, MA, 1978).
[9] I. M. Gel'fand and L. A. Dikii, Funk. Anal. **10**, 1 (1976).
[10] A. N. Kaufman, Phys. Scr. **T2**, 17 (1982).
[11] J. E. Marsden, T. Ratiu, and A. Weinstein, Trans. Am. Math. Soc. (to be published).
[12] J. E. Marsden, A. Weinstein, T. Ratiu, R. Schmid, and R. Spencer, *Proceedings of the IUTAM-ISIMM, Symposium on Modern Developments in Analytical Mechanics* (Academy of Sciences, Turin, 1982), p. 289.
[13] Y. I. Manin, J. Sov. Math. **11**, 1 (1979).
[14] P. J. Morrison, in *Mathematical Methods in Hydrodynamics and Integrability in Dynamical Systems*, edited by M. Tabor and Y. Treve (AIP, New York, 1982), AIP Conf. Proc. No. 88, p. 13.
[15] P. J. Morrison and R. D. Hazeltine, presented at the Sherwood Theory Meeting, Arlington, Virginia, 1983, 3Q14.
[16] R. G. Littlejohn, in *Mathematical Methods in Hydrodynamics and Integrability in Dynamical Systems*, edited by M. Tabor and Y. Treve (AIP, New York, 1982), AIP Conf. Proc. No. 88, p. 47.
[17] J. D. Meiss and W. Horton, Phys. Fluids **26**, 990 (1983).
[18] D. Holm, J. E. Marsden, T. Ratiu, and A. Weinstein, to appear in Phys. Lett. **98A**, 15 (1983).
[19] P. J. Morrison and R. D. Hazeltine (work in progress).
[20] P. J. Morrison, J. D. Meiss, and J. R. Cary, Institute for Fusion Studies Report #80, Austin, Texas 1983 (submitted to Physica D).
[21] F. S. Henyey, Phys. Rev. A **26**, 480 (1982).
[22] J. E. Marsden and A. Weinstein, Physica **4D**, 394 (1982).

[23]P. J. Morrison, Princeton Plasma Physics Laboratory Report No. PPPL-1783, 1981.
[24]J. B. Taylor, Phys. Rev. Lett. **35**, 1139 (1974).
[25]A. Bhattacharjee, R. L. Dewar, and D. A. Monticello, Phys. Rev. Lett. **45**, 347 (1980); **45**, 1217 (1980).
[26]J. E. Marsden and P. J. Morrison, in *Fluids and Plasmas: Geometry and Dynamics*, edited by J. E. Marsden, P. Holms, and A. Weinstein (AMS, Boulder, CO, 1983).
[27]P. J. Morrison and J. M. Greene, Phys. Rev. Lett. **45**, 790 (1980); **48**, 569 (1982).
[28]See D. P. Stern, Am. J. Phys. **38**, 494 (1970).
[29]A. Clebsch, J. Reine Angew. Math. **54**, 293 (1857); **56**, 1 (1859).
[30]D. Holm and B. Kupershmidt, Physica **6D**, 347 (1983).
[31]P. J. Morrison, Princeton Plasma Physics Laboratory Report No. PPPL-1788, 1981.
[32]O. Buneman, Phys. Fluids **23**, 1716 (1980); in *Mathematical Methods in Hydrodynamics and Integrability in Dynamical Systems*, edited by M. Tabor and Y. Treve (AIP, New York, 1982), AIP Conf. Proc. No. 88, p. 137.
[33]R. Salmon, J. Phys. Oceanogr. **12**, 1458 (1982).