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MAGNETIC ISLANDS IN TOROIDALLY CONFINED PLASMA

M. KOTSCHENREUTHER, A.Y. AYDEMIR,
D.C. BARNES, J.R. CARY, J.D. HANSON,
R.D. HAZELTINE, P.J. MORRISON
Institute for Fusion Studies,
University of Texas at Austin,
Austin, Texas,
United States of America

Abstract

MAGNETIC ISLANDS IN TOROIDALLY CONFINED PLASMA.

The production of magnetic islands in toroidal confinement systems is examined analytically and numerically in several contexts: (1) The resistive dynamics of magnetic islands is examined analytically, including curvature and pressure. A Grad-Shafranov equation is derived to describe the MHD equilibria of thin islands. The resistive evolution is then obtained. Interchange effects are very important for small islands and progressively less so for larger ones. (2) A numerical method for eliminating stochasticity in vacuum magnetic fields is introduced. Application of this method shows that stochasticity can be made negligible by proper choice of the coil configuration. It is possible to increase the equilibrium β-limit by factors of two or more over that of a simple, 'straight' coil winding law. (3) The production of magnetic islands by the introduction of plasma pressure into non-axisymmetric confinement configurations is analysed, assuming scalar pressure. Far from the rational surfaces a procedure based on linearization in β applies. Singularities at the resonant surfaces are resolved with a non-linear analysis. Scaling is found by using the approximation of nearly circular flux surfaces. Island size depends dramatically on whether or not a magnetic well is present. If a magnetic hill is present, islands overlap for arbitrarily low pressure.

1. RESISTIVE DYNAMICS OF MAGNETIC ISLANDS

Here we analytically derive the resistive nonlinear dynamics of thin magnetic islands including the effects of pressure and curvature. This combines and extends the nonlinear island calculation of Rutherford[1] and the linear stability results of Glasser, Greene and Johnson [2].

A principal result is that there is a critical island width Δx_c. Islands wider than Δx_c are dominated by Δ', measuring the magnetic free energy, while narrower islands are dominated by pressure and curvature in the island vicinity. Δx_c is given by

$$\Delta x_c^s \Delta_*' \sim k_2(E+F) \tag{1}$$

where E, F, H and D_I = E+F+H are standard measures of magnetic curvature[2], $s = \sqrt{1-4D_I}$, and Δ'_* is the finite-β generalization[2] of Δ'. k_2 is roughly 6.3.

To obtain this result, a Grad-Shafranov equation is derived for thin islands which describes the resonant magnetic field in the island vicinity. Interestingly, the curvature enters this nonlinear equation through an expression proportional to the Mercier ideal interchange criterion.

The resistive evolution is then obtained. H is relatively unimportant in the resistive dynamics, similarly to the linear theory [2].

An expansion in the tokamak inverse aspect ratio ε is used for simplicity, using either $\beta \sim \varepsilon^2$ or $\beta \sim \varepsilon$. We believe the essential physics for general geometries is similar.

For tokamaks with stabilizing curvature and destabilizing Δ', Equation (1) gives the minimum island width for growth. This matches linear theory in the following sense. An island just barely into the Rutherford regime, whose width equals the linear layer width, requires a Δ' to overcome curvature stabilization, which scales as the critical Δ' of linear theory. Only for high β, $\beta \sim \varepsilon$, can Δx_c be a significant fraction of the minor radius. For destabilizing curvature and stabilizing Δ', Equation (1) gives the saturated island width for a single resistive interchange.

Nonlinear islands grow relatively slowly, so following Rutherford we neglect inertia. Therefore, our starting equations are

$$\underset{\sim}{j} \times \underset{\sim}{B} = \nabla p \tag{2}$$

$$\nabla \cdot \underset{\sim}{j}_{\parallel} = \underset{\sim}{B} \cdot \nabla p \times \nabla(-1/B^2) \tag{3}$$

and Ohm's law $E_{\parallel} = \eta j_{\parallel}$. The right-hand side of Equation (3) was neglected in Rutherford's treatment.

The island grows on an equilibrium field $\underset{\sim}{B}_0 = \nabla\chi\times\nabla(q(\chi)\vartheta - \zeta)$, with safety factor $q(\zeta)$ and toroidal angle ζ. We focus attention on the region near a rational surface with $\ell = m/n = q_0$. We define a periodic angle coordinate $\alpha = \vartheta - \zeta/q_0$. All quantities f have resonant and nonresonant parts. Since these parts generally behave differently, we define an averaging operator to select out the resonant part, $\bar{f}(\chi,\alpha) = \oint d\zeta\, f(\chi,\alpha,\zeta)/\oint d\zeta$; the nonresonant part is $\tilde{f} = f - \bar{f}$. We also define

$$[A,B] = \nabla\zeta \cdot (\nabla A \times \nabla B) \tag{4}$$

Equations (2) and (3) are now ordered for the case of islands which are thin compared to the minor radius. Also, an aspect ratio expansion is used. However, we keep terms one additional order higher[3] than usual in ε, but not in β. This is accurate enough to give the lowest-order average curvature for $\beta \sim \varepsilon$ or $\beta \sim \varepsilon^2$.

The total average magnetic flux, ψ_h, through a helical ribbon α = const, plays the role of a flux function for the island. In the island region, the perturbed flux ψ_1 is given by the covariant ζ component of the vector potential.

ψ_1 is obtained from Ampère's law, which for thin islands is

$$|\nabla\chi|^2 \frac{\partial^2\psi_1}{\partial\chi^2} = I \qquad (5)$$

where I is the covariant ζ current. $\bar{\psi}_1$ is found by multiplying Equation (5) by $|\nabla\chi|^{-2}$ and averaging. I is found by considering the average and nonresonant parts of Equations (2) and (3). To requisite order $\bar{p}$ is a function of ψ_h. $\bar{I}$ is given by

$$[\psi_h, \bar{I}] = [\bar{p}, \bar{h}] + \overline{[\tilde{p}, \tilde{h}]} - \overline{[\tilde{\psi}, \tilde{I}]} \qquad (6)$$

where h gives the effect of curvature, $h = -1/B_0^2 + 2P_0/B_0^4$. Without pressure and curvature, $\bar{I}$ is a function of ψ_h; $\bar{h}$ gives the effect of average normal curvature, and the last two terms give the effect of geodesic curvature and Pfirsch–Schlüter currents.

Explicit expressions for the last two terms are obtained using the nonresonant parts of Equations (2), (3) and (5). The calculation is lengthy but similar to linear theory, since the magnetic nonlinearity is negligible for the nonresonant component. The sum of these terms can be written in the form $[\bar{p}, h_g]$, with h_g depending on the geodesic curvature. With this form, Equation (7) is easily solved:

$$\bar{I} = J(\psi_h) + p'(\psi_h)(h+h_g) \qquad (7)$$

where J is an arbitrary function.

The results for I are substituted into Equation (5), and equilibrium quantities are Taylor-expanded about the rational surface $\chi = \chi_0$. We thus obtain a Grad–Shafranov equation describing the flux function ψ_h in the island region:

$$\frac{\partial^2\psi_h}{\partial\chi^2} = J_*(\psi_h) + (\chi-\chi_0)(G_1+G_2)\frac{\partial p(\psi_h)}{\partial\psi_h} \qquad (8)$$

where J_* is an arbitrary function, where G_1 and G_2 are (to relevant order in ε) E+F and H divided by $(\partial p/\partial\chi_0)(q_0/(\partial q/\partial\chi_0))^2$. The Mercier instability criterion is E+F+H > 1/4.

It remains to determine $J_*(\psi_h)$ and the dynamics. The average of Ohm's law and Faraday's law give to lowest order

$$\frac{\partial\overline{\psi}_1}{\partial t} + \frac{1}{q_0}[\psi_h,\overline{\varphi}] = \eta\overline{I} \tag{9}$$

where φ is the electrostatic potential, and η is the resistivity. As in Rutherford's[1] analysis, Equation (9) is flux-averaged (denoted by < >) at constant ψ_h, thus determining $\langle\overline{I}\rangle$ and eliminating $\overline{\varphi}$. J_* is determined from $\langle\overline{I}\rangle$. Matching to the exterior region far from the island introduces Δ_*'.

To solve Equation (8) analytically we assume that: (1) a single harmonic m dominates in $\overline{\psi}_h$ (e.g. the most unstable one), so that $\overline{\psi}_h \cong q_0^1(\chi-\chi_0)^2/q_0^2 + A(\chi,t)\cos m\alpha$; (2) $A(\chi,t)$ is nearly constant in χ near the island. This requires a subsidiary expansion in which G_1 and G_2 are small.

The pressure profile $p(\psi_h)$ near the island is assumed to be dominated by diffusion. The pressure gradient is maintained by sources deep in the plasma interior and is determined for slowly growing islands by the condition that the flux of pressure is constant. In view of the large anomalous transport inferred from experiments, we simply assume a large constant diffusion coefficient in the island region.

The resulting evolution equation for A(0,t) is best expressed in terms of the island width $\Delta\chi = 4\sqrt{q_0^2 A/(\partial q_0/\partial\chi)}$.

$$\frac{k_1}{\eta}\frac{\partial\Delta\chi}{\partial t} = \Delta_*'\Delta\chi^{s'} + k_2\frac{(E+F)}{\Delta\chi} \tag{10}$$

where $s' = -1 + \sqrt{1-4D_I}$, $k_1 \simeq 3$, $k_2 \simeq 6.3$. For $\beta=0$, this agrees with Rutherford's result (the island width grows linearly with time). The driving term from pressure dominates for $\Delta\chi < \Delta\chi_c$ given by Equation (1).

2. ELIMINATION OF STOCHASTICITY IN STELLARATOR VACUUM FIELDS

We describe here a numerical method of optimizing the coil winding law in stellarators to greatly reduce magnetic stochasticity in the vacuum magnetic field. For numerous

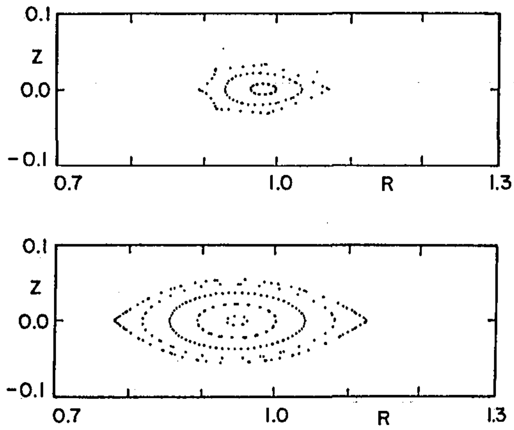

FIG.1. Surfaces of section of unoptimized and optimized stellarators.

examples it has led to much higher usable plasma volume, rotation number $\iota\!\!-$ and inverse aspect ratio.

The crucial ingredient of this method is a measure of the nonintegrability (i.e. size of island or amount of stochasticity), called the residue. First, the coil winding law is given as a function of several parameters. Those parameters are then varied to minimize the stochasticity.

Consider a field line through the points (R_0, z_0, ζ) and $(R, z, \zeta+2\pi/m)$, where R, z and ζ are cylindrical coordinates, and m is the field periodicity. The qth iterated return map is $N^q(R_0, z_0) = (R, z)$. The residue[4] of a fixed point of N^q is related to the eigenvalues of the linearization of N^q. For an integrable system without island structure, the eigenvalues are unity and r=0. We thus expect, and have found in practice, that reducing the residue decreases the stochasticity near the corresponding rational surface.

In the following example, the coils are wound on a torus with major radius $R_0 = 1$ and minor radius $r = 0.3$. The coils are wound on a function $\eta(\zeta)$, where η is the ordinary poloidal angle about $R_0 = 1$, $z=0$. The parameters we vary are the Fourier coefficients of η.

The top of Figure 1 shows the surface of section for an unoptimized stellarator with $\ell_0 = 2$, $m_0 = 5$, $B_0 = 1$, two helical coils, and $\iota\!\!-_{axis} = 0.72$.

The bottom of Figure 1 shows the results of minimizing the residues of the fixed points with rotation numbers 1/3, 1/4, 1/5. The usable area is clearly substantially larger. ε was increased by 80%, $\langle\iota\!\!-\rangle$ by 15%. A rough estimate[5] of the equilibrium β limit for a stellarator is $\beta_{eq} \sim \langle\iota\!\!-^2\rangle\varepsilon$. Here, β_{eq} was increased by a factor of 2.4.

3. PRESSURE-INDUCED EQUILIBRIUM ISLANDS IN 3-DIMENSIONAL PLASMAS

Here, we consider the plasma equilibrium resulting from the introduction of a small amount of plasma pressure into a nonaxisymmetric vacuum magnetic field with good flux surfaces, free of islands. We examine the pressure-induced islands within the MHD model. As in the resistive island dynamics section, local interchange effects are analysed (but without an aspect ratio expansion) and found to be important.

Newcomb[6] noted that there is a singularity at each rational surface in the equilibrium MHD equations which leads to the requirement that $\int d\ell/B$ must be constant on each surface. This is not generally true for vacuum magnetic fields. The islands at the rational surface resolve the singularity and relax this constraint.

An analytic formula for the induced island width is derived under the approximation of nearly circular flux surfaces. Magnetic perturbations arise both from currents far from the resonant surface and from those in the island region. Once β exceeds a threshold value β_t, island size depends strongly on whether there is a magnetic well. β_t is typically quite small. Island widths in regions of good average curvature do not increase for $\beta > \beta_t$. This occurs because the external magnetic perturbation is attempting to drive the island against stabilizing interchange forces.

In the case of bad curvature, the island width scales linearly with β for $\beta > \beta_t$. The island in this case is a saturated resistive interchange (as in the previous section), and the exterior currents play the role of an initiating perturbation. It is significant that then the island scales with mode number ℓ' as $1/\ell'$. Since the mean spacing of rational surfaces with rotation number $\iota = m/\ell$ (for both $m, \ell < \ell'$) scales as $1/\ell'^2$, island overlap always occurs for sufficiently high ℓ.

We choose flux coordinates χ, ϑ, ζ of the vacuum magnetic field, $\underset{\sim}{B}_0 = \nabla\chi\times\nabla\vartheta + \iota(\chi)\nabla\zeta\times\nabla\chi$. ζ is chosen to be a constant times the scalar potential[7] of $\underset{\sim}{B}_0$, and the Jacobian $J = \gamma/B_0^2$.

The currents in the exterior region are computed by linearization in the plasma pressure, $\underset{\sim}{j}_1 \times \underset{\sim}{B}_0 = \nabla p_1$.

The perpendicular current is obtained directly. $\nabla\cdot j = 0$ gives the parallel current $j_\parallel$. Define $j_\parallel = Q\underset{\sim}{B}$. After Fourier-transforming in ϑ and ζ, $Q = \sum_{\ell m} Q_{\ell m}\, e^{i(\ell\vartheta - m\zeta)}$, we obtain

$$Q_{\ell m} = \frac{-\ell P_1'(\chi) J_{\ell m}(\chi)}{\ell\iota(\chi) - m} + \hat{Q}\zeta(\chi - \chi_{\ell m}) \tag{11}$$

where $\iota(\chi_{\ell m}) = m/\ell$. Note that $J_{\ell m}$ is proportional to the variation of $\int d\ell/B$ on a flux surface. The singularity in the linearized equations is resolved by islands, thus determining $\hat{Q}$.

Since the singularities are integrable, the vector potential $\underset{\sim}{A}_1$ of the perturbed magnetic field can be found by a Green's function integration. This gives a formal solution:

$$\underset{\sim}{A}_1(\chi,\ell,m) = \underset{\sim}{C}_1(\chi,\ell,m) + \sum_{\ell',m'} \underset{\sim}{D}(\chi,\ell,m,\ell',m')\hat{Q}_{\ell',m'} \qquad (12)$$

To make further analytical progress, we introduce the approximation of nearly circular flux surfaces. This decouples the harmonics. Expressions can then be obtained in terms of the equilibrium profiles. For large ℓ, these depend only on local quantities at the resonant surface, and the covariant ζ components of Equation (12) (which produce the islands) are

$$C_\zeta = \frac{-R_0}{\partial \ell^2}\left(\frac{dP}{dr} J_{\ell m}\right)^{-1} \frac{d}{dr}\left[r\left(\frac{dp}{dr} J_{\ell m}\right)^2 \Big/ \frac{d\iota}{dr}\right]$$

$$D_\zeta = \frac{R_0}{\partial \ell}\left[1 + \left(\frac{mr}{\ell R_0}\right)^2\right]^{1/2} \qquad (13)$$

where R_0 is the mean major radius of the magnetic axis, and r is the minor radius of the magnetic surface.

The solution for the interior region is formally almost identical to that in the case of resistive islands, except that ζ is now proportional to the magnetic potential. Since we are considering here low β, h is taken as $1/B^2$, and the last two terms of Equation (6) can be neglected. Again for low-β, the constant ψ approximation[1] is made. The arbitrary function $J(\psi)$ is determined from the condition that the average toroidal current must vanish (corresponding to a resistive steady state).

Boozer[7] has pointed out that resistive diffusion near a resonant surface becomes singular for nonvanishing $J_{\ell m}$ and tends to flatten the pressure profile. This diffusion has been computed for the present case with an island at the resonant surface. Again, the island resolves the singularity. For sufficiently large islands (corresponding to $\beta > \ell^2 J_{\ell m}/J_{00}$, which is extremely low-β) this diffusion is smaller than the nonresonant diffusion and causes little flattening. There is some island-induced resonant diffusion, but this virtually never reduces the space-integrated current from the interior region by more than a factor $U \sim 1 - 1/2$.

We again assume a single harmonic dominates in $\bar{\psi}_1$ with coefficient $A_{\ell m}$. Having obtained $\hat{Q}$ as a nonlinear function of the island width, Equation (12) becomes, for circular flux surfaces:

$$0 = C_\zeta(\chi_0, \ell, m) - A_{\ell m} - \frac{\partial p}{\partial r}\frac{\partial J_{00}}{\partial r} UD_\zeta(\chi_0, \ell, m)\,|A_{\ell m}|^{1/2}$$

$$\times\ \mathrm{sign}(A_{\ell m})/|\iota_r'| \qquad (14)$$

where $\partial J_{00}/\partial r$ gives the magnetic well. For low-β, $A_{\ell m} \sim \beta$, the island width $\sim \beta^{1/2}$, and the island current term $\sim \beta^{3/2}$ is negligible. All terms are comparable at $\beta \sim \beta_t$. For $\beta >> \beta_t$ the third term becomes large.

For the good curvature case, the second and third have the same sign. Thus, the third balances the first, resulting in slower island growth with β. For the bad curvature case, the second and third nearly cancel, leading to a larger island, almost independent of C_ζ. This is the saturated state of a resistive interchange.

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