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Nonaxisymmetric, Sharp Boundary, Toroidal Equilibria

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One of the most intriguing problems in ideal magnetohydrodynamics (MHD) concerns the existence of stationary equilibrium states with nested magnetic surfaces in three dimensional toroidal geometry. Although this problem has been under scrutiny for over two decades, an existence proof for such states has been accomplished only under simplifying assumptions regarding the spatial symmetry of the geometry. Although toroidal magnetic surfaces may exist in the absence of spatial symmetry, the topology of these surfaces is generally complex because of the presence of magnetic islands and regions of space where magnetic field lines are ergodic.

To unravel the complex mathematical issues which are intrinsic to three-dimensional geometry, it is useful to explore equilibrium states of simplified configurations. One such example is the ideal MHD equilibrium problem of a sharp boundary, nonaxisymmetric toroidal plasma. In the sharp boundary configuration, a toroidal surface, S, separates a constant pressure, current free region of plasma, R, from an external vacuum region, V. The MHD solvability conditions for sharp boundary equilibria take on a different form than that required for smooth pressure profiles with rational and irrational magnetic surfaces. Existence of equilibria in this state requires that two conditions be satisfied on S: (i) the plasma-vacuum interface be a magnetic surface for both the internal plasma magnetic field, $\mathbf{B}_i$, and the external vacuum magnetic field, $\mathbf{B}_e$, and ii) pressure balance be maintained, i.e., $2P = B_e^2 - B_i^2$, where P is the plasma pressure. Given P and $\mathbf{B}_i$, the problem is to determine single-valued solutions for $\mathbf{B}_e$ which span the equilibrium interface. Writing $\mathbf{B}_e = \nabla\Phi_e$, the pressure balance equation acquires the form of the Hamilton-Jacobi equation, which is a first order, nonlinear, partial differential equation for the magnetic potential Φ_e on the interface. The characteristics of this equation are the lines of force of $\mathbf{B}_e$ on the interface. Because the Hamilton-Jacobi equation governs the magnetic potential on the plasma interface, the problem for $\mathbf{B}_e$ can be associated with integrals of an equivalent Hamiltonian system, where the magnetic pressure difference, $(B_e^2 - B_i^2)/2$, is the Hamiltonian function. In general, it is difficult to determine when the single-valued condition on $\mathbf{B}_e$ is exactly satisfied, as in all problems connected with integrability of classical Hamiltonian systems. Single-valued solutions for B_e appear explicitly only if spatial symmetry is present. In the absence of symmetry, resonance phenomena affect integrability of the system, and numerical solutions or expansions are required to identify acceptable solutions. Perturbation studies of the Hamiltonian system, in terms of either small pressure or weak asymmetry, have been carried out to obtain series solutions

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for $\mathbf{B}_e$. Although these approximate solutions satisfy the required periodicity constraints of toroidal geometry, the convergence properties of the generated series remains to be established. However, subject to the integrability conditions provided by the KAM theorem, exact solutions for $\mathbf{B}_e$ do exist. It can be demonstrated that the exact solutions can be approached by adopting an iteration scheme based on renormalizations of the canonical coordinates.

In this project, modeling of sharp boundary equilibria is accomplished in terms of a vacuum magnetic field, $\mathbf{B}_v$, which is generated by a suitable distribution of external currents. The plasma-vacuum interface, S, of this configuration is identified with a closed toroidal magnetic surface of $\mathbf{B}_v$, $\alpha_v(\mathbf{r}) = \text{const.}$, while the internal magnetic field, $\mathbf{B}_i$, is proportional to $\mathbf{B}_v$, i.e., $\mathbf{B}_i = \sigma \mathbf{B}_v$, where σ is a constant. The poloidal currents in region V determine the value of σ.

It is straightforward to find vacuum magnetic fields that have at least one closed, toroidal, magnetic surface. Given some toroidal surface, S, the magnetic field in the enclosed region, R, is derivable from a scalar potential, Φ_v, that satisfies Laplace's equation, subject to the constraint that the normal component $(\nabla \Phi_v)$ vanishes on S. These conditions define a well-posed Neumann problem for Φ_v: $\nabla^2 \Phi_v = 0$ in R; $\hat{n} \cdot \nabla \Phi_v = 0$ on S, where $\hat{n}$ is a normal vector on S. A more difficult problem is to find vacuum magnetic fields that have a distribution of toroidal magnetic surfaces. Although a distribution of magnetic surfaces is not required to construct sharp boundary plasmas, it may be desirable for modeling equilibria with steep, but continuous, profiles.

The coordinate system adopted for our analysis is the triad, (q_1, q_2, α_v). The radial-like coordinate, referenced with respect to the toroidal magnetic axis of $\mathbf{B}_v$, is the magnetic surface function, $\alpha_v(\mathbf{r})$, which satisfies the expression, $\mathbf{B}_v \cdot \nabla \alpha_v = 0$. The angle-like coordinates, which label points on the magnetic surfaces, $\alpha_v(\mathbf{r}) = \text{const.}$, are the variables, (q_1, q_2). There is obviously considerable freedom in the choice of the q_1 and q_2. Convenient variables for certain applications are β_v and φ_v, based on the Clebsch representation of $\mathbf{B}_v$, $\mathbf{B}_v = \nabla \varphi_v = \nabla \alpha_v \times \nabla \beta_v$, where φ_v designates the scalar magnetic potential for $\mathbf{B}_v$. In terms of the coordinates, $(\beta_v, \varphi_v, \alpha_v)$, the exterior field has the form, $\mathbf{B}_e = \nabla \Phi_e = (\partial \Phi_e / \partial \beta_v) \nabla \beta_v + (\partial \Phi_e / \partial \varphi_v) \nabla \varphi_v + (\partial \Phi_e / \partial \alpha_v) \nabla \alpha_v$. Imposing the condition, $(\nabla \alpha_v) \cdot \mathbf{B}_e = 0$ on S yields an expression for $(\partial \Phi_e / \partial \alpha_v)$; $(\partial \Phi_e / \partial \alpha_v) = -|\nabla \alpha_v|^{-2} (\nabla \alpha_v \cdot \nabla \beta_v)(\partial \Phi_e / \partial \beta_v)$. With this result, pressure balance across S acquires the form,

$$p_1^2 + |\nabla \alpha_v|^{-2} p_2^2 - 2P \, B_v^{-2} = \sigma^2, \tag{1}$$

where p_1, and p_2 are respectively the derivatives, $(\partial \Phi_e / \partial \beta_v)$ and $(\partial \Phi_e / \partial \varphi_v)$. Equation (1) is a first order, nonlinear, partial differential equation for the magnetic potential, $\Phi_e(\beta_v, \varphi_v)$, on the surface S, which is labeled by some value of α_v. The characteristics of this equation are the lines of force of $\mathbf{B}_e$ on S. Designating the left-hand side of Eq.

(1) as $H(p_1, p_2, q_1, q_2)$, where q_1 and q_2 are respectively the coordinates β_v and φ_v, the characteristic lines, represented parametrically as $q_i = q_i(\tau)$ for $i = 1$ and 2, satisfy a two-degree of freedom, autonomous, Hamiltonian system,

$$\frac{dp_i}{d\tau} = -\frac{\partial H}{\partial q_i}, \quad \frac{dq_i}{d\tau} = \frac{\partial H}{\partial p_i}, \tag{2}$$

which shows that q_i and p_i are conjugate variables. Because of this connection to Hamiltonian theory, canonical transformations and renormalization methods are useful in order to solve the system for p_1 and p_2 that are single-valued over S. Another application of Eq. (1) is to plasma configurations with steep but continuous pressure profiles. The only required modification of Eq. (1) is that P is replaced by the pressure differences between a point in the profile and the plasma center. The resulting expression and the equation, $\nabla \cdot \mathbf{B} = 0$, determine $\mathbf{B}$ in the profile.