A four-field model for tokamak plasma dynamics

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A generalization of reduced magnetohydrodynamics is constructed from moments of the Fokker–Planck equation. The new model uses familiar aspect-ratio approximations but allows for (i) evolution as slow as the diamagnetic drift frequency, thereby including certain finite Larmor radius effects, (ii) pressure gradient terms in a generalized Ohm's law, thus making accessible the adiabatic electron limit, and (iii) plasma compressibility, including the divergence of both parallel and perpendicular flows. The system is isothermal and surprisingly simple, involving only one additional field variable, i.e., four independent fields replace the three fields of reduced magnetohydrodynamics. It possesses a conserved energy. The model's equilibrium limit is shown to reproduce not only the large-aspect-ratio Grad–Shafranov equation, but also such finite Larmor radius effects as the equilibrium ion parallel flow. Its linearized version reproduces, among other things, crucial physics of the long mean-free-path electron response. Nonlinearly, the four-field model is shown to describe diffusion in stochastic magnetic fields with good qualitative accuracy.

I. INTRODUCTION

The success of reduced magnetohydrodynamics (RMHD)[1-3] in simulating the nonlinear dynamics of tokamak plasmas is well documented.[4] Yet it is clear that certain RMHD approximations, such as the neglect of finite Larmor radius (FLR) terms, are not always realistic. Especially as tokamak experiments enter higher-temperature regimes, various aspects of the model can appear less convincing—a circumstance that has stimulated the development of numerous extended and generalized versions.[5-8]

It should be emphasized that the weaknesses of RMHD are most apparent in its linear limit, where comparison to more elaborate (e.g., kinetic) treatments is straightforward. But criticism based strictly on the linear predictions of RMHD is not appropriate. Observed plasma behavior is rarely linear, and the model is specifically intended to study nonlinear processes. Thus any reduced fluid model is to be judged by the degree to which it has captured the dominant nonlinear physics. RMHD is a preeminent success by this criterion; an alternative model is interesting insofar as it includes distinct physical processes of plausible importance in the nonlinear regime.

The present generalization of RMHD addresses the following specific considerations.

A. Low-frequency evolution

The characteristic MHD flow velocities are

$$V_{\rm MHD} \sim v_{{\rm th}i} \quad \text{or} \quad v_{\rm A}, \tag{1}$$

where $v_{\rm A}$ is the Alfvén speed and $v_{{\rm th}i} \equiv (2T_i/m_i)^{1/2}$ is the ion thermal velocity. The corresponding MHD frequencies are

$$\omega_{\rm MHD} \sim k v_{{\rm th}i} \quad \text{or} \quad k v_{\rm A}, \tag{2}$$

where k represents a typical wavenumber. Equations (1) and (2) are pertinent under certain conditions, such as those characterizing the final phase of a tokamak plasma disruption. However, the prevalent disturbances of interest evolve more slowly. Specifically, if we assume

$$k\rho_i < 1, \tag{3}$$

where ρ_i is the ion Larmor radius,

$$\rho_i \equiv v_{{\rm th}i}/\Omega_i, \quad \Omega_i \equiv eB/m_i c, \tag{4}$$

then typical speeds and frequencies of interest are estimated by

$$V \sim (k\rho_i) v_{{\rm th}i} < V_{\rm MHD}, \quad \omega \sim kV < \omega_{\rm MHD}. \tag{5}$$

The orderings, (5), are appropriate in particular when

$$e\phi/T \sim 1 \tag{6}$$

and

$$\omega \sim \omega_*, \tag{7}$$

where ϕ is the electrostatic potential and ω_* is the diamagnetic drift frequency. Of course ω_* terms importantly affect a large variety of linear disturbances; they are missing from RMHD because of Eqs. (1) and (2). The inclusion of drift effects, in a *nonlinearly* consistent manner, is a major objective of the present work.

B. Long mean-free-path electron response

Dissipation enters RMHD through the parallel Ohm's law,

$$J_\parallel = \sigma_S E_\parallel, \tag{8}$$

where $J_\parallel$ and $E_\parallel$ are, respectively, the parallel components of the current density and electric field ("parallel" means in the direction of the confining magnetic field) and σ_S is the conductivity of Spitzer and Harm.[9] It has long been recognized that Eq. (8) cannot be justified under experimental conditions of present interest. The "dc" conductivity pertains only to processes that are collision-dominated in both the temporal and spatial senses:

$$\omega \ll \nu, \tag{9}$$

$$k_\parallel \lambda \ll 1, \tag{10}$$

where ν is the electron collision frequency, $k_\parallel$ is the parallel wavenumber, and λ is the collisional mean free path. The most important disturbances often satisfy Eq. (9), but they are consistent with Eq. (10) only in a very narrow neighborhood of the mode-rational magnetic surface. Furthermore, even when Eqs. (9) and (10) are satisfied, the linear response is not described by Eq. (8) unless $\omega\nu \gg (k_\parallel v_{\rm the})^2$. Thus the long mean-free-path "ac" conductivity, $\sigma_*(\omega,k_\parallel,\nu)$, is typically more pertinent than σ_S. The inclusion of long mean-free-path effects can dramatically alter both linear[10] and nonlinear evolution. In particular, the model derived here provides a good qualitative description of electron diffusion in a stochastic magnetic field (see Sec. IV).

Nonlinearly, one cannot expect any simple Ohm's law to be valid: one must generalize the *form* of Eq. (8), not just the coefficient. But the crucial physical processes that determine σ_*—electron inertia at large ω and electron adiabaticity at large $k_\parallel$—are amenable to straightforward nonlinear generalization.

C. Compressibility

The RMHD ordering scheme leads to an effectively incompressible plasma flow, even at high beta. Such simplification is internally consistent, but it omits physical effects of proven qualitative importance. The model derived here includes (parallel, perpendicular, and diffusive) compressibility terms in nonlinear form.

These comments summarize the weaknesses of RMHD that our generalized model is intended to remedy. Of course the new model has weaknesses as well. We believe the following omissions, which also pertain to RMHD, are the most serious.

(*i*) *Temperature gradients.* We assume both plasma species to be isothermal, despite the fact that several linear instabilities, such as tearing modes, can be critically affected by temperature variation.[10–12] A nonlinear model that emphasizes electron temperature gradient effects (with cold ions) has been presented previously.[8]

(*ii*) *True kinetic effects.* Here the qualifier refers to the fact that our model *does* include crucial features of the electron response that are often labeled kinetic. For example, the gross dependence of σ_* on $k_\parallel$ and ω is usually obtained from Vlasov–Krook theory, but it results here from the linear limit[13] of a fluid description that is qualitatively accurate even for $k_\parallel \lambda \gg 1$.[7] Examples of "true" kinetic effects are Landau damping and magnetic trapping of nearly collisionless electrons; the latter seems by far the more important in parameter regimes of experimental interest.

(*iii*) *Higher-order finite-Larmor-radius* (*FLR*) *effects.* Our model assumes the gyroradius to be small, in the sense of Eq. (3). Although $k\rho_i$ is not an explicit ordering parameter, the procedures adopted leave out a number of $(k\rho_i)^2$ terms. (Other FLR terms are retained.) Note that for some disturbances of interest Eq.(3) is at best marginally satisfied.

While fluid truncation may be problematic in the strong FLR case, $k\rho_i \sim 1$, we see no fundamental difficulty in retaining temperature gradients or trapped particles in a nonlinear fluid description. For example, trapping can be manifested through anisotropy in the stress tensor of the Chew–Goldberger–Low[14] form. Thus the omissions (i) and (ii) are primarily intended to simplify the final result, rather than to expedite the derivation. The point is that any fluid model represents a compromise between simplicity and realism. A prominent virtue of the compromise represented by (i)–(iii) is that it yields a surprisingly general model that nonetheless contains only one independent field component not already present in (high-beta) RMHD. In other words we present a four-field model, generalizing the three-field model of RMHD.

The derivation of the four-field model is the subject of Sec. II. It is based on the familiar large aspect ratio ordering, but begins with exact moments of the Boltzmann equation rather than MHD equations. The stress tensor (for each plasma species) is truncated in a manner consistent with the three simplifications listed above; it is the sum of a scalar pressure and the gyroviscosity (or magnetoviscosity) tensor. One then finds the lowest-order perpendicular flow velocity to be given by the sum of $\mathbf{E}\times\mathbf{B}$ and diamagnetic drifts. It is well known that this flow, together with the gyroviscosity tensor, yields the correct ω_* terms in the linear limit; its nonlinear manifestation is rather more complicated.

The four-field system of equations is summarized at the end of Sec. II. Significantly, all its nonlinearities are quadratic and expressible in terms of conventional Poisson brackets. As in RMHD, the ordering parameter does not appear in the final system, all terms of which are formally comparable. New (non-RMHD) physics appears in conjuction with two parameters, β and δ, which roughly measure compressibility and FLR effects, respectively. Thus RMHD is recovered in the limit $\beta = \delta \to 0$.

Finite compressibility couples pressure evolution to, in particular, the parallel plasma flow, thus requiring the fourth field. The compressibility terms also provide qualitative agreement between the four-field model and nonreduced MHD theory; for example, the critical value of Δ', in both its linear[15] and nonlinear[16] contexts, is reproduced.

Finite δ is responsible for various diamagnetic effects, as well as the corrections to Ohm's law. The corrections provide, in particular, a description of the adiabatic (Maxwell–Boltzmann) limit of the electron response.

The dissipationless version of the four-field model is shown to conserve energy, although the form of the energy functional differs from that of conventional fluid theory.

The four-field equilibrium, studied in Sec. III, is characterized by the same Pfirsch–Schlüter return currents and approximate Grad–Shafranov equation found in RMHD. It also contains equilibrium return flows in the parallel mass flux, consistent with those emphasized in neoclassical transport theory, but omitted by RMHD.

Section III also considers some linear consequences of the model. As an example, we calculate the four-field version of the generalized conductivity, σ_*, which is shown to include crucial features of the corresponding kinetic result.

A nonlinear application of the model is presented in Sec. IV. We consider, using the direct interaction approximation, the diffusion of particles in a stochastic magnetic field. The results for both the collisional and collisionless limits are in qualitative accord with those of previous theory.

II. DERIVATION

A. Normalized variables

Aspect-ratio reduction assumes that

$$\epsilon \equiv a/R_0 \ll 1, \tag{11}$$

where a is a characteristic scale length in the poloidal plane and R_0 is the major radius of the magnetic axis. In other words, scale lengths in the toroidal direction, parallel to the dominant component of the confining magnetic field, are assumed to be much longer than those in the radial or poloidal directions. Small ϵ is exploited not only in the obvious geometrical sense—to simplify various differential operators—but also dynamically: the dependent field variables, as well as time, are scaled with ϵ. After expressing the dynamical equations in terms of the scaled fields and coordinates, one obtains the reduced system by neglecting terms of order ϵ^3. Because this reduction scheme has been discussed in previous literature,[3,7] the present treatment will be brief.

It is convenient to use dimensionless variables. For spatial coordinates, we choose

$$x \equiv (R - R_0)/a, \quad y \equiv Z/a, \quad z \equiv -\zeta,$$

where (R,ζ,Z) are cylindrical coordinates, with ζ being the toroidal angle and Z being measured along the "vertical" symmetry axis. Note that the normalized coordinates (x,y,z) form a right-handed triplet. For any scalar S, we find, from the conventional cylindrical gradient,

$$a\nabla S = \hat{x}\frac{\partial S}{\partial x} + \hat{y}\frac{\partial S}{\partial y} + \hat{z}\left(\frac{\epsilon}{(1+\epsilon x)}\right)\frac{\partial S}{\partial z}. \tag{12}$$

For an arbitrary vector $\mathbf{A}$, we write

$$\mathbf{A} = \mathbf{A}_\perp + \hat{z}A_z, \quad \mathbf{A}_\perp \equiv \hat{x}A_x + \hat{y}A_y$$

and express the divergence and curl as

$$a\nabla\cdot\mathbf{A} = \nabla_\perp\cdot\mathbf{A}_\perp + \left(\frac{\epsilon}{(1+\epsilon x)}\right)\left(\frac{\partial A_z}{\partial z} + A_x\right), \tag{13}$$

$$a\nabla\times\mathbf{A} = -\hat{z}\times\nabla_\perp A_z + \hat{z}(\hat{z}\cdot\nabla_\perp\times\mathbf{A}_\perp) + \left(\frac{\epsilon}{(1+\epsilon x)}\right)\left[-\hat{x}\frac{\partial A_y}{\partial z} + \hat{y}\left(\frac{\partial A_x}{\partial z} - A_z\right)\right]. \tag{14}$$

Note that the dimensionless gradient $\nabla_\perp$ is computed as if the unit vectors were Cartesian; the curvature terms arising from spatial variation of $\hat{x}$ and $\hat{z}$ have been made explicit. Thus

$$\nabla_\perp\cdot\mathbf{A}_\perp = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y},$$

$$\hat{z}\cdot\nabla_\perp\times\mathbf{A}_\perp = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}.$$

The magnetic field is decomposed into a vacuum contribution,

$$\mathbf{B}_{\text{vacuum}} = \hat{z}B_T/(1+\epsilon x),$$

where B_T is a constant measuring the toroidal field, and a contribution from plasma currents, $\mathbf{B}_{\text{plasma}} = \nabla\times\mathbf{A}_{\text{plasma}}$. It is characteristic of large aspect ratio tokamak confinement that the vacuum toroidal field dominates all other components of $\mathbf{B}$; we therefore assume

$$\mathbf{A}_{\text{plasma}} = O(\epsilon) \tag{15}$$

and find, from Eq. (14), that the magnetic field can be expressed as

$$\mathbf{B}/B_T = \hat{z}(1+\epsilon x)^{-1} + \epsilon\hat{z}b - \epsilon\hat{z}\times\nabla_\perp\psi + O(\epsilon^2). \tag{16}$$

Here

$$\psi = (\epsilon B_T a)^{-1}A_z, \quad A_z = \hat{z}\cdot\mathbf{A}_{\text{plasma}} \tag{17}$$

measures the poloidal flux while b, coming from $\hat{z}\cdot\nabla\times\mathbf{A}_{\text{plasma}}$, allows for diamagnetic corrections to $\mathbf{B}$. Equations (12) and (16) imply that, for any S,

$$(a/B_T)\mathbf{B}\cdot\nabla S = \epsilon\nabla_\parallel S + O(\epsilon^2), \tag{18}$$

where

$$\nabla_\parallel S \equiv \frac{\partial S}{\partial z} - \hat{z}\times\nabla_\perp\psi\cdot\nabla S \tag{19}$$

is the normalized, nonlinear parallel gradient.

We use B_T, together with a constant measure of the plasma density, n_c, to define the Alfvén speed,

$$v_{\text{A}}^2 = B_T^2/(4\pi m_i n_c),$$

and the Alfvén time,

$$\tau_{\text{A}} = a/v_{\text{A}}.$$

Since a represents a transverse dimension, it is clear that τ_{A} measures the compressional Alfvén time; the time scale of interest corresponds to shear-Alfvén motion, which is slower by a factor of roughly $k_\parallel/k = O(\epsilon)$. We therefore define the normalized time variable τ by

$$\tau = \epsilon t/\tau_{\text{A}}. \tag{20}$$

Consider next the electric field,

$$\mathbf{E} = -\nabla\phi - c^{-1}\frac{\partial\mathbf{A}}{\partial t}. \tag{21}$$

The normalized electrostatic potential is defined by

$$\varphi = c\phi/(\epsilon v_{\text{A}}B_T a), \tag{22}$$

whence

$$\mathbf{E}_\perp = -\epsilon(v_{\text{A}}/c)B_T\nabla_\perp\varphi + O(\epsilon^2) \tag{23}$$

and

$$E_\parallel = -\epsilon^2\frac{v_{\text{A}}}{c}B_T\left(\nabla_\parallel\varphi + \frac{\partial\psi}{\partial\tau}\right) + O(\epsilon^3) \tag{24}$$

in view of Eqs. (17), (18), and (20). Notice that while $\mathbf{E}$ is dominated by the electrostatic field, the electrostatic and electromagnetic terms make comparable contributions to $E_\parallel$.

The fields φ and ψ are defined above in a manner equivalent to RMHD. We depart from RMHD in our choices for the remaining two field variables, p and v. Let n be the plasma density, $V_\parallel$ the plasma (ion) parallel flow velocity, and

$$\beta_e = 8\pi n_c T_e/B_T^2, \tag{25}$$

where T_e is the constant electron temperature. Then we define

$$p = (\beta_e/\epsilon)(n/n_c - 1), \tag{26}$$

$$v = (\epsilon v_{\text{A}})^{-1}V_\parallel. \tag{27}$$

Equation (26) differs from the RMHD version essentially because of the constant term, the purpose of which will become clear presently. The variable v does not appear in RMHD.

Finally we take note of certain dimensionless parameters that enter the four-field model. These are β_e, defined by Eq. (25), the normalized resistivity,

$$\eta = \tau_A / \epsilon \tau_S, \tag{28}$$

and the gyroradius parameter,

$$\delta = (2\Omega \tau_A)^{-1}. \tag{29}$$

Here τ_S is the usual skin time,

$$\tau_S = 4\pi a^2 \sigma_S / c^2,$$

and Ω is a constant measure of the ion gyrofrequency,

$$\Omega = eB_T / m_i c.$$

While the parameter η is standard, it may not be obvious that δ measures FLR effects. Note in particular that the definitions of τ_A and Ω imply

$$\delta = c/(2\omega_{pi} a), \tag{30}$$

where ω_{pi} is the ion plasma frequency; this quantity is hardly proportional to $k\rho_i$. In fact the gyroradius is measured by the product of δ and β_e. One finds that

$$\omega_* \tau_A \sim \beta_e \delta, \tag{31}$$

$$\rho_i^2 / a^2 \sim (T_i/T_e) \beta_e \delta^2. \tag{32}$$

B. Equations of motion

The $m\mathbf{vv}$ moment of the Boltzmann equation can be written as

$$mn \frac{\partial \mathbf{V}}{\partial t} + mn \mathbf{V} \cdot \nabla \mathbf{V} + \nabla \cdot \mathsf{P} - en(\mathbf{E} + c^{-1} \mathbf{V} \times \mathbf{B}) = \mathbf{F}, \tag{33}$$

where P is the pressure tensor and $\mathbf{F}$ is the collisional friction force. A species subscript has been suppressed for simplicity. Our basic closure assumption is an ansätz for the form of P:

$$\mathsf{P} = \mathsf{I} P + \mathbf{\Pi}, \tag{34}$$

where $P = nT$ is the scalar pressure and $\mathbf{\Pi}$ denotes the gyroviscosity tensor. The explicit form of $\mathbf{\Pi}$ will not be needed[17]; it is estimated by

$$\Pi_s \sim (P_s / \Omega_s) \nabla V_s, \tag{35}$$

where $s = i,e$ is the species label. As noted in Sec. I, Eq. (34) omits various kinetic processes and cannot be justified in general.

Regarding the friction force, it is consistent with isothermality and Eq. (34) to write, for the electrons,[17]

$$\mathbf{F}_e = en\eta_S \mathbf{J}, \tag{36}$$

where $\eta_S = \sigma_S^{-1}$ and

$$\mathbf{J} = (c/4\pi) \nabla \times \mathbf{B} \tag{37}$$

is the plasma current density. For simplicity, no distinction is made between parallel and perpendicular resistivities. Collisional momentum conservation provides the ion friction force,

$$\mathbf{F}_i = -\mathbf{F}_e. \tag{38}$$

As a first application of Eq. (33), we solve its ion version for the perpendicular flow. Assuming that

$$P \sim \epsilon, \quad \mathbf{V} \sim \epsilon, \tag{39}$$

as in RMHD, we see that the inertial and gyroviscous terms are $O(\epsilon^2)$. Furthermore the resistive term, although not explicitly ordered in ϵ, is very small. Hence, after computing the cross product of Eq. (33) with $\mathbf{B}$, we find

$$\mathbf{V} = \mathbf{V}_\parallel + \mathbf{V}_E + \mathbf{V}_D + O(\epsilon^2) + O(\eta), \tag{40}$$

with

$$\mathbf{V}_E = c\mathbf{E} \times \mathbf{B}/B^2, \quad \mathbf{V}_{Ds} = (nm_s \Omega_s B)^{-1} \mathbf{B} \times \nabla P_s. \tag{41}$$

In principle, the next step is to compute a more accurate equation of motion, including $O(\epsilon^2)$ terms, by iteratively substituting Eq. (40) into both Eq. (33) and the exact version of Eq. (34). Fortunately, this rather complicated step has been accomplished previously by, among others, Stringer[18] and Hinton and Horton.[19] (Details of the calculation are presented in Ref. 20.) The main point is that most of the gyroviscous terms cancel with terms from $d\mathbf{V}_D/dt$, leaving

$$\begin{aligned} m_s n \Big[\Big(\frac{\partial}{\partial t} + \mathbf{V}_s \cdot \nabla \Big) \mathbf{V}_E + \Big(\frac{\partial}{\partial t} + \mathbf{V}_E \cdot \nabla \Big) \mathbf{V}_{\parallel s} \Big] \\ + \nabla \{ P_s [1 - (2\Omega_s B)^{-1} \mathbf{B} \cdot \nabla \times \mathbf{V}_s] \} \\ - e_s n (\mathbf{E} + c^{-1} \mathbf{V}_s \times \mathbf{B}) = \mathbf{F}_s + O(\epsilon^3). \end{aligned} \tag{42}$$

Note in particular that, as noted by Mikhailovskii,[21] the parallel flow is advected only by $\mathbf{V}_E$. The FLR pressure correction, involving $\nabla \times \mathbf{V}$, will not enter our final results, for two reasons: because it enters in order ϵ^2, it cannot affect lowest-order force balance; and because it occurs inside a gradient, it can't affect vorticity evolution. [We also point out that the coefficient of this correction is, strictly speaking, anisotropic;[22] the form shown in Eq. (42) pertains for the perpendicular components of ∇P.] We have used quasineutrality to omit the species subscript on n.

The species sum of Eq. (42) can be expressed as

$$\begin{aligned} m_i n \Big[\Big(\frac{\partial}{\partial t} + \mathbf{V}_i \cdot \nabla \Big) \mathbf{V}_E + \Big(\frac{\partial}{\partial t} + \mathbf{V}_E \cdot \nabla \Big) \mathbf{V}_{\parallel i} \Big] \\ + \nabla P [1 - (2\Omega_i B)^{-1} \mathbf{B} \cdot \nabla \times \mathbf{V}_i] = c^{-1} \mathbf{J} \times \mathbf{B}. \end{aligned} \tag{43}$$

Here we omitted several $O(m_e/m_i)$ terms, and used Eq. (38). Of course P denotes the total scalar pressure:

$$P = nT_e (1 + T_i/T_e). \tag{44}$$

Our first application of Eq. (43) is familiar from RMHD. After writing the equation of motion in terms of our normalized variables, we identify the $O(\epsilon)$ terms, coming from ∇P and the lowest-order contribution to $\mathbf{J} \times \mathbf{B}$. Equations (14) and (37) show that the latter involve only the field magnitude perturbation, b; thus we obtain

$$\nabla_\perp b = -\tfrac{1}{2} (1 + T_i/T_e) \nabla_\perp p, \tag{45}$$

the familiar consequence of lowest-order equilibration.

Our second application of Eq. (43) involves the parallel terms. After taking the dot product of the equation with $\mathbf{B}$, inserting the normalized variables, and neglecting $O(\epsilon^3)$, we find

$$\frac{\partial v}{\partial \tau} + [\varphi, v] + \frac{1}{2} \Big(1 + \frac{T_i}{T_e} \Big) \nabla_\parallel p = 0, \tag{46}$$

an unsurprising result for the evolution of the parallel flow.

Equation (46) uses the conventional Poisson bracket notation,

$$[f,g] \equiv \hat{z}\cdot\nabla_\perp f\times\nabla_\perp g, \tag{47}$$

for any functions f and g. Thus $[\varphi,f]$ is the reduced expression for $\mathbf{V}_E\cdot\nabla f$. Similarly, it can be seen from Eq. (19) that

$$\nabla_\parallel f = \frac{\partial f}{\partial z} - [\psi,f]. \tag{48}$$

Our final application of Eq. (43) is the derivation, from the parallel component of its curl, of the shear-Alfvén law. Considering first the right-hand side, we find that[3]

$$c^{-1}\mathbf{B}\cdot\nabla\times(\mathbf{J}\times\mathbf{B}) = -\epsilon^2(B_T^3/4\pi a)(2[b,x] + \nabla_\parallel\nabla_\perp^2\psi) + O(\epsilon^3). \tag{49}$$

Here the term involving ψ is the line-bending term, coming from $\nabla_\parallel J_\parallel$; we emphasize this by means of the abbreviation

$$J \equiv \nabla_\perp^2\psi. \tag{50}$$

(More strictly, J is proportional to the negative of the parallel current.) The term involving b can be rewritten as

$$2[b,x] = (1 + T_i/T_e)[x,p],$$

in view of Eq. (45). It is thus recognized as an interchange term, resulting from the interaction of pressure gradients with toroidal field curvature.

Consider next the left-hand side of Eq. (43). We write $\mathbf{V}_E = \epsilon v_A \mathbf{u}_E$, $\mathbf{V}_{Ds} = \epsilon v_A \mathbf{u}_{Ds}$, and $\mathbf{u} = \mathbf{u}_E + \mathbf{u}_{Ds}$, noticing from Eqs. (23), (26), and (41) that

$$\mathbf{u}_E = \hat{z}\times\nabla_\perp\varphi + O(\epsilon), \tag{51}$$

$$\mathbf{u}_{Di} = \delta(T_i/T_e)(\beta_e/\epsilon)\hat{z}\times\nabla_\perp \ln(1 + \epsilon p/\beta_e) + O(\epsilon). \tag{52}$$

The transcendental nonlinearity in Eq. (52) compels us at this point to depart, in a conventional manner,[7,23] from our strict ϵ ordering: we assume

$$\epsilon p/\beta_e < 1 \tag{53}$$

and retain only the linear term in Eq. (52). Thus

$$\mathbf{u}_{Di} = \delta(T_i/T_e)\hat{z}\times\nabla_\perp p + O(\epsilon)$$

and

$$\mathbf{u} = \hat{z}\times\nabla_\perp[\varphi + \delta(T_i/T_e)p]. \tag{54}$$

An approximation equivalent to Eq. (53) is also made in RMHD, with the motivation of retaining only quadratic nonlinearities. It can be justified by the assumption that density *gradients* are rather small, i.e., that $n \approx n_c$ in Eq. (26).

The reduced shear-Alfvén law is obtained by substituting Eqs. (45), (49), and (54) into Eq. (43):

$$\hat{z}\cdot\nabla_\perp\times\left[\frac{n}{n_c}\left(\frac{\partial\mathbf{u}_E}{\partial\tau} + \mathbf{u}\cdot\nabla\mathbf{u}_E\right)\right] = -\nabla_\parallel J - \left(1 + \frac{T_i}{T_e}\right)[x,p]. \tag{55}$$

Here the left-hand side can be written more explicitly by noting that

$$\hat{z}\cdot\nabla_\perp\times\mathbf{u}_E = \nabla_\perp^2\varphi \equiv U, \tag{56}$$

and using the identity

$$\hat{z}\cdot\nabla\times[\mathbf{u}\cdot\nabla_\perp\mathbf{u}_E] = \delta(T_i/2T_e)\{[p,U] + [\varphi,\nabla_\perp^2 p] + \nabla_\perp^2[p,\varphi]\} + [\varphi,U],$$

which can be straightforwardly verified from Eq. (54) and the Cartesian nature of ∇. We also use Eq. (53) to replace the n/n_c factor by unity, as in RMHD. Then Eq. (55) becomes

$$\frac{\partial U}{\partial\tau} + [\varphi,U] + \nabla_\parallel J + (1 + T_i/T_e)[x,p] = -\delta(T_i/2T_e)\{[p,U] + [\varphi,\nabla_\perp^2 p] + \nabla_\perp^2[p,\varphi]\}. \tag{57a}$$

An alternative version is

$$\frac{\partial U}{\partial\tau} + \left[\varphi + \delta\left(\frac{T_i}{T_e}\right)p, U\right] + \nabla_\parallel J + \left(1 + \frac{T_i}{T_e}\right)[x,p] = \delta(T_i/T_e[\nabla_\perp\varphi;\nabla_\perp p], \tag{57b}$$

where the vector bracket is defined by

$$[\mathbf{A};\mathbf{B}] = \sum_j [A_j,B_j]. \tag{58a}$$

It is not hard to verify that

$$2[\nabla_\perp\varphi;\nabla_\perp p] = [p,U] - [\varphi,\nabla_\perp^2 p] - \nabla_\perp^2[p,\varphi]. \tag{58b}$$

Hence Eqs. (57a) and (57b) are equivalent.

Equation (57b) is easily understood: the second term on the left-hand side shows that U is advected by the total drift $\mathbf{u}$ as in Eq. (43) [recall also Eq. (54)]; the term on the right-hand side reflects spatial variation of the diamagnetic drift, which obviously enters the curl.

Equations (46) and (57) describe evolution of the ion flow. Only the parallel dynamics of the electrons is needed. Thus we return to Eq. (42), set $s = e$, and take the dot product with $\mathbf{B}$. For the parallel electron speed we deduce from Eq. (27) and $\mathbf{J} = en(\mathbf{V}_i - \mathbf{V}_e)$ that

$$V_{\parallel e} = \epsilon v_A(v + 2\delta J) + O(\epsilon^2), \tag{59}$$

and for the parallel electric field we use Eq. (24). Then, after straightforward normalization and use of Eq. (53), we find

$$\left(\frac{2\delta m_e}{m_i}\right)\left(\frac{\partial}{\partial\tau} + \mathbf{u}_E\cdot\nabla_\perp\right)(v + 2\delta J) = \nabla_\parallel\varphi + \frac{\partial\psi}{\partial\tau} - \delta\nabla_\parallel p - \eta J.$$

Of course the left-hand side of this relation, describing electron inertia, is very small, and it is consistent with our approximations elsewhere to neglect it. This yields the generalized, reduced Ohm's law,

$$\frac{\partial\psi}{\partial\tau} + \nabla_\parallel\varphi = \eta J + \delta\nabla_\parallel p, \tag{60a}$$

which is most pertinent to tokamak experimental conditions. However, for the sake of a pedagogical application in Sec. III, we also take note of the form that retains electron inertia:

$$\frac{\partial\psi}{\partial\tau} + \nabla_\parallel\varphi = \eta J + \delta\nabla_\parallel p + 4\delta^2\left(\frac{m_e}{m_i}\right)\left(\frac{\partial J}{\partial\tau} + [\varphi,J]\right). \tag{60b}$$

Here the terms involving v have been computed from Eq. (46) and, because they merely add a $O(m_e/m_i)$ correction to the pressure, neglected.

C. Pressure evolution

In an isothermal plasma, pressure evolution is determined by density evolution, i.e., by the particle conservation law,

$$\frac{\partial n}{\partial t} + \mathbf{V}_e \cdot \nabla n = -n\nabla \cdot \mathbf{V}_e. \tag{61}$$

Our reason for using the electron velocity here will become clear presently. The electron version of Eq. (54) is

$$\mathbf{V}_e = \epsilon v_A \mathbf{u}_e, \quad \mathbf{u}_e = \hat{z}\times\nabla(\varphi - \delta p) + O(\epsilon), \tag{62}$$

so that straightforward normalization of Eq. (61) yields

$$\frac{\partial p}{\partial \tau} + [\varphi, p] = -\left(\frac{\beta_e}{\epsilon}\right) a\nabla \cdot \mathbf{u}_e. \tag{63}$$

It is clear from Eqs. (13) and (62) that the right-hand side of Eq. (63) is $O(\beta_e) = O(\epsilon)$ and therefore consistently neglected. In other words the strictly consistent four-field model would be, like RMHD, incompressible. Therefore, recognizing the special qualitative importance of compressibility, we depart at this point from the formal $\beta_e \sim \epsilon$ ordering: the $O(\beta_e)$ terms on the right-hand side of Eq. (63) are retained, while terms of explicitly higher order in ϵ are neglected as usual.

In order to compute the $O(\epsilon)$ contribution to $\nabla \cdot \mathbf{u}_e$ we need the unnormalized electron velocity $\mathbf{V}_e$ through $O(\epsilon^2)$. Because electron inertia is small, this quantity is easily determined:

$$\mathbf{V}_e = \mathbf{V}_{\|e} + \mathbf{V}_E + \mathbf{V}_{De} + \mathbf{V}_\eta + O(\epsilon^3) + O(m_e/m_i),$$

where the first term is given by Eq. (59), the second and third terms are given by Eqs. (41), and the last term,

$$\mathbf{V}_\eta \equiv (c/eB^2)\mathbf{B}\times\mathbf{F}_e = \epsilon v_A \eta B^{-2}[\mathbf{B}\times(\nabla\times\mathbf{B})],$$

corresponds to resistive diffusion. It is worth noting that an analysis based on the compressibility of the *ion* flow, although equivalent, is more complicated because ion inertia is not negligible in the order of interest.

Faraday's law provides a well-known expression for the divergence of the $\mathbf{E}\times\mathbf{B}$ drift,

$$\nabla \cdot \mathbf{V}_E = c\mathbf{E}\times\mathbf{B}\cdot\nabla B^{-2} - B^{-1}\frac{\partial B}{\partial t} - 4\pi B^{-2} J \cdot E,$$

which reduces to

$$\nabla \cdot \mathbf{V}_E = \left(\frac{\epsilon^2 v_A}{a}\right)\left(2[\varphi, x] - [\varphi, b] - \frac{\partial b}{\partial \tau}\right) + O(\epsilon^3). \tag{64}$$

Similarly, one finds

$$\nabla \cdot \mathbf{V}_{De} = (e^2 v_A/a)(-2\delta[p, x]) + O(\epsilon^3),$$

and, after computing the perpendicular current from Eqs. (14) and (16),

$$\nabla \cdot \mathbf{V}_\eta = (\epsilon^2 v_A/a)(\eta \nabla_\perp^2 b).$$

We next use Eq. (45) to eliminate b. Notice that the sum of the two terms involving b in Eq. (64) then becomes proportional to the left-hand side of Eq. (63). It follows that, after introducing an alternative measure of beta,

$$\beta \equiv [1 + \tfrac{1}{2}(1 + T_i/T_e)\beta_e]^{-1}\beta_e, \tag{65}$$

we can express the conservation law as

$$\frac{\partial p}{\partial \tau} + [\varphi, p] = \beta\{2[x, \varphi - \delta p] - \nabla_\|(v + 2\delta J) + \tfrac{1}{2}(1 + T_i/T_e)\eta\nabla_\perp^2 p\}. \tag{66}$$

The interpretation of this result should be clear. Considering the terms on the right-hand side, we note that those involving x describe perpendicular compressibility, resulting from the variation of B on a magnetic surface. (Only the vacuum variation, corresponding to $B \propto R^{-1}$, enters explicitly; the diamagnetic contribution simply changes β_e to β.) The terms involving v and J reflect parallel compressibility and the final term corresponds to resistive diffusion.

Equations (46), (57), (60), and (66) yield a closed system for the four fields φ, ψ, p, and v. This system accomplishes the goals outlined in the introduction, including drift effects, a physical description of the long mean-free-path electron response, and so on (see Sec. III). It is also reasonably simple. However, there is a deficiency—failure to conserve energy in the limit $\eta \to 0$—which we now proceed to remedy.

Energy conservation is considered explicitly in the following subsection. Here we note that any conservation law can be lost when terms of a certain order are retained in one context and neglected in others. Thus the problem is that Eqs. (46), (57), and (60) omit $O(\beta)$ terms while Eq. (66) does not. The omitted terms can be evaluated[22]; most of them are proportional, effectively, to $(\rho_i/a)^2$ and therefore represent effects beyond the scope of the present formalism. But there also occur $O(\beta)$ terms, in the parallel acceleration law, that are exceptional in involving relatively few spatial derivatives. These terms have been calculated from the gyroviscous tensor by C. T. Hsu[22]; an equivalent and somewhat simpler derivation uses the parallel-velocity moment of the ion drift-kinetic equation. One finds that the terms in question simply represent advection of the parallel flow by the curvature and grad-B drift velocities:

$$\int d\mathbf{v}\, v_\| \mathbf{v}_d \cdot \nabla f = n_c\left(\frac{\epsilon^2 v_A^2}{a}\right)\delta\beta_e\left(\frac{T_i}{T_e}\right)[v, b - 4x] + O(\epsilon^3), \tag{67}$$

where $v_\|$ is the parallel particle velocity, f is the ion distribution function, and $\mathbf{v}_d$ is the guiding-center drift caused by curvature and ∇B. After inserting the right-hand side of Eq. (67) into Eq. (46), eliminating b in the usual way, and using

$$\beta_e = \beta + O(\beta^2), \tag{68}$$

we obtain the modified parallel acceleration law,

$$\frac{\partial v}{\partial \tau} + [\varphi, v] + \frac{1}{2}\left(1 + \frac{T_i}{T_e}\right)\nabla_\| p = \delta\beta(T_i/T_e)\{\tfrac{1}{2}(1 + T_i/T_e)[p, v] + 4[x, v]\}. \tag{69}$$

We show in the next subsection that the system consisting of Eqs. (57), (60), (66), and (69) possesses an exact energy invariant when resistivity is neglected.

D. Summary and discussion

Before summarizing the four-field model we make one minor change in notation, renaming the quantity x by

$$h \equiv (R - R_0)/a. \tag{70}$$

The point is that (x,y,z) coordinates, while helpful in the derivation, are rarely useful in applications. Then we have the vorticity equation,

$$\frac{\partial U}{\partial \tau} + \left[\varphi + \delta \frac{T_i}{T_e} p, U\right] + \nabla_\parallel J + \left(1 + \frac{T_i}{T_e}\right)[h,p]$$

$$= \delta(T_i/T_e)[\nabla_\perp \varphi; \nabla_\perp p]; \tag{71}$$

the generalized Ohm's law,

$$\frac{\partial \psi}{\partial \tau} + \nabla_\parallel \varphi = \eta J + \delta \nabla_\parallel p; \tag{72}$$

the parallel acceleration law,

$$\frac{\partial v}{\partial \tau} + [\varphi, v] + \frac{1}{2}\left(1 + \frac{T_i}{T_e}\right)\nabla_\parallel p$$

$$= \partial\beta(T_i/T_e)\{\tfrac{1}{2}(1 + T_i/T_e)[p,v] + 4[h,v]\}; \tag{73}$$

amd the particle (or internal energy) conservation law,

$$\frac{\partial p}{\partial \tau} + [\varphi, p] = \beta\{2[h, \varphi - \delta p]$$

$$- \nabla_\parallel(v + 2\delta J) + \tfrac{1}{2}(1 + T_i/T_e)\eta \nabla_\perp^2 p\}. \tag{74}$$

The current and vorticity are defined by

$$J = \nabla_\perp^2 \psi, \quad U = \nabla_\perp^2 \varphi,$$

where $\nabla_\perp$ is a normalized gradient in the poloidal plane; the bracket is given by

$$[f,g] = \hat{z} \cdot \nabla_\perp f \times \nabla_\perp g;$$

[the vector bracket in Eq. (71) is defined in Eq. (58)] and the nonlinear parallel gradient is defined by

$$\nabla_\parallel f = \frac{\partial f}{\partial z} - [\psi, f].$$

The four independent field quantities ψ, φ, v, and p measure, respectively, the poloidal magnetic flux [Eq. (17)], the electrostatic potential [Eq. (22)], the parallel flow velocity [Eq. (26)], and the electron pressure [Eq. (27)]; Eqs. (25), (28)–(30), and (65) define the constant parameters β, δ, and η. The quantity h, whose gradient measures the vacuum field curvature, is given by Eq. (70). Notice that h is constant in time, and that

$$\nabla_\perp^2 h = 0. \tag{75}$$

Two limiting versions of the four-field model are of interest. First, when β and δ are both neglected, one obtains the high-beta RMHD equations of Strauss.[3] Notice that v evolves autonomously in this limit, so that RMHD involves only three coupled fields. A more interesting limit has $\delta = 0$ and $T_i = T_e$, but nonzero β; it describes compressible, reduced MHD (CRMHD).[23] The CRMHD equations are given by

$$\frac{\partial U}{\partial \tau} + [\varphi, U] + \nabla_\parallel J + 2[h,p] = 0, \tag{76}$$

$$\frac{\partial \psi}{\partial \tau} + \nabla_\parallel \varphi = \eta J, \tag{77}$$

$$\frac{\partial v}{\partial \tau} + [\varphi, v] + \nabla_\parallel p = 0, \tag{78}$$

$$\frac{\partial p}{\partial \tau} + [\varphi, p] = \beta\{2[h,\varphi] - \nabla_\parallel v + \eta \nabla_\perp^2 p\}. \tag{79}$$

Turning now to energy conservation, we take note of the well-known identities

$$\int d\mathbf{x}\, K[L,M] = \int d\mathbf{x}\, L[M,K] = \int d\mathbf{x}\, M[K,L] \tag{80}$$

for any functions $K(\mathbf{x})$, $L(\mathbf{x})$, and $M(\mathbf{x})$. Of course these relations depend upon the neglect of surface contributions (homogeneous boundary conditions). In the present case one also needs the identity

$$\int d\mathbf{x}\, h[K, \nabla_\perp^2 K] = 0, \tag{81}$$

where K is arbitrary and h is given by Eq. (70). This result can be seen to follow from Eqs. (58), (75), and (80), since ∇h is constant in lowest order. It is then straightforward to verify that the four-field energy,

$$H = \frac{1}{2}\int d\mathbf{x}\Big[\,|\nabla_\perp \varphi|^2 + |\nabla_\perp \psi|^2 + \frac{p^2}{2\beta}$$

$$- 2\frac{T_i}{T_e} hp - \frac{\psi v}{2\delta}$$

$$+ \left(\frac{1}{2(1 + T_i/T_e)} + \frac{T_i \beta}{4T_e}\right)v^2\Big], \tag{82}$$

is exactly conserved by the dissipationless ($\eta = 0$) version of Eqs. (71)–(74):

$$\frac{dH}{d\tau} = O(\eta). \tag{83}$$

The first two terms in H coincide with terms in the RMHD energy and the fourth term, involving hp, also has an RMHD counterpart.[3] The term proportional to p^2 is familiar from electrostatic theory,[7,24] and the occurrence of a v^2 term is unsurprising, although we have not interpreted its coefficient. The remaining term in H, involving ψv, is new. It resembles the Lagrangian interaction $(e/c)\mathbf{V} \cdot \mathbf{A}$, and in fact can be combined with part of the v^2 term to produce

$$\tfrac{1}{2}(v - \psi/2\delta)^2 \propto (m_i V_\parallel - eA_\parallel/c)^2,$$

the square of the canonical parallel momentum. [Here we use the fact that the spatial integral of any function of ψ alone is conserved, as easily follows from Eq. (72).] Thus Eq. (82) combines features of the fluid and particle energies.

It is noteworthy that the $(\beta \to 0, \delta \to 0)$ limit of the four-field energy is singular. Nonetheless, the relation between H and the conserved RMHD energy H_{RMHD} is quite simple:

$$H_{\mathrm{RMHD}} \equiv \frac{1}{2}\int d\mathbf{x}\left[\,|\nabla_\perp \psi|^2 + |\nabla_\perp \varphi|^2 - 2\left(1 + \frac{T_i}{T_e}\right)hp\right]$$

$$= H(v = 0) - (4\beta)^{-1}\int d\mathbf{x}(\mathrm{p} + 2\beta \mathrm{h})^2. \tag{84}$$

Finally we note that the conserved energy for CRMHD, Eqs. (76)–(79), is given by

$$H_{\mathrm{CRMHD}} = \frac{1}{2}\int d\mathbf{x}\left(|\nabla_\perp \psi|^2 + |\nabla_\perp \varphi|^2 + \frac{p^2}{\beta} + v^2\right). \tag{85}$$

III. EQUILIBRIUM AND LINEAR CONSEQUENCES

A. Four-field equilibrium

In this section we consider some relatively simple predictions of the four-field model, beginning with its description of the tokamak equilibrium. Thus, in Eqs. (71)–(74), we neglect time derivatives, neglect z derivatives (because of tokamak axisymmetry), and suppose for simplicity that $T_i = T_e, \varphi = 0$. Equation (73) then implies that $[\psi,p] = O(\delta\beta)$, or

$$p = p(\psi) + O(\delta\beta)$$

and Eq. (71),

$$[\psi,J] + 2[h,p] = 0, \tag{86}$$

is solved by the equilibrium current[25]

$$J = \nabla_\perp^2 \psi = \bar{J}(\psi) - 2\left(\frac{dp}{d\psi}\right) h + O(\delta\beta), \tag{87}$$

where $\bar{J}(\psi)$ is an integration constant. Equation (87) is the reduced Grad–Shafranov equation, which is also obtained in RMHD.

A more interesting result follows from equilibrium particle conservation. Equation (74) implies that

$$[\psi,v] + 2\delta([\psi,J] - [h,p]) = 0, \tag{88}$$

or, after use of Eq. (86),

$$[\psi,v] - 2\frac{dp}{d\psi}[\psi,h] = 0.$$

Hence

$$v = \bar{v}(\psi) + 2\delta h\frac{dp}{d\psi}. \tag{89}$$

Thus toroidal curvature produces an equilibrium parallel flow. The second term on the right-hand side of Eq. (89), like that in Eq. (87), is a return flow, balancing the divergence of the diamagnetic drift. Its existence is well known from neoclassical theory,[26] and in fact Eq. (89) is simply the normalized, large aspect ratio version of the neoclassical result. It is missing from RMHD because it depends upon finite gyroradius, as indicated by the factor δ. The first term on the right-hand side of Eq. (89) is an integration constant, determined by processes outside the four-field model (such as external momentum inputs).

Two other applications of the four-field equilibrium system are worth mentioning. First, one can use the equilibrium versions of Eqs. (73) and (74), including $O(\delta\beta)$ terms, to iteratively evaluate p for small $\delta\beta$. The resulting FLR corrections to the equilibrium pressure are not constant on flux surfaces. Second, by including equilibrium diffusion terms, as well as the electrostatic potential, one can compute the Pfirsch–Schlüter modification to equilibrium radial diffusion.[25] The result is standard, although of course it omits the correction due to temperature gradients.[26]

B. Linear electron response

The linear predictions of the four-field model are in good accord with those obtained from more elaborate theories, including kinetic treatments. For example, Eq. (71) reproduces the same (ω-ω_{*i}) factors, in the ion equation of motion, that are conventionally found from the ion kinetic equation—a circumstance that depends upon the inclusion of gyroviscosity. Similarly, the special case of CRMHD ($\delta = 0$) is easily applied to linear tearing mode theory in toroidal geometry. One finds the resistive interchange and modified tearing-mode stability criteria of Glasser, Greene, and Johnson,[15] with "Δ critical." Of course these results involve large aspect ratio and high-beta ($\beta \sim \epsilon$) approximations; the low-beta, near-axis Mercier criterion is inaccurately reproduced. The four-field dispersion relations for ballooning instability also agree well with known results, even when the latter include "kinetic" effects.[20]

Here we consider the four-field linearized electron response as an example with broad relevance in linear theory. We restrict attention to cylindrical geometry by setting $h = 0$ and using cylindrical coordinate (r,θ,z), with $x = r\cos\theta, x = r\sin\theta$. Notice that Eq. (47) implies

$$[f,g] = r^{-1}\left(\frac{\partial f}{\partial r}\frac{\partial g}{\partial \theta} - \frac{\partial g}{\partial r}\frac{\partial f}{\partial \theta}\right) \tag{90}$$

in this case.

Our notation distinguishes equilibrium quantities with a 0 subscript, while omitting the subscript on the linear perturbation: $p \rightarrow p_0 + p$, etc.. Because h is absent, all equilibrium fields depend on position only through r; radial derivatives are indicated by primes. Since perturbed quantities depend on φ only through the conventional $\exp(im\theta)$ factor, Eq. (90) implies, in particular,

$$[f_0,g] = ikf_0' g,$$

where $k = m/r$, with m the poloidal mode number and g the r-dependent amplitude of a linear perturbation.

It is easily seen that the reduced tokamak safety factor, q, is given by

$$\frac{1}{q} = -r^{-1}\frac{\partial\psi_0}{\partial r}.$$

Thus, from Eq. (19), the parallel gradient linearizes to yield

$$\nabla_\parallel f \rightarrow ik_\parallel f + ikf_0'\psi.$$

Here $k_\parallel \equiv n - m/q$, where n is the toroidal mode number. Finally, we write $\partial f/\partial\tau = -i\omega f$, defining the normalized mode frequency, and introduce the normalized drift frequencies,

$$\omega_{*e} = -\delta k p_0', \quad \omega_{*i} = -(T_i/T_e)\omega_{*e}.$$

These quantities are related to the conventional, unnormalized drift frequencies by the expected factor of τ_A/ϵ.

We now consider the linearized versions of Eqs. (72)–(74). For simplicity we let $T_i = T_e$, omit equilibrium current gradients, and neglect the resistive diffusion term in Eq. (74). Then we have

$$E \equiv -i\omega\psi + ik_\parallel\varphi = \eta J + i(k_\parallel\delta p - \omega_{*e}\psi), \tag{91}$$

$$\delta(\omega + \beta\omega_{*i})v = k_\parallel\delta p - \omega_{*e}\psi, \tag{92}$$

$$\omega\delta p - \omega_{*e}\varphi = \delta\beta k_\parallel(v + 2\delta J). \tag{93}$$

Note that E, in Eq. (91), is the *normalized* parallel electric field. The desired response function is the four-field conductivity σ_4 defined by

$$J = \sigma_4 E. \tag{94}$$

The existence of such a conductivity cannot be guaranteed *a priori*; in general, it might be necessary to allow for additional terms on the right-hand side of Eq. (94). [In fact, the inclusion of equilibrium current gradient terms in Eq. (93) would contradict Eq. (94).] Yet kinetic treatments—at least those that neglect non-Maxwellian contributions to the equilibrium distribution function—invariably yield such a generalized Ohm's law, even when temperature gradients are included.[10]

Equation (94) results in the four-field case because the parallel speed can be expressed as a linear combination of E and J. From Eqs. (92) and (93),

$$\delta(\omega + \beta\omega_{*_i})v = \omega^{-1}(1+\Delta_i)(-i\omega_{*_e}E + 2\delta^2\beta k_\parallel^2 J), \tag{95}$$

where we have introduced the abbreviation

$$1+\Delta_i = \left[1 - \beta k_\parallel^2/\omega(\omega+\beta\omega_{*_i})\right]^{-1}. \tag{96}$$

The physical interpretation of Δ_i will be considered presently. We next combine Eqs. (91) and (92) to write

$$E = \eta J + i\delta(\omega + \beta\omega_{*_i})v,$$

and use Eq. (95) to eliminate v from the right-hand side. The result is Eq. (94) with

$$\sigma_4 = \left[\omega - \omega_{*_e}(1+\Delta_i)\right]/\left[\eta\omega + 2i\delta^2\beta k_\parallel^2(1+\Delta_i)\right]. \tag{97}$$

The four-field conductivity is pertinent under prevailing tokamak experimental conditions. A mild generalization of Eq. (97), which has pedagogical interest, is obtained by keeping electron inertia, i.e., by using Eq. (60b) instead of (60a). Since J_0 is presumed constant, this modification simply replaces η by η_*, where

$$\eta_* = \eta - 4i\delta^2(m_e/m_i)\omega. \tag{98}$$

After making this replacement, it is convenient to undo our various normalizations, rewriting σ_4 in more conventional notation. The unnormalized conductivity is denoted by σ_*, and related to σ_4 by

$$\sigma_*(\omega,k_\parallel,\nu_e) = c^2\tau_A\sigma_4/(4\pi a^2),$$

as can be seen from Eq. (28). We also introduce the electron and ion thermal speeds, $v_s = (2T_s/m_s)^{1/2}(s = i,e)$, and the electron collision frequency ν_e which satisfies

$$\sigma_S = 2e^2n/m_e\nu_e.$$

Here σ_S is the Spitzer conductivity of Eq. (8); from Eq. (30) we notice that

$$4\pi a^2/(c^2\sigma_S) = 2(m_e/m_i)\delta^2\nu_e.$$

Thus one finds that Eq. (97) can be expressed as

$$\sigma_* = \sigma_S\left[1 - (\omega_{*_e}/\omega)(1+\Delta_i)\right]\left[1 - 2i(\omega/\nu_e) + i(k_\parallel v_e)^2(1+\Delta_i)/(\omega\nu_e)\right]^{-1}, \tag{99}$$

with

$$1+\Delta_i = \left[1 - (k_\parallel v_i/\omega)^2\right]^{-1}. \tag{100}$$

Here we used Eq. (68) to neglect $O(\beta^2)$ terms.

Because of the radial dependence of $k_\parallel$, which vanishes at the mode-rational surface, $\sigma_*(\omega,k_\parallel,\nu_e)$ is a strong function of radial position. Very close to the rational surface,

$$\sigma_* \approx \sigma_*(\omega,0,\nu_e) = \sigma_S\left[1 - (\omega_{*_e}/\omega)\right]/\left[1 - 2i(\omega/\nu_e)\right], \tag{101}$$

which reduces to σ_S when $\nu_e \gg \omega \gg \omega_*$, as it should. This is the limit described by RMHD. As noted in Sec. I, few disturbances are so localized as to have ω (or ν_e)$\gg k_\parallel v_e$ throughout their radial extent. Equation (101) is difficult to justifiy under typical experimental conditions.

Proceeding to larger values of $k_\parallel$, we next suppose that $\omega \gg k_\parallel v_i$, without constraining the size of $(\omega/k_\parallel v_e)$. In this regime, σ_* is given by the $\Delta_i = 0$ limit of Eq. (99), a formula first derived by Rutherford and Furth.[13] Its most striking feature is that σ_* becomes small,

$$\sigma_* \approx -2ie^2n(\omega - \omega_{*_e})(k_\parallel v_e)^{-2} \tag{102}$$

for $k_\parallel v_e \gg \omega \gg k_\parallel v_i$. It is not hard to trace this behavior to the δp and δJ terms in Eqs. (72) and (74); Eq. (102) is essentially an artifact of electron adiabaticity at large $k_\parallel v_e$ (Ref. 7).

Finally we suppose that $\omega \ll k_\parallel v_i$; when $\omega \sim \omega_*$, this ordering pertains at quite moderate distances from the rational surface. There σ_* achieves its asymptotic limit,

$$\sigma_* \to ie^2n/(m_i\omega), \quad \text{for } k_\parallel \to \infty. \tag{103}$$

Equations (99)–(103) may be compared to the corresponding results of small-gyroradius kinetic theory. The kinetic conductivity, σ_k, emphatically differs from the right-hand side of Eq. (97): usually expressed in terms of Z functions, σ_k is a transcendental function of ω, ν_e, and $k_\parallel$, with imaginary contributions corresponding to Landau damping.[10] Since a fluid description, involving truncated moments of the distribution function, necessarily misses microscopic resonance effects, this disagreement is not surprising. One would expect agreement only in the "cold" limit, $k_\parallel v_s \to 0$, where truncation is essentially exact. In fact σ_k does agree with σ_*, i.e., with Eq. (101), in this limit.

What is noteworthy is that the similarity between σ_k and σ_* extends beyond Eq. (101); the asymptotic limits of Eq. (102) and (103), in which $k_\parallel v_s$ is large, are also in agreement. Thus σ_* reproduces the gross spatial features of its more accurate kinetic counterpart, missing only the Landau-resonance structure near $\omega = k_\parallel v_s$. The verification of this statement is straightforward for the case of Eq. (102); one computes σ_k for stationary ions, and then considers the appropriate limit of the resulting Z function. The case of Eq. (103) is more complicated (and less frequently important), because the kinetic treatment must include not only ion parallel streaming but also perpendicular inertia, which enters kinetic theory through the polarization drift. [This fact is easily appreciated from Eq. (71), which shows that $\partial U/\partial\tau$ is comparable to $\nabla_\parallel J$.] Here we refer to the kinetic analysis of Coppi *et al.*,[11] which includes the full ion dynamics and agrees with Eq. (103) when particle conservation is taken into account.

The omission of Landau damping in σ_* is rarely important in applications. Instabilities of practical interest are often sensitive to the asymptotic structure of σ_*—its decay for increasing $k_\parallel$—but rarely sensitive to the details of wave–particle resonance. For example, when Eq. (99) is applied to the collisionless tearing mode,[27] the resulting dispersion relation agrees with that obtained from σ_k except for a minor difference in numerical coefficients. Similarly, σ_* gives a physical description of the semicollisional regime in the isothermal case.

Finally we point out that the four-field conductivity is not σ_*, since the four-field model, neglecting electron inertia, contains η rather than η_*. The difference is important only for small $\kappa_\|$, and only when $\omega \gg \nu_e$. It obviously rules out application of the strict four-field model to linear collisionless tearing modes. Fortunately the majority of observed tokamak disturbances have $\omega \sim \omega_* < \nu_e$.

IV. STOCHASTIC DIFFUSION

We show here that the four-field model contains sufficient physics to describe the diffusion of plasma in a stochastic magnetic field. Rechester and Rosenbluth[28] have computed the plasma diffusion in a given, static magnetic field when self-consistency and curvature are neglected. We show that similar results apply in the present model with similar simplifications. We do not attempt to provide a self-consistent description of the perturbed fields φ and ψ.

We will see that the crucial terms to keep in this analysis are the parallel pressure gradient in Ohm's law and the parallel compressibility term, proportional to δ, in the pressure evolution law. It is also instructive initially to keep the electron inertia in Ohm's law. Neglecting the self-consistent fields φ and $\partial\psi/\partial t$, and curvature effects, Eqs. (60b), (73), and (74) become

$$\frac{\partial p}{\partial \tau} - \frac{1}{2}\left(1 + \frac{T_i}{T_e}\right)\eta \nabla_\perp^2 p = -\beta \nabla_\| (2\delta J + v) = -\frac{\beta}{\epsilon v_A} \nabla_\| v_{\|e}, \tag{104}$$

$$\frac{\partial}{\partial \tau} 2\delta J = -\frac{1}{2}\frac{m_i}{m_e} \nabla_\| p - \frac{\eta}{2\delta}\frac{m_i}{m_e} J, \tag{105}$$

$$\frac{\partial v}{\partial \tau} = -\frac{1}{2}\left(1 + \frac{T_i}{T_e}\right) \nabla_\| p. \tag{106}$$

Since $m_i/m_e \gg 1$, we neglect v relative to $2\delta J$ in Eq. (104). Then the system above is mathematically equivalent to a conceptually more transparent system: two beams of electrons traveling along field lines in opposite directions at the electron thermal velocity. If we define the density in each of these beams as n_+ and n_-, where

$$n_+ + n_- = p\sqrt{2m_e/m_i}, \quad n_+ - n_- = 4\delta J\sqrt{\beta},$$

then, in terms of unnormalized space and time variables, $n_\pm$ satisfy

$$\frac{\partial n_\pm}{\partial t} \pm v_e \nabla_\| n_\pm + \nu(n_\pm - n_\pm) = D_c \nabla_\perp^2 (n_+ + n_-). \tag{107}$$

Here $v_e = \sqrt{T_e/m_e}$; $\nu(n_+ - n_-)$ gives the Coulomb collisional equilibration rate (i.e., friction) and D_c gives the classical electron diffusion across field lines. Equation (107) is therefore equivalent to the situation examined by Rechester and Rosenbluth.[28] Thus n_+ and n_- diffuse at the rates corresponding to whichever collisionality regime is appropriate for the given stochastic magnetic field and the size of ν.

In a stochastic magnetic field, the average nonlinear terms $\nabla_\| J$ and $\nabla_\| p$ produce diffusion operators. We now verify that these operators have the correct scaling in the two collisionality regimes by applying the direct interaction approximation to Eqs. (104)–(106) (neglecting classical particle diffusion).

We use the notation of Dupree,[29] and consider the evolution of perturbations $p^{(1)}$ and $J^{(1)}$ about the equilibrium $\psi^{(0)}$. When multiple-helicity perturbations $\psi^{(1)}$ are present, we have

$$\frac{\partial p^{(1)}}{\partial \tau} + 2\beta\delta\left(\frac{\partial}{\partial z} J^{(1)} - [\psi^{(0)}, J^{(1)}]\right) + c_{11} p^{(1)} + c_{12} J^{(1)} = 0, \tag{108}$$

$$\frac{\partial J^{(1)}}{\partial \tau} + \eta^1 J^{(1)} + \frac{1}{4\delta}\frac{m_i}{m_e}\left(\frac{\partial p^{(1)}}{\partial z} - [\psi^{(0)}, p^{(1)}]\right) + c_{21} J^{(1)} + c_{22} p^{(1)} = 0, \tag{109}$$

where $\eta^1 \equiv \eta m_i / r\delta^2 m_e$, and the c_{ij} are the diffusion operators, which are defined recursively by

$$c_{11} p^{(1)} + c_{12} J^{(1)} = -2\delta\beta \langle [\psi^{(1)}, J^{(2)}] \rangle, \tag{110}$$

$$c_{21} J^{(1)} + c_{22} p^{(1)} = -(1/4\delta)(m_i/m_e) \langle [\psi^{(1)}, p^{(2)}] \rangle. \tag{111}$$

The quantities $p^{(2)}$ and $J^{(2)}$ are driven by $p^{(1)}$ and $J^{(1)}$; i.e.,

$$\frac{\partial p^{(2)}}{\partial t} + 2\beta\delta\left(\frac{\partial J^{(2)}}{\partial z} - [\psi^0, J^{(2)}]\right) + c_{11} p^{(2)} + c_{12} J^{(2)} = 2\beta\delta [\psi^{(1)}, J^{(1)}], \tag{112}$$

$$\frac{\partial J^{(2)}}{\partial t} + \eta^1 J^{(2)} + \frac{m_i}{4\delta m_e}\left(\frac{\partial}{\partial z} p^{(2)} - [\psi^{(0)}, p^{(2)}]\right) + c_{21} p^{(2)} + c_{22} J^{(2)} = (m_i/4\delta m_e)[\psi^{(1)}, p^{(1)}]. \tag{113}$$

The operators c_{11} and c_{21} are the parts of $J^{(2)}$ and $p^{(2)}$ driven by $p^{(1)}$, and c_{12} and c_{22} are the parts driven by $J^{(1)}$. This decomposition of $J^{(2)}$ and $p^{(2)}$ is justified in the DIA, since Eqs. (112) and (113) are linear in $J^{(2)}$ and $p^{(2)}$.

The presence of the c_{ij} in Eqs. (112) and (113) distinguishes the present analysis from quasilinear theory. Without these operators, agreement with previous results is not obtained in either collisionality regime.

To simplify the analysis we take $\psi^{(1)}$ to be constant around each rational surface (as is appropriate for tearing mode turbulence). We also take the shear to be constant so $\nabla_\| A = m[(r - r_s)/L_s]A \equiv ik_\| A$, where r_s is the minor radius of the rational surface and m is the poloidal mode number. Then c_{12} and c_{21} vanish by parity considerations. Furthermore, for nearly constant $\psi^{(1)}$,

$$[\psi^{(1)}, A] = -\delta B_r^{(1)} \frac{\partial A}{\partial r},$$

where $\delta B_r = \mathrm{i}(m/r_s)\psi^{(1)}$.

We then obtain coupled nonlinear equations for c_{11} and c_{22},

$$c_{11} = -\frac{\beta m_i}{2m_e}\frac{\partial}{\partial r}\left\langle \delta B^{(1)} \left[\frac{\partial}{\partial t} + \eta^1 + c_{22} - \frac{m_i}{4\delta m_e} ik_\| \left(\frac{\partial}{\partial t} + c_{11}\right)^{-1} 2\beta\delta ik_\|\right]^{-1} \delta B^{(1)} \right\rangle \frac{\partial}{\partial r},$$

$$c_{22} = -\frac{\beta m_i}{2m_e}\frac{\partial}{\partial r}\Big\langle \delta B^{(1)}\Big[\frac{\partial}{\partial t} + c_{11} - 2\beta\delta ik_\parallel\Big(\frac{\partial}{\partial t} + \eta^1 + c_{22}\Big)^{-1}\frac{m_i ik_\parallel}{4\delta m_c}\Big]^{-1}\delta B^{(1)}\Big\rangle\frac{\partial}{\partial r}.$$

These are diffusion-like operators. We will Markovianize in both space and time for simplicity; that is, we assume that the nonlinear decorrelation time is short compared to the evolution time of $p^{(1)}$ and $J^{(1)}$, and that the decorrelation scale length is short.

We then have $c_{11} = D_{11}\,\partial^2/\partial r^2$ and $c_{22} = D_{22}\,\partial^2/\partial r^2$, where

$$D_{11} = \frac{\beta m_i}{2m_e}\sum_k |\delta B_k|^2 \times\Big(\eta^1 + D_{22}\frac{\partial^2}{\partial r^2} + \frac{m_i}{4\delta m_e}k_\parallel c_{11}^{-1}2\beta\delta k_\parallel\Big)^{-1}$$

and similarly for c_{22}.

For the present purposes it is adequate to compute only the scalings of the D's. For this we need the nonlinear decorrelation width Δx, which is the scale of the broadened propagator. Given this width, the number of Fourier harmonics contributing at any point in space is $\sim\Delta x/\lambda$, where λ is the typical spacing between rational surfaces. To find Δx, we replace $\partial/\partial x \rightarrow 1/\Delta x$ and $k_\parallel \rightarrow k_y\Delta x/L_s$ and balance the terms in the propagator obtaining

$$\eta^1 + \frac{D_{22}}{\Delta x^2}\sim\frac{\beta m_i}{m_e}\frac{k_y^2}{L_s^2}\frac{\Delta x^4}{D_{11}}$$

and

$$D_{11}\sim\frac{(\beta m_i/m_e)|\delta B_k^{(1)2}|}{\eta^1 + D_{22}/\Delta x^2}\frac{\Delta x}{\lambda},$$

$$D_{22}\sim\frac{\beta m_i}{m_e}\frac{|\delta B_k^{(1)2}|}{(D_{11}/\Delta x^2)}\frac{\Delta x}{\lambda}.$$

We thus have three equations for D_{11}, D_{22}, and Δx.

We begin with the case of low collisionality, $\eta^1 \ll D_{22}/\Delta x^2$. Surprisingly, we find that the above expressions are degenerate for $\eta\rightarrow 0$, and only the product $D_{11}D_{22}$ can be determined. However, for $\eta\rightarrow 0$, this difficulty can be avoided by decoupling the equations first as in Eq. (107), and then applying the DIA. We then can determine the diffusion coefficients of Eqs. (107) similarly to the above, but without degeneracy. The uncoupled analysis shows that $D_{11} = D_{22}$, so that $D_{11}\sim D_{22}\sim(\beta m_i/m_e)^{1/2}D_m$, where $\sqrt{\beta m_i/m_e}$ is the electron thermal velocity in Alfvénic units, and

$$D_m = L_s|\delta D_k|^2/k_y\lambda$$

is the magnetic field line diffusion coefficient in the quasilinear limit.

The boundary between the collisional and collisionless regime, $D_{22}/\Delta x^2\sim\eta^1$, can be restated using the Δx above as

$$L_n \equiv\Big(\frac{\Delta x k_y}{L_s}\Big)^{-1}\sim\frac{(\beta m_i/m_e)^{1/2}}{\eta^1}.$$

Note that $(\Delta x k_y/L_s)^{-1}$ can be interpreted as the parallel nonlinear correlation length of the magnetic field. The right-hand side is the normalized collisional mean free path. Krommes[30] has shown that, with Δx expressed in terms of δB^2, $\Delta x k_y/L_s$ scales like the Lyapanov exponentiation length; thus this criterion is similar to that found previously.[28]

For $\eta^1 \gtrsim D_{22}/\Delta x^2$, the above expressions give

$$D_{11}\sim(D_m/L_n)(\beta m_i/m_e\eta^1),$$

where $\beta m_i/(m_e\eta^1)$ is the normalized collisional parallel heat conductivity. This is the collisional result of Rechester and Rosenbluth, but missing a logarithmic factor (which is rarely much different from unity). A similar discrepancy has been found by authors[30] with different formulations of the collisional diffusion coefficient, and is attributed to the fact that the DIA by its nature often underestimates correlations in a system.

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Erratum: "A four-field model for tokamak plasma dynamics" [Phys. Fluids 28, 2466 (1985)]

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Equation (83) is not in general correct: the energy defined in Eq. (82) is not generally conserved by the four-field equations of motion in the dissipationless limit. Instead H is invariant only in two-dimensional (axisymmetric or helically symmetric) theory. The discussion following Eq. (83) is correct; in particular, the compressible, reduced magnetohydrodynamic (CRMHD) energy of Eq. (85) is indeed conserved. Also, the "cold-ion" CRMHD energy $H = \frac{1}{2}\langle|\nabla_\perp\psi|^2 + |\nabla_\perp\varphi|^2 + v^2 + p^2/2\beta\rangle$ is conserved for the three-dimensional four-field model in the limit $T_i = 0$. Discussion of energy conservation together with additional constants of motion for the complete four-field model will be the subject of a future publication.

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