Nonlinear dynamics of magnetic islands with curvature and pressure

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Curvature and finite pressure are known to have a dramatic influence on linear magnetic tearing stability. An analytic theory of the nonlinear resistive growth of magnetic islands in tokamaks that includes the interchange driving term in presented here. A Grad-Shafranov equation to describe the magnetohydrodynamic (MHD) equilibrium of thin islands is derived. The resistive evolution of these islands is then obtained. Interchange effects are found to become progressively less important with increasing island width.

I. INTRODUCTION

Magnetic islands caused by resistive instabilities are important in many areas of plasma physics. For example, tearing modes in tokamaks are implicated in major disruptions and are otherwise detrimental to confinement. The analytic theory of the resistive, nonlinear growth of these islands was first given Rutherford.[1] He considered modes driven by magnetic-free energy, measured by Δ'; plasma pressure and expansion-free energy caused by curvature were neglected. However, in the linear theory of these modes, magnetic curvature and pressure have been found to be important.[2] Specifically it is known that Δ' must exceed a critical value Δ_c for linear instability. Here we give an analytic theory of the nonlinear dynamics of magnetic islands including curvature and pressure. This is a nonlinear generalization of the linear theory results of Glasser, Greene, and Johnson[2] (henceforth referred to as GGJ). We use an aspect ratio expansion for simplicity, but believe that the essential physics for more general geometries is quite similar.

A principal result of this calculation is that there is a critical island width Δx_c. Islands wider than Δx_c are dominated by Δ', while those narrower than Δx_c are dominated by pressure and curvature in the island vicinity. The critical width is given by

$$\Delta x_c^p \, \Delta'_* \sim k_2(E+F), \tag{1}$$

where the quantities E, F, H, and $D_I \equiv E+F+H$ are standard measures of magnetic curvature (obtained here to relevant order in ϵ, the inverse aspect ratio) obtained by GGJ for linear interchange stability, $p=\sqrt{1-4D_I}$, and Δ'_* is the finite pressure generalization[2] of Δ'. For low β (i.e., $D_I \to 0$), $\Delta'_* \to \Delta'$. The quantity k_2 is a numerical constant roughly equal to six. For low β, e.g., $\beta \sim \epsilon^2$, this width is small. Hence in low-β tokamaks, favorable curvature would have little stabilizing influence on robust islands. But for high β, $\beta \sim \epsilon$, the island width above can be a substantial fraction of the minor radius. Thus, curvature stabilization of magnetic islands is of potentially major importance.

In order to obtain this result, a nonlinear Grad–Shafranov equation is derived that is valid for thin islands, and that describes the resonant magnetic field in the vicinity of the resonant surface. The pressure is constant along flux surfaces of the magnetic field distorted by the island structure; it appears in the Grad–Shafranov equation along with an expression for the average effects of curvature. This expression for the average curvature is proportional to that in the Mercier linear interchange stability criterion. It contains the effects of the average magnetic well, the diamagnetic corrections to the well, and the geodesic curvature (all of which are typically comparable for tokamaks with moderate to high β).

The criterion derived here, Eq. (1), agrees with linear theory in the following sense. Consider islands that are just barely into the Rutherford regime, that is, whose width just exceeds the linear tearing layer width. Then the Δ' needed to overcome the stabilizing effect of curvature and pressure, according to Eq. (1), essentially agrees with Δ_c derived by GGJ. (The slight differences are explained in Sec. II.) Since the stabilizing influence of curvature decreases as the island grows, linear theory estimates of stability are overly optimistic for nonlinear instabilities.

Finally, we note that the aspect ratio expansion used here is more accurate than standard high-β reduced magnetohydrodynamics (MHD). Thus, the average curvature expression is accurate enough to obtain the low-β Mercier interchange criterion for $\beta \sim \epsilon^2$.

The remainder of this paper is oganized as follows. In Sec. II, a qualitative, physical explanation of the results of the calculation is given. The detailed derivation of the Grad–Shafranov equation for thin islands is presented in Sec. III. The resistive evolution of these islands is described in Sec. IV. We summarize our results in Sec. V and indicate their application to related problems.

II. HEURISTIC INTERPRETATION OF RESULTS

We examine here a simple slab model with gravity. Gravity simulates the role of the proper average of the curvature, which suffices to demonstrate the qualitative features of the case with curvature. We also indicate the physical content of the various terms in the average of the curvature, which is computed in the next section.

A slab geometry with constant gravity and islands of one helicity still has one symmetry direction. Moreover, MHD equilibrium is described by the Grad–Shafranov equation, which is particularly simple for thin islands and $\beta \ll 1$. In the vicinity of the island, the magnetic field can be written

$$\mathbf{B} = B_0\hat{z} + \hat{z}\times\nabla\psi,$$

and B_0 can be taken as constant for $\beta \ll 1$. Gravity $\mathbf{g}$ is in the $\hat{x}$ direction and $\hat{z}$ is the symmetry direction. In the absence of islands, this is a sheared slab geometry with $\psi = B_0x^2/2L_s$, L_s being the shear length. More generally, ψ is determined

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by Ampere's law, $(\partial^2/\partial x^2 + \partial^2/\partial y^2)\psi = j_z$, where j_z is the current in the $\hat{z}$ direction. For thin islands, $\partial^2/\partial x^2 \gg \partial^2/\partial y^2$ in the island region.

The equilibrium satisfies $\mathbf{j}\times\mathbf{B} = \rho\mathbf{g}$, and ρ is a flux function, $\rho(\mathbf{x}) = \rho(\psi)$. The the Grad–Shafranov equation in the island vicinity for this case can be written

$$\frac{\partial^2}{\partial x^2}\psi = \mathscr{J}(\psi) + xg\frac{\partial\rho}{\partial\psi}, \tag{2}$$

where $\mathscr{J}(\psi)$ is an arbitrary function needed to specify the current on a flux surface. The second term describes parallel currents caused by ρg as follows. Because of quasineutrality, $\nabla_\parallel\cdot\mathbf{j}_\parallel = -\nabla_\perp\cdot\mathbf{j}_\perp$. The equilibrium condition gives $\mathbf{j}_\perp \cong \rho\mathbf{g}\times\hat{z}/B_0$. Then, $\nabla\cdot\mathbf{j}_\perp \cong \nabla\rho\cdot\mathbf{g}\times\hat{z}/B_0 = (\partial\rho/\partial\psi)\nabla\psi\cdot\mathbf{g}\times\hat{z}/B$. Solving for $j_\parallel \cong j_z$ gives the second term just described.

The detailed analysis with curvature produces a Grad–Shafranov equation [Eq. (57)] that is similar to Eq. (2), but g is replaced by an expression proportional to the pressure-driving terms in the Mercier criterion.

If the plasma has small but finite resistivity, these equilibria can evolve in time, but slowly enough so that MHD equilibrium is maintained. This evolution comes about because an induction electric field $\partial\psi/\partial t$ drives a current. The appropriate average [defined by Eq. (62)] of the current on a flux surface is given by

$$\eta\langle j_z\rangle = \left\langle\frac{\partial\psi}{\partial t}\right\rangle, \tag{3}$$

where η is the resistivity. The basic features of the dynamics can be obtained from a qualitative analysis of Eqs. (2) and (3).

With the island, ψ is modified from the equilibrium value to $\psi = B_0x^2/2L_s + A\cos(k_y y)$. To obtain the evolution of A, the parameter Δ' must be introduced. For simplicity, A is assumed to be nearly constant in the island region, so $\Delta' \equiv [(dA/dx)|_{+\infty} - (dA/dx)|_{-\infty}]/A(0)$.

After integrating Ampere's law, we have

$$A(0)\Delta' = \int_{-\infty}^{+\infty} j_\Delta \cos(k_y y)\,dy\,dx,$$

where j_Δ denotes the difference between j_z and the equilibrium current. Since the relevant currents are localized to the island region,

$$A(0)\Delta' \sim j_\Delta\Delta x, \tag{4}$$

where $\Delta x \sim \sqrt{L_sA/B_0}$ is the island width.

Also, $\langle j\rangle = \mathscr{J}(\psi) + \langle x\rangle g\rho'(\psi)$; then with Eq. (3), the unknown function $\mathscr{J}(\psi)$ can be eliminated from the j_z to obtain

$$j_\Delta = \frac{1}{\eta}\left\langle\frac{\partial A}{\partial t}\cos(k_y y)\right\rangle + (x - \langle x\rangle)g\rho'(\psi). \tag{5}$$

In the island region, $\langle\partial A\cos(k_y y)/\partial t\rangle \sim \partial A/\partial t$ and $x - \langle x\rangle \sim \Delta x$. Now assume that $\partial\rho/\partial x \sim \rho/L_n$, where L_n is the density scale length. Then

$$\frac{\partial\rho}{\partial\psi} = \frac{\partial\rho/\partial x}{\partial\psi/\partial x} \sim \frac{\rho/L_n}{\Delta x B_0/L_s},$$

so that

$$(x - \langle x\rangle)g\rho'(\psi) \sim g\rho L_s/(L_nB_0). \tag{6}$$

With Eq. (4),

$$\frac{A\Delta'}{\Delta x} \sim \frac{1}{\eta}\frac{\partial A}{\partial t} + \rho g\frac{L_s}{L_nB_0}.$$

This can be written more transparently by eliminating A in terms of the island width, $A \sim B_0\Delta x^2/L_s$. Also, ρg plays the role of pressure times curvature, $p\kappa$ (and has the same dimensions). Making this replacement, the evolution equation is finally obtained;

$$\frac{1}{\eta}\frac{\partial\Delta x}{\partial t} \sim \Delta' + \frac{1}{\Delta x}\beta\frac{\kappa L_s^2}{L_n}. \tag{7}$$

The detailed mathematical analysis gives an essentially equivalent expression. That analysis shows that the pressure-driving term has contributions only from outside the separatrix, but still from the region near the island.

With only the first term on the right, the island width grows linearly in time, which is Rutherford's result. The sign of the curvature term is such that for favorable (stabilizing) curvature it tends to make the island shrink, while for unfavorable curvature it contributes to growth. A crucial feature is that it has a $1/\Delta x$ dependence relative to the Δ' term. Thus, for large Δx, the island growth is determined only by Δ', whereas for small Δx its growth is dominated by curvature and pressure. Here Δx_c in Eq. (1) gives the boundary between these two regimes for the finite β generalization of Δ'.

The actual expression for growth has $\beta\kappa L_s^2$ replaced by $E + F$, which is made up of two different types of terms. The first type is an average, over the equilibrium field lines, of the curvature normal to a flux surface, κ_N. This is also proportional to the derivative with respect to the poloidal flux of the equilibrium value of $\int dl/B$, along with corrections from equilibrium currents and diamagnetic corrections. These terms account for all of E, but the diamagnetic corrections are part of F.

The second type of term, arising from the geodesic curvature, constitutes the remainder of F. This term has a simple interpretation, if the corrections to $\int dl/B$ in the first term are ignored. Then, the sum of this term plus the first terms give the normal gradient of $\int dl/B$ taken along the field lines *perturbed* by the local disturbance. In the thin island ordering used here, the relevant field perturbations turn out to be those which are nonresonant. These field perturbations are produced by changes in the local Pfirsh–Schluter currents, because the pressure is altered from its equilibrium value (i.e., the pressure is constant on flux surfaces associated with the island). The terms obtained because of this effect depend on geodesic curvature because the Pfirsh–Schluter currents are produced by geodesic curvature. We therefore refer to these second types of terms as geodesic terms. They are also proportional to one higher power of β relative to the normal curvature terms. They are generally comparable to the first type of terms except at very low β.

There is a third type of term, H, different than those above which appears in the Grad–Shafranov equation but not in the expression for the resistive island growth. This type arises from the local toroidal coupling, via Ampere's law, of nonresonant Pfirsh–Schluter currents to produce a resonant ψ. GGJ also found that the H is rarely significant for resistive instabilities in tokamaks, though they make a significant contribution to the pressure-driving terms for ideal modes (corresponding to our Grad–Shafranov equation for ideal equilibria). GGJ evaluate E, F, and H for shifted circle Shrafranov equilibrium with $\beta \sim \epsilon^2$, and find that they are all of order ϵ^2. For high-β equilibria, $\beta \sim \epsilon$, those terms are all ~ 1.

Note that to solve the the Grad–Shafranov equation analytically and obtain the evolution equation, we must use a subsidiary expansion in which $E \sim F \sim H$ are assumed small. The resistive criterion of GGJ contains $E + F + H^2$. In the subsidiary expansion this is indistinguishable from $E + F$, and the main point is that H does not affect resistive growth as much as $E + F$. However, note that H^2 arises in the GGJ calculation because in the thin linear tearing layer, resistive diffusion is as important in the pressure response as the terms tending to make the pressure respond adiabatically. For nonlinear islands whose width exceeds the linear theory layer, the adiabatic terms dominate and the pressure becomes a function of the perturbed flux. We would therefore expect that the H^2 term would be absent in the nonlinear case.

III. THE GRAD–SHAFRANOV EQUATION FOR THIN ISLANDS

The fundamental equations needed to describe the islands are the vorticity equation (or equivalently, quasineutrality), Ohm's law, and a relation to determine the pressure. Nonlinear islands grow relatively slowly, so following Rutherford we neglect inertia in the vorticity equation. Also, we assume that the island growth rate is slow compared to the parallel propagation time for sound waves, so that there are no parallel pressure gradients. Therefore, our starting equations are quasineutrality,

$$\nabla_\parallel \cdot \mathbf{j}_\parallel = -\nabla \cdot \mathbf{j}_\perp , \tag{8}$$

and, because of the neglect of inertia, pressure balance,

$$\mathbf{j} \times \mathbf{B} = \nabla p. \tag{9}$$

Inserting $j_\perp$ from Eq. (9) into Eq. (8) gives

$$\nabla_\parallel \cdot \mathbf{j}_\parallel = \mathbf{B} \cdot \nabla p \times \nabla(-1/B^2), \tag{10}$$

and $\mathbf{B}\cdot$ Eq. (9) gives

$$\mathbf{B} \cdot \nabla p = 0. \tag{11}$$

In this section the consequences of Eqs. (10) and (11) are considered. These results are combined with Ohm's law in the next section to obtain the dynamics.

A. Mathematical preliminaries

First, coordinates are chosen. Consider the flux coordinates of the equilibrium magnetic field B_0,

$$\mathbf{B}_0 = \nabla\chi \times \nabla[q(\chi)\vartheta - \zeta], \tag{12}$$

where ζ is the toroidal symmetry angle, χ and ϑ are the poloidal flux and angle, and $q(\chi)$ is the safety factor. We suppose the islands under consideration are centered on some surface χ_0 with a particular rational value of q, say q_0. These islands are caused by magnetic perturbations that are harmonics of the helicity angle $\alpha \equiv \vartheta - \zeta/q_0$. Our coordinates will be α, χ, ζ. For any quantity, f,

$$\mathbf{B}_0 \cdot \nabla f(\zeta,\alpha,\chi) = J^{-1}\left[q\left(\frac{\partial f}{\partial \zeta}\right)_\alpha + \left(1 - \frac{q}{q_0}\right)\frac{\partial f}{\partial \alpha}\right], \tag{13}$$

where

$$J = [\nabla\zeta \cdot (\nabla\chi \times \nabla\vartheta)]^{-1} = q/\mathbf{B}_0 \cdot \nabla\zeta$$

is the Jacobian.

Also, an averaging procedure must be defined. Rutherford's analysis was essentially two-dimensional because of symmetry in the third dimension. The third dimension corresponds here to ζ at constant α. If ζ varies at fixed α, ϑ varies as well, so ζ at constant α is not a symmetry coordinate, and $(\partial/\partial\zeta)_\alpha$ does not vanish even for equilibrium quantities. We therefore define the ζ average of a quantity f by

$$\bar{f} \equiv \frac{\oint d\zeta\, f(\zeta,\alpha,\chi)}{\oint d\zeta}, \tag{14a}$$

and the ζ varying part of f by

$$\tilde{f} = f - \bar{f}. \tag{14b}$$

Average quantities $\bar{f}$ are functions of α and χ alone and are resonant at q_0. Quantities $\tilde{f}$ are nonresonant. Of course, two nonresonant quantities can beat together to yield a resonant quantity.

Finally, it will be convenient to introduce the following notation:

$$[A,B] \equiv J\, \nabla\zeta \cdot (\nabla A \times \nabla B). \tag{15}$$

We also define the helical flux ψ_{h0} through a ribbon of constant α by

$$\frac{\partial \psi_{h0}}{\partial \chi} \equiv 1 - \frac{q}{q_0}. \tag{16}$$

In terms of this bracket,

$$\mathbf{B}_0 \cdot \nabla f = J^{-1}\left[q\left(\frac{\partial f}{\partial \zeta}\right)_\alpha + [\psi_{h0}, f]\right]. \tag{17}$$

Note that $[\psi_{h0}, f]$ vanishes at the rational surface, $q(\chi) = q_0$, and in linear theory, $J^{-1}[\psi_{h0}, f] = ik_\parallel f$, where $k_\parallel$ is the parallel wavenumber. Also, note that

$$[A,B] = \frac{\partial A}{\partial \chi}\left(\frac{\partial B}{\partial \alpha}\right)_\zeta - \left(\frac{\partial A}{\partial \alpha}\right)_\zeta \frac{\partial B}{\partial \chi}, \tag{18a}$$

and that $[A,B]$ behaves like a Poisson bracket, i.e.,

$$[A,B] = -[B,A], \tag{18b}$$

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0, \tag{18c}$$

and

$$[AB,C] = A[B,C] + B[A,C]. \tag{18d}$$

Equation (18c) is known as the Jacobi identity, while (18d) implies that $[A,B]$ is a derivation.

Also, note that $[A,B]$ acts like a typical quadratic form under the average Eq. (14):

$$\overline{[A,B]} = [\bar{A},\bar{B}] + \overline{[\tilde{A},\tilde{B}]}. \tag{19}$$

Of course, the equilibrium magnetic field is perturbed by the instability. This perturbation can be written as

$$\mathbf{B}_1 = \nabla \times \mathbf{A} = \nabla\zeta \times \nabla\psi_1 + \mathbf{B}_{T1}, \tag{20}$$

where $\psi_1 = -R\hat{\zeta} \cdot \mathbf{A}$, with R the radius from the symmetry axis, and $\mathbf{B}_{T1} = \nabla \times (\mathbf{A} - \hat{\zeta}\hat{\zeta} \cdot \mathbf{A})$. The total poloidal magnetic flux, $\psi_1 + \chi$, is related to the total toroidal current $\hat{\zeta} \cdot j_{\rm tot}$ by Ampere's law,[3]

$$R\nabla \cdot 1/R^2 \nabla(\psi_1 + \chi) = \hat{\zeta} \cdot j_{\rm tot}, \tag{21a}$$

and the perturbed flux satisfies

$$R\nabla \cdot 1/R^2 \nabla \psi = \hat{\zeta} \cdot (\mathbf{j}_{\rm tot} - \mathbf{j}_{\rm eq}) = \hat{\xi} \cdot \mathbf{j}_1. \quad (21\text{b})$$

B. Ordering procedure for thin islands and large aspect ratio

After the above expressions for $\mathbf{B}_0$ and $\mathbf{B}_1$ are substituted into Eqs. (10) and (11), the result is split into average parts and parts which vary in ζ. It is simplified by keeping only terms relevant for thin islands, and for small inverse aspect ratio ϵ.

The island results from the part of ψ_1 with the resonant helicity, namely $\bar{\psi}_1$. In order for the calculation to be tractable, we must consider islands that are thin compared to the minor radius. Thus, $\bar{\psi}_1$ is taken to be small $\bar{\psi}_1 \sim \delta \ll 1$. The width of the island is measured by $\bar{\psi}_1^{1/2} \sim \delta^{1/2}$. This $\delta^{1/2}$ serves as a localization parameter, analogous to the thin layer parameter of linear theory.[2]

The pressure p is perturbed away from the equilibrium pressure p_0 by the presence of the island. Because of the topology change and distortion of the local flux surfaces, the local pressure profile is substantially altered. The magnitude of the pressure perturbation $p_1 = p - p_0$ is ordered so that $\nabla p_1 \sim \nabla p_0$. However, since the flux surface modification progressively diminishes away from the island, p_1 is localized to the island region. Thus, $\nabla p_1 \to 0$ for distances from the rational surface which greatly exceed the island width:

$$p_1 \sim \delta^{1/2} \nabla p_1 \sim \delta^{1/2} \nabla p_0.$$

For the aspect ratio expansion, we take $\hat{\zeta} \cdot \mathbf{B}_0 \sim 1$, $\hat{\vartheta} \cdot \mathbf{B}_0 \sim \epsilon$, the minor radius scale ~ 1, $R \sim 1/\epsilon$, and $\chi \sim 1$. In the ϵ expansion, we will keep terms to lowest nontrivial order in ϵ and additionally those of one higher order in ϵ. This is done to treat the curvature more accurately than in high-β reduced MHD. For example, the present calculation includes sufficient terms in ϵ to obtain the low-β ($\beta \sim \epsilon^2$) Mercier interchange criterion.

While corrections of order ϵ are kept, for simplicity additional corrections of order β are not. This is quite consistent for the case where, say, $\beta \sim \epsilon^2$ or $\beta \sim \epsilon^{3/2}$, and for this case it is also necessary to compute the curvature quite accurately. For $\beta \sim \epsilon$ the next order corrections in ϵ are not strictly needed to correctly obtain the average curvature to lowest order, and they can be dispensed with for this case.

We now examine the relations between the small parameters δ, ϵ, and β for the problem at hand.

We wish to have an expression for the island evolution that includes both the interchange driving term (caused by pressure and curvature) and the Δ' driving term. Therefore, an ordering should be chosen in which these are comparable. The discussion in Sec. II shows that this implies $\delta^{1/2} \sim \beta L_s^2 \bar{\kappa}/\Delta'$, where $\bar{\kappa}$ is the proper average of the curvature, $L_s \sim R \sim 1/\epsilon$, $\kappa \sim \epsilon$, and $L_n \sim 1$. Thus, we take $\delta^{1/2} \sim \beta/\epsilon\Delta'$ to be small.

The parameter Δ' is often numerically rather large for current profiles of interest (sometimes 20–30 times an inverse equilibrium scale length). We therefore take $\Delta' \gg 1$. This is evidently consistent with small island width and equivalent to conventional orderings in linear theory.[2]

Because finite islands are being considered, the part of $\mathbf{B}_1 \cdot \nabla f$ caused by ψ_1 is comparable to $\mathbf{B}_0 \cdot \nabla \bar{f}$ for resonant quantities, $\bar{f}$, associated with the island. To see this, recall Eq. (13),

$$\mathbf{B}_0 \cdot \nabla \bar{f} = J^{-1}\left(1 - \frac{q}{q_0}\right)\frac{\partial \bar{f}}{\partial \alpha}.$$

We take $q \sim \partial q/\partial \chi \sim 1$ and also $\partial/\partial \alpha \sim 1$. Therefore, in the island vicinity,

$$\left(1 - \frac{q}{q_0}\right)\frac{\partial f}{\partial \alpha} \sim J^{-1}\delta^{1/2} f.$$

Also

$$\mathbf{B}_1 \cdot \nabla \bar{f} \simeq J^{-1}[\psi_1, \bar{f}] = J^{-1}\left(\frac{\partial \psi_1}{\partial \chi}\frac{\partial \bar{f}}{\partial \alpha} - \frac{\partial \psi_1}{\partial \alpha}\frac{\partial \bar{f}}{\partial \chi}\right).$$

For quantities associated with the island, $\partial \bar{f}/\partial \chi \sim \bar{f}\delta^{-1/2}$, so $\mathbf{B}_1 \cdot \nabla \bar{f} \sim J^{-1}\delta^{1/2}\bar{f}$.

Finally, we estimate the size of average current $\hat{\zeta} \cdot \bar{j}_1$ from Ampere's law, Eq. (21b). Here Δ' is defined so that $d\bar{\psi}_1/dx \sim \Delta' \bar{\psi}_1 \sim \Delta' \delta^{1/2} \cdot d\bar{\psi}_1/dx$ changes on the island scale, so $R^{-1} d^2\bar{\psi}_1/dx^2 \sim \epsilon\delta^{-1/2}\Delta'\bar{\psi}_1$. Thus $\hat{\zeta} \cdot \bar{j}_1 \sim \epsilon\delta^{1/2}\Delta'$.

Collecting all the fundamental island orderings, we have

$$\begin{gathered} \bar{\psi}_1 \sim \delta, \quad \bar{p} \sim \delta^{1/2}\beta, \quad \delta^{1/2} \sim \beta/\epsilon\Delta', \\ \hat{\zeta} \cdot \bar{\mathbf{j}}_1 \sim \epsilon\delta^{1/2}\Delta', \quad [\bar{\psi}_1, \bar{f}] \sim \delta^{1/2}\bar{f}. \end{gathered} \quad (22)$$

These orderings are now applied to Eqs. (10) and (11).

C. Simplification of the curvature term

The curvature term in Eq. (10),

$$\mathbf{B} \cdot \nabla p \times \nabla(-1/B^2),$$

is simplified in several ways.

(1) Here $\mathbf{B}$ is replaced by the equilibrium toroidal field. This is appropriate since the ζ component of $\mathbf{B}$ dominates by $O(1/\epsilon)$, and because the perpendicular components of ∇p and $\nabla(-1/B^2)$ dominate the toroidal components by $O(1/\epsilon)$. Thus $(\mathbf{B} \cdot \hat{\vartheta})\hat{\vartheta} \cdot \nabla p \times \nabla(-1/B^2)$ is relatively small by $O(\epsilon^2)$. Furthermore, $\hat{\zeta} \cdot \mathbf{B}$ can be replaced by its equilibrium value. The change in B caused by p can be found from the well-known fact[3] that the two remain in approximate pressure equilibrium, $(B^2/2) + p = \text{const}$. Therefore, $B_1 = p_1/B \sim \beta\delta^{1/2}$ is small. Finally, for axisymmetric equilibria, $\hat{\zeta} \cdot \mathbf{B} = f(\chi)/R$, where f is a particular function of χ alone. For $\beta/\epsilon \sim 1$, the relative variation in f over the minor radius is approximately ϵ, and thus its variation in the island is approximately $\delta^{1/2}$, so $f(\chi)$ may be taken as a constant $f_0 \equiv f(\chi_0)$. Thus, the right-hand side of Eq. (10) is consistently approximated by

$$f_0 \nabla \zeta \cdot \nabla p \times \nabla(-1/B^2) = J^{-1}[p, -f_0/B^2]. \quad (23\text{a})$$

(2) The $1/B^2$ term on the right-hand side in Eq. (23a) is the total magnetic field. The important change in the equilibrium magnetic field magnitude caused by the perturbation is because of p_1, giving the diamagnetic correction to the equilibrium field $B_{\rm eq}$: $B_1 B_{\rm eq} \cong -p_1$. Hence $1/B^2 \equiv 1/B_{\rm eq}^2 + 2p_1/B_{\rm eq}^4$, and the bracket in Eq. (23a) is $[p, 1/B_{\rm eq}^2 + 2p_1/B_{\rm eq}^4]$. To requisite order the $B_{\rm eq}^4$ can be taken to be a constant. Since $[p,p] = 0$, we can subtract $[p, 2p/B_{\rm eq}^4]$ from the above. The curvature term can thus be written

$$J^{-1}[p,h],\tag{23b}$$

where $h \equiv f_0(-1/B^2_{eq} + 2p_0/B^4_{eq})$.

The second term in h subtracts off the part of the magnetic well from diamagnetic currents.

D. Simplification of MHD equations

It is convenient to define

$$I = j_\parallel/B,\tag{24}$$

so that $\mathbf{j}_\parallel = \mathbf{B}I$, and $\nabla_\parallel \cdot \mathbf{j}_\parallel = \mathbf{B}\cdot\nabla I$. With the total magnetic field, $\mathbf{B}\cdot\nabla I$ is

$$J^{-1}\left[q\left(\frac{\partial I}{\partial \zeta}\right)_\alpha + [\psi_{h0}+\psi_1, I]\right] + \mathbf{B}_{T1}\cdot\nabla I.\tag{25}$$

The terms from B_{T1} can be shown to be higher order. Thus, from Eqs. (23b) and (25) it is clear that

$$q\left(\frac{\partial}{\partial\zeta}\right)_\alpha I + [\psi_{h0}+\psi_1, I] = [p,h],\tag{26}$$

$$q\left(\frac{\partial}{\partial\zeta}\right)_\alpha p + [\psi_{h0}+\psi_1, p] = 0.\tag{27}$$

All quantities, e.g., I, are now split into δ averaged parts, $\bar I$, and ζ varying parts, $\tilde I = I - \bar I$. These quantities are then determined from the average and ζ varying parts of Eqs. (26) and (27). The total helical flux, $\psi_{h0}+\bar\psi_1$, acts like a flux function for the island. We denote this important quantity by

$$\psi_h \equiv \psi_{h0} + \bar\psi_1.\tag{28}$$

Then the averages of Eqs. (26) and (27) are

$$[\psi_h, \bar I] = [\bar p, \bar h] = \overline{[\tilde p, \tilde h]} - \overline{[\tilde\psi_1, \tilde I]},\tag{29}$$

$$[\psi_h, \bar p] = -\overline{[\tilde\psi_1, \tilde p]},\tag{30}$$

upon making use of Eq. (19).

Equation (29) is of central importance. In the absence of pressure and toroidicity, it becomes $[\psi_h, \bar I] = 0$. This implies that the current is a function of ψ_h, which is a crucial element of Rutherford's argument. The main effect of pressure is to modify this result. The last two terms of Eq. (29) are determined by the ζ varying part of Eqs. (26) and (27). Upon subtracting Eqs. (29) and (30) from Eqs. (26) and (27), respectively, we obtain

$$q\left(\frac{\partial \tilde I}{\partial\zeta}\right)_\alpha + [\tilde\psi_1, \bar I] + [\psi_h, \tilde I] + [\tilde\psi_1, \tilde I] - \overline{[\tilde\psi_1, \tilde I]}$$
$$= [\bar p, \tilde h] + [\tilde p, \bar h] + [\tilde p, \tilde h] - \overline{[\tilde p, \tilde h]},\tag{31}$$

$$q\left(\frac{\partial \tilde p}{\partial\zeta}\right)_\alpha + [\psi_h, \tilde p] + [\tilde\psi, \bar p] + [\tilde\psi_1, \tilde p] - \overline{[\tilde\psi_1, \tilde p]} = 0.\tag{32}$$

The terms in these equations that dominate for thin islands are easily recognized. The left-hand side of Eq. (31) comes from the $\mathbf{B}\cdot\nabla I$. For the nonresonant I, the term involving the equilibium magnetic field is dominant; this is the term $q(\partial\tilde I/\partial\zeta)_\alpha$. Also, p is constant on the surfaces describing the island; thus $\tilde p$ is small compared to $\bar p$, and $[\bar p, \tilde h]$ dominates on the right-hand side. In Eq. (32), the term with $\bar p$ is large and the largest of the remaining terms is the one caused by the equilibrium toroidal field, $q(\partial\tilde p/\partial\zeta)_\alpha$. Thus, we expect in lowest order,

$$q_0\left(\frac{\partial \tilde I}{\partial\zeta}\right)_\alpha = [\bar p, \tilde h],\tag{33}$$

$$q_0\left(\frac{\partial \tilde p}{\partial\zeta}\right)_\alpha = -[\tilde\psi_1, \bar p].\tag{34}$$

The size of $\tilde I$, $\tilde p$, and $\tilde\psi_1$ can be estimated from Eqs. (33) and (34).

One can check that the remaining terms of Eqs. (31)–(32) are negligible. For this, note that $\tilde h \sim 1$. Therefore, Eq. (33) implies $\tilde I \sim \beta$, and Eq. (34) implies $\tilde p \sim \delta^{-1/2}\tilde\psi_1\bar p \sim \tilde\psi_1\beta$. In the next section we solve for $\tilde\psi_1$ using Amperes's law [Eq. (47)] and find $\partial\tilde\psi_1/\partial\chi \sim \beta\delta^{1/2}\epsilon$. With these results and Eq. (22), the validity of Eqs. (33) and (34) can be readily verified. It can be shown that the right-hand side of Eq. (30) is negligible.

Equations (33) and (34) are easily interpreted physically. The first says that $\tilde j$ is the Pfirsh-Schlüter current; the second that $\tilde p$ arises from the tilt of the ζ varying magnetic field into gradients of $\bar p$. We have

$$\tilde I = q_0^{-1}\left[\bar p, \int d\zeta\, \tilde h\right],\tag{35}$$

$$\tilde p = q_0^{-1}\left[\bar p, \int d\zeta\, \tilde\psi\right].\tag{36}$$

The sum of the last two terms of Eq. (29) becomes

$$\frac{xq_0^{-1}}{\oint d\zeta}\int d\zeta\left(\left[\left[\bar p, \int d\zeta\,\tilde\psi_1\right], \tilde h\right] - \left[\tilde\psi_1, \left[\bar p, \int d\zeta\,\tilde h\right]\right]\right).\tag{37}$$

Integrating by parts and using the Jacobi identity [cf. Eq. (18c)], this is

$$\left[\bar p, \overline{\left[q_0^{-1}\int d\zeta\,\tilde h, \tilde\psi_1\right]}\right],\tag{38}$$

so that

$$[\psi_h, \bar I] = \left[\bar p, \bar h + \overline{\left[\frac{1}{q_0}\int d\zeta\,\tilde h, \tilde\psi_1\right]}\right].\tag{39}$$

The right-hand side of Eq. (39) gives the appropriate average of curvature, $\bar h + (1/q_0)\overline{[\int d\zeta\,\tilde h, \tilde\psi_1]}$. To the order of the calculation, this can be shown to be the field line average of the curvature κ,

$$\oint \frac{dl}{B}\kappa,$$

taken over the field line perturbed by $\tilde\psi$ (the perturbations caused by ψ_h contribute only in higher order). This perturbation is accounted for by the second term involving h and $\tilde\psi$ in Eq. (39) and it brings in the geodesic curvature, whereas the first term gives the contribution from the normal curvature. Both terms in the average curvature are generally the same order.

E. Computation of geodesic curvature terms

Here $\tilde\psi$ is now computed using Ampere's law Eq. (21a).

$$R\nabla\cdot(1/R^2)\nabla\psi_1 = \hat{\zeta}\cdot(\mathbf{j}_{tot} - \mathbf{j}_{eq}).\tag{40}$$

We consider here only currents present in the island region. In the interior, ψ_1 must be matched by exterior solutions in the usual way, thereby introducing Δ'. For islands thin compared to a perpendicular wavelength, the gradient operators simplify to

$$\frac{1}{R}|\nabla\chi|_0^2\,\frac{\partial^2}{\partial\chi^2}\psi_1 = \hat\zeta\cdot(\mathbf{j}_{tot} - \mathbf{j}_{eq}),$$

where $|\nabla\chi|_0$ is $|\nabla\chi|$ evaluated at $\chi=\chi_0$; it is generally a function of α and ζ. Recalling that $I \equiv j_\parallel/B$ and $f \equiv \hat{\zeta}\cdot B_0 R$, we find to requisite order in ϵ and β,

$$\frac{\partial^2}{\partial\chi^2}\psi_1 = f_0|\nabla\chi|_0^{-2}(I_{\rm tot} - I_{\rm eq}). \qquad (41)$$

The varying part of Eq. (41) is

$$\frac{\partial^2}{\partial\chi^2}\tilde{\psi}_1 = f_0[\,|\nabla\chi|_0^{-2}\tilde{I}_\Delta - \overline{|\nabla\chi|_0^{-2}\tilde{I}_\Delta} - |\tilde{\nabla}\chi|_0^{-2}\bar{I}_\Delta\,], \qquad (42)$$

where $I_\Delta = I_{\rm tot} - I_{\rm eq}$.

The averaged part is

$$\frac{\partial^2}{\partial\chi^2}\bar{\psi}_1 = f_0[\,\overline{|\nabla\chi|_0^{-2}\bar{I}_\Delta} + \overline{|\tilde{\nabla}\chi|_0^{-2}\tilde{I}_\Delta}\,]. \qquad (43)$$

Eliminating the $\bar{I}$ from Eq. (42) using Eq. (43), we find

$$\begin{aligned}\frac{\partial^2}{\partial\chi^2}\tilde{\psi}_1 = f_0[\,&|\nabla\chi|_0^{-2}\tilde{I}_\Delta - \overline{|\tilde{\nabla}\chi|_0^{-2}\tilde{I}_\Delta} \\ &- |\tilde{\nabla}\chi|_0^{-2}\overline{|\nabla\chi|_0^{-2}\tilde{I}_\Delta}\,(\,\overline{|\nabla\chi|_0^{-2}}\,)\,]^{-1} \\ &+ |\tilde{\nabla}\chi|_0^{-2}\frac{\partial\bar{\psi}_1}{\partial\chi^2}(\,\overline{|\nabla\chi|_0^{-2}}\,)^{-1}.\end{aligned} \qquad (44)$$

Equation (35) is now used for $\tilde{I}_\Delta$. With $\bar{p}_1 = \bar{p} - \bar{p}_0$,

$$\tilde{I}_\Delta = \left[\bar{p}_1, \frac{1}{q_0}\int d\zeta\,\tilde{h}\right].$$

Recall Eq. (18a). Since $\bar{p}_1$ has the scale length of the island width,

$$\tilde{I}_\Delta \cong \frac{\partial\bar{p}_1}{\partial\chi}\frac{\partial}{\partial\alpha}\frac{1}{q_0}\int d\zeta\,\tilde{h}_0. \qquad (45a)$$

where h_0 is h evaluated at $\chi=\chi_0$.

The term in $\tilde{h}$ can be simplified for axisymmetric equilibria, for which $\tilde{h}$ is a function of ϑ only. The $\int d\zeta$ is taken at constant α, so upon writing $\tilde{h}(\vartheta) = \tilde{h}(\alpha + \zeta/q_0)$, we see that, $\partial/\partial\alpha$ is the inverse of $q_0^{-1}\int d\zeta$. Hence

$$\tilde{I}_\Delta = \frac{\partial\bar{p}_1}{\partial\chi}\tilde{h}_0. \qquad (45b)$$

Inserting this into Eq. (44) yields

$$\begin{aligned}\frac{\partial^2}{\partial\chi^2}\tilde{\psi}_1 = f_0\frac{\partial\bar{p}_1}{\partial\chi}[\,&|\nabla\chi|_0^{-2}\tilde{h}_0 - \overline{|\tilde{\nabla}\chi|_0^{-2}\tilde{h}} \\ &- |\tilde{\nabla}\chi|_0^{-2}\overline{|\nabla\chi|_0^{-2}\tilde{h}}\;\overline{|\nabla\chi|_0^{-2}}\,] \\ &+ \frac{|\nabla\chi|_0^{-2}\,(\partial^2\bar{\psi}_1/\partial\chi^2)}{\overline{|\nabla\chi|_0^{-2}}}.\end{aligned} \qquad (46)$$

After integrating in χ,

$$\begin{aligned}\frac{\partial\tilde{\psi}_1}{\partial\chi} = C &+ \frac{(2\bar{\psi}_1/\partial\chi)|\nabla\chi|_0^{-2}}{\overline{|\nabla\chi|_0^{-2}}} \\ &+ \bar{p}_1 f_0[\,|\nabla\chi|_0^{-2}|\tilde{h}_0 - \overline{|\nabla\chi|_0^{-2}\tilde{h}_0} \\ &- |\tilde{\nabla}\chi|_0^{-2}\overline{|\nabla\chi|_0^{-2}\tilde{h}_0}/\,\overline{|\nabla\chi|_0^{-2}}\,],\end{aligned} \qquad (47)$$

where C is a constant in χ. Recall that $\tilde{\psi}_1$ is needed to evaluate (38), where $\tilde{\psi}_1$ enters in a bracket. In this bracket, the $\partial\tilde{\psi}_1/\partial\chi$ terms dominate. This is true for the second on the right-hand side in Eq. (47) since Δ' is large. The third term, which is the part of $\tilde{\psi}_1$ driven by $\tilde{I}_\Delta$, is slightly more subtle.

First, note that I_Δ is a Pfirsh–Schlüter current and so is proportional to a gradient of $\bar{p}_1$. Since $\bar{p}_1$ is localized to the island region, the total integrated current, $\int d\chi\,\tilde{I}_\Delta$, vanishes (at least to this lowest order in $\delta^{1/2}$). Usually, a given $\tilde{I}_\Delta$ would produce a $\tilde{\psi}_1 \sim f_0 \int \tilde{I}_\Delta\, d\chi/\tilde{\Delta}' \sim f_0\tilde{I}_\Delta\delta^{1/2}/\tilde{\Delta}'$. The $\tilde{I}_\Delta$ in Eq. (45) produces a $\tilde{\psi}_1$ smaller by $\delta^{1/2}$, $\tilde{\psi}_1 \sim f_0\tilde{I}_\Delta\delta$. The second property of $\tilde{I}_\Delta$ is that it is not small for small islands; from Eq. (45a), $\tilde{I}_\Delta \sim \beta$ and is independent of δ. Since $f_0 \sim 1/\epsilon$, the $\tilde{I}_\Delta$ in Eq. (45a) produces $\tilde{\psi}_1 \sim \delta\beta/\epsilon$. Note however, that this $\tilde{\psi}_1$ has a scale length of $\delta^{1/2}$. Thus $\partial\tilde{\psi}_1/\partial\chi \sim \delta^{1/2}\beta/\epsilon$. The second term is of order $\delta\Delta'$, which is equivalent to the first since $\delta^{1/2} \sim \beta/\epsilon\Delta'$. The constant does not contribute to (38) for axisymmetry. Thus,

$$\left[\bar{p}, \overline{\left[\frac{1}{q_0}\int d\zeta\,\tilde{h}, \tilde{\psi}\right]}\right] = [\bar{p}, h_g], \qquad (48)$$

where

$$\begin{aligned}h_g &= -\frac{1}{q_0}\left(\frac{\partial}{\partial\alpha}\int d\zeta\,\tilde{h}\right)\frac{\partial\tilde{\psi}}{\partial\chi} \\ &\cong -\bar{p}_1 f_0[\,\overline{|\nabla\chi|_0^{-2}\tilde{h}_0^2} - (\overline{\tilde{h}_0|\nabla\chi|_0^{-2}})^2/\overline{|\nabla\chi|_0^{-2}}\,] \\ &\quad - \frac{\partial\bar{\psi}_1}{\partial\chi}\overline{\tilde{h}_0|\nabla\chi|_0^{-2}}(\overline{|\nabla\chi_0|_0^{-2}})^{-1}.\end{aligned} \qquad (49)$$

Note that h_g depends only on the variations of h within a flux surface, and is therefore related to the geodesic curvature.

F. Grad–Shafranov equation

Having found the right-hand side of Eq. (39), we now determine its effect on $\bar{I}$ and $\bar{\psi}_1$. We have

$$[\psi_h, \bar{I}] = [\bar{p}, \bar{h} + h_g]. \qquad (50)$$

As discussed in part D, the left-hand side of Eq. (30) is negligible. Thus, for thin islands, $\bar{p}$ is a function only $\bar{I}$ and $\bar{\psi}_1$. We have

$$\bar{p} = \bar{p}(\psi_h). \qquad (51)$$

Equation (50) can be easily solved using this result. In view of Eq. (18a) and (18d),

$$[\bar{p}(\psi_h), \bar{h}\,h_g] = \frac{\partial\bar{p}}{\partial\psi_h}[\psi_h, \bar{h} + h_g] = \left[\psi_h, (\bar{h} + h_g)\frac{\partial\bar{p}}{\partial\psi_h}\right]. \qquad (52)$$

Therefore, Eq. (50) becomes $[\psi_h, \bar{I} - (\bar{h} + h_g)\times(\partial\bar{p}/\partial\psi_h)] = 0$, which has the solution

$$\bar{I} = \mathscr{I}(\psi_h) + (\bar{h} + h_g)\frac{\bar{p}}{\partial\psi_h}, \qquad (53)$$

for any arbitrary $\mathscr{I}$.

Many of the terms on the right-hand side in Eq. (53) are essentially functions of ψ_h, so we lump them into $\mathscr{I}$. In the vicinity of thin islands, $\bar{h}(\chi) = \bar{h}(\chi_0) + (\chi - \chi_0)\partial\bar{h}/\partial\chi_0$. Here $\bar{h}(\chi_0)$ can be lumped into $\mathscr{I}$ and similarly for $p_0(\chi)$ upon expansion. Also, $(\partial\bar{p}/\partial\psi_h)\bar{p}(\psi_h)$ can be lumped into $\mathscr{I}$. Thus,

$$\begin{aligned}\bar{I} = \mathscr{I}(\psi_h) &+ (\chi - \chi_0)\frac{\partial\bar{p}}{\partial\psi_h}g_1 \\ &- \frac{\partial\bar{p}}{\partial\psi_h}\overline{\tilde{h}_0|\tilde{\nabla}\chi|_0^{-2}}(\,\overline{|\nabla\chi|_0^{-2}}\,)\,\frac{\partial\bar{\psi}_1}{\partial\chi}^{-1},\end{aligned} \qquad (54)$$

where

$$g_1 = \frac{\partial \bar{h}}{\partial \chi_0} + \frac{\partial p_0}{\partial \chi} f_0 \overline{[\,|\nabla\chi|_0^{-2}\tilde{h}_0^2} - \overline{|\nabla\chi|^{-2}\tilde{h}_0^2}\,(\,\overline{|\nabla\chi_0^2|}\,)\,]^{-1}.$$

We now determine the equation for ψ_h, which is the Grad–Shafranov equation. Ampere's law gives

$$\frac{\partial^2}{\partial\chi^2}\bar{\psi}_1 = f_0\,\overline{|\nabla\chi|_0^{-2}}(\bar{I} - I_{\rm eq}) + \overline{f_0|\tilde{\nabla}\chi|_0^2\tilde{I}_\Delta}. \qquad (55)$$

Equation (45b) gives $\tilde{I}_\Delta$, so the second term on the right is

$$\overline{f_0|\tilde{\nabla}\chi_0|_0^{-2}\tilde{h}_0}\left(\frac{\partial\bar{\psi}_h}{\partial\chi}\frac{\partial\bar{p}}{\partial\psi_h} - \frac{\partial p_0}{\partial\chi}\right). \qquad (56)$$

To the requisite order, $\partial p_0/\partial\chi$ and $I_{\rm eq}$ may be taken to be constants, p_0' and I_0. Also, $\partial\psi_h/\partial\chi = \partial\psi_{h0}/\partial\chi + \partial\bar{\psi}_1/\partial\chi$. Recall $\partial\psi_{h0}/\partial\chi = 1 - q(\chi)/q_0 \cong -(\chi-\chi_0)q_0'/q_0$. The term involving $\partial\bar{\psi}_1/\partial\chi$ in Eq. (56) cancels the $\partial\bar{\psi}_1/\partial\psi$ term from Eq. (54). Finally, to obtain an expression for ψ_h, $\partial^2\psi_{h0}/\partial\chi^2$ is added to both sides, which can also be regarded as a constant. Defining $\chi' = \chi - \chi_0$, and lumping all the constants into a function $\mathscr{F}_*(\psi)$ yields

$$\frac{\partial^2}{\partial\chi'^2}\psi_h = \mathscr{F}_*(\psi_h) + \chi'(G_1+G_2)\frac{\partial\bar{p}}{\partial\psi_h}, \qquad (57)$$

where

$$G_1 = f_0\,\overline{|\nabla\chi|_0^{-2}}g_1, \quad G_2 = -f_0q_0'/q_0\overline{|\tilde{\nabla}\chi|_0^{-2}\tilde{h}_0}.$$

This equation is similar to the the Grad–Shafranov equation with gravity. Note that G_1 and G_2 correspond to the expressions in GGJ (to requisite order to ϵ),

$$E + F = p_0'(q_0/q_0')^2G_1, \quad H = p_0'(q_0/q_0')^2G_2,$$

with $p_0' = \partial p_0/\partial\chi$. The Mercier criterion for instability is $E+F+H > 1/4$.

IV. RESISTIVE ISLAND EVOLUTION

A. Determination of average island current

As in Rutherford's analysis, the abitrary function $\mathscr{F}$ in Eq. (54) is determined using Ohm's law,

$$E_\parallel = \eta j_\parallel, \qquad (58)$$

and Faraday's law,

$$\mathbf{E} = -\frac{\partial\mathbf{A}}{\partial t} - \nabla\phi, \qquad (59)$$

where ϕ is the electrostatic potential. In the aspect ratio expansion, and localized about the rational surface, Eqs. (58) and (59) yield

$$\frac{\partial\psi_1}{\partial t} + \left(\frac{\partial\phi}{\partial\zeta}\right)_\alpha + \frac{1}{q_0}[\psi + \psi_{h0},\phi\,] = f_0\eta I_\Delta. \qquad (60)$$

The average of this equation is

$$\frac{\partial\psi_h}{\partial t} + \frac{1}{q_0}([\,\psi_h,\bar{\phi}\,] + \overline{[\tilde{\psi},\tilde{\phi}\,]}) = f_0\eta\bar{I}_\Delta. \qquad (61)$$

The dominant terms of the ζ varying part give

$$\frac{\partial\tilde{\psi}_1}{\partial t} + \left(\frac{\partial\tilde{\phi}}{\partial\zeta}\right)_\alpha = f_0\eta\tilde{I}_\Delta. \qquad (62)$$

Using this, the $[\tilde{\psi},\tilde{\phi}\,]$ term in Eq. (61) can be seen to be small. The discussion in Sec. II indicated that $\partial\psi_h/\partial t$ balanced $\eta\bar{I}$ so $\partial/\partial t \sim \eta\delta^{-1/2}\Delta'$. This implies $\tilde{\phi}$ balances $\tilde{I}_\Delta$ in Eq. (62), so $\tilde{\phi} \sim \eta\beta/\epsilon$. Thus, $[\tilde{\psi},\tilde{\phi}\,] \sim \eta(\beta/\epsilon)^2\delta^{1/2}$, which is negligible compared to $\partial\bar{\psi}_1/\partial t$. Therefore, Eq. (61) becomes

$$\frac{\partial\bar{\psi}_1}{\partial t} + \frac{1}{q_0}[\,\psi_h,\bar{\phi}\,] = \eta f_0\bar{I}_\Delta. \qquad (61')$$

This is a two-dimensional equation which is essentially equivalent to Rutherford's expression. To eliminate the term $[\,\psi_h,\bar{\phi}\,]$, "flux" averages over surfaces of constant ψ_h are defined:

$$\langle f(\chi,\alpha)\rangle = \frac{\oint d\alpha\,[(\partial\psi_h/\partial\chi')^{-1}f\,]}{\oint d\alpha(\partial\psi_h/\partial\chi')^{-1}}. \qquad (62')$$

It is easily shown that $\langle[\,\psi_h,\bar{\phi}\,]\rangle = 0$ for any function $\bar{\phi}$, and $\langle f(\psi_h)\rangle = f(\psi_h)$. Averaging Eq. (61') and using Eq. (54) gives $\mathscr{F}(\psi)$, or equivalently,

$$\bar{I} = \frac{1}{f_0\eta}\left\langle\frac{\partial\bar{\psi}_1}{\partial t}\right\rangle + (\chi' - \langle\chi'\rangle)g_1\frac{\partial\bar{p}}{\partial\psi_h} - \frac{\partial\bar{p}}{\partial\bar{\psi}_h}\left(\frac{\partial\bar{\psi}_1}{\partial\chi'} - \left\langle\frac{\partial\bar{\psi}_1}{\partial\chi'}\right\rangle\right)\overline{\tilde{h}_0|\nabla\chi|_0^{-2}}(\,\overline{|\nabla\chi_0|^{-2}}\,)^{-1} \qquad (63)$$

[compare Eq. (63) and Eq. (5)]. Using Eqs. (63) and (55) we see that the island portion of the the Grad–Shafranov equation [Eq. (57)] becomes

$$\frac{\partial\bar{\psi}_1}{\partial\chi'^2} = \overline{|\nabla\chi|^{-2}}\,\frac{1}{\eta}\left\langle\frac{\partial\bar{\psi}_1}{\partial t}\right\rangle + (\chi' - \langle\chi'\rangle)G_1\frac{\partial\bar{p}}{\partial\psi_h} - G_2\frac{q_0}{q_0'}\frac{\partial\bar{p}}{\partial\psi_h}\left\langle\frac{\partial\bar{\psi}_1}{\partial\chi}\right\rangle + G_2\left(\chi'\frac{\partial\bar{p}}{\partial\psi_h} - \frac{q_0}{q_0'}\frac{\partial p_0}{\partial\chi_0}\right). \qquad (64)$$

Note the difference between the G_1 and G_2 terms which arises because only G_1 appears in Eq. (54). This distinction leads in the next part to the fact that G_2 does not contribute to resistive growth.

B. Approximate solution of island Grad–Shafranov equation

Equation (64) is now solved using two conventional approximations:

(1) We assume one harmonic in ψ_{h1} dominates (e.g., the most unstable one), so that [recall Eq. (28)]

$$\psi_h = \chi + \bar{\psi}_1 = -\chi'^2(q_0'/2q_0) + A\,(\chi',t\,)\cos m\alpha. \qquad (65)$$

(2) We assume that the "constant ψ" approximation is valid. This requires a subsidary expansion where $\Delta'\delta^{1/2} \sim G_1 \sim G_2$ is taken to be small.

Here A is obtained by operating on both sides of Eq. (64) with $(1/\pi)\oint d\alpha\cos m\alpha$ to obtain

$$\frac{\partial^2 A}{\partial\chi'^2} = \overline{|\nabla\chi|_0^{-2}}\,\frac{1}{\pi\eta}\oint d\alpha\cos m\alpha\left\langle\frac{\partial A}{\partial t}\cos m\alpha\right\rangle + \frac{G_1}{\pi}\oint d\alpha\cos m\alpha(\chi' - \langle\chi'\rangle)\frac{\partial\bar{p}}{\partial\psi_h} + \frac{G_2}{\pi}\oint d\alpha\cos m\alpha\left(\chi'\frac{\partial\bar{p}}{\partial\psi_h} - \frac{q_0p_0'}{q_0'}\right). \qquad (66)$$

Note that $\Delta'\delta^{1/2} \ll 1$ implies that the $\langle\partial\bar{\psi}_1/\partial\chi\rangle$ term in Eq. (64) can be dropped, and that $\partial\psi_h/\partial\chi \cong \chi'q_0'/q_0$. The island width is $\sim\delta^{1/2}$, so $A\sim\delta$. The right-hand side of Eq. (64) is $\sim\delta^{1/2}$. Thus, to lowest order, $A=$ const. The solubility condition for A in next order gives the evolution equation.

This will require matching to the exterior solution. In the exterior, $\psi_{h0}\gg\bar{\psi}_1$, and the flux surfaces are only slightly perturbed from the equilibrium surfaces. The right-hand side of Eq. (64) vanishes for large χ', but the pressure-driven terms vanish least slowly. Those terms can be straightforwardly evaluated for large χ', and they depend on the asymptotics of $\partial\bar{p}/\partial\psi_h$. Recalling that the pressure gradient $\partial p/\partial\chi$ must approach its equilibrium value $\partial p_0/\partial\chi_0$, the right-hand side of Eq. (66) approaches

$$-D_I A/\chi'^2,$$

where

$$D_I=p_0'(q_0/q_0')^2(G_1+G_2)=E+F+H.$$

Therefore, the solution of Eq. (64) in the exterior is

$$A\sim C_1\chi^{\alpha_1}+C_2\chi^{\alpha_2}, \tag{67}$$

where

$$\alpha_{1,2}=(1\pm\sqrt{1-4D_I})/2.$$

These are simply the Mercier interchange solutions, and this agrees with the resistive linear exterior solutions of GGJ.

The exterior solutions that satisfy the boundary conditions far from the island are characterized by a given value of C_1/C_2 on each side of $\chi'=0$. The relevant matching parameter is

$$\Delta'_*\equiv C_{1+}/C_{2+}-C_{1-}/C_{2-}. \tag{68}$$

Since G_1 and G_2 are taken small, $p_1\simeq D_I$ and $p_2\simeq 1-D_I$.

With $D_1=0$, Δ'_* becomes the discontinuity of the slope. For small D_I, keeping terms only to order D_I,

$$A\sim A_0(\chi'^{D_I}+C_1/C_2\chi'^{1-D_I}),$$

and

$$\frac{dA}{d\chi'}\sim A_0\chi'^{D_I}\left(\frac{D_1}{\chi'}+\frac{C_1}{C_2}(1-D_I)\chi'^{-2D_I}\right). \tag{69}$$

The lowest-order solution is a constant. The evolution equation is obtained by matching the next-order solution of Eq. (66) to the remainder of Eq. (69). The next-order solution A_1 is obtained by inserting A_0 into the right-hand side of Eq. (66) and integrating

$$\frac{\partial A_1}{\partial\chi'}=\int^{\chi'}d\chi'\left[\frac{1}{\eta\pi}\frac{\partial A_0}{\partial t}\,\overline{|\nabla\chi_0|^{-2}}\int d\alpha\cos m\alpha\langle\cos m\alpha\rangle\right.$$
$$+\frac{G_1}{\pi}\int d\alpha\cos m\alpha(\chi'-\langle\chi'\rangle)\frac{\partial\bar{p}}{\partial\psi_h}$$
$$\left.+\frac{G_2}{\pi}\int d\alpha\cos m\alpha\left(\chi'\frac{\partial\bar{p}}{\partial\psi_h}-\frac{q_0}{q_0'}\frac{\partial p_0}{\partial\chi_0}\right)\right]. \tag{70}$$

The asymptotic behavior of the G_1 and G_2 terms will give agreement with the D_1/χ term in Eq. (69). The matching to the C_1/C_2 term requires that $A_0\Delta'_*$ equal the integral from $-\infty$ to $+\infty$ of the right-hand side. These terms can be written explicitly using Eq. (65), and the G_2 term can be shown to vanish. It is clearer to write the result in terms of $\Delta\chi=4\sqrt{q_0/q_0'}\,A_0^{1/2}$, which is the island width:

$$\frac{k_1}{\eta}\frac{\partial}{\partial t}\Delta\chi=\Delta'_*\Delta\chi^{-2D_I}+\frac{k_2(E+F)}{\Delta\chi}, \tag{71}$$

where k_1 and k_2 are numerical constants. Explicitly,

$$k_1=\frac{1}{\sqrt{2}}\int_{-1}^{\infty}dw$$
$$\times\left[\left(\oint d\alpha\frac{\cos\alpha}{\sqrt{w-\cos\alpha}}\right)^2\left(\oint d\alpha\frac{1}{\sqrt{w-\cos\alpha}}\right)^{-1}\right], \tag{72}$$

which agrees with Rutherford, and

$$k_2=\frac{16}{\sqrt{2}}\int_1^{\infty}\frac{dW}{W^{1/2}}\left[\frac{\partial\bar{p}}{\partial\psi_h}(A_0W)\left(\frac{2q_0}{q_0'}A_0W\right)^{1/2}(p_0')^{-1}\right]$$
$$\times\int_0^{2\pi}d\alpha\frac{\cos\alpha}{\sqrt{W-\cos\alpha}}\left(\int_0^{2\pi}d\alpha\frac{1}{\sqrt{W-\cos\alpha}}\right)^{-1}. \tag{73}$$

In these expressions, W is ψ_h normalized so that $W=1$ corresponds to the separatrix. Note that only the region outside the separatrix contributes to the pressure-driving term. The expression in k_2 in [ ] depends on the pressure profile near the island, but it approaches 1 as $w\to\infty$ and is typically ~ 1 in the island region. In Appendix A, $\partial\bar{p}/\partial\psi_h$ is computed under the assumption that the pressure gradient is maintained by a constant pressure source and diffusion coefficient, with the result that the expression in [ ] is

$$\frac{2\pi W^{1/2}}{\int_0^{2\pi}d\alpha(\sqrt{W-\cos\alpha})}, \tag{74}$$

in which case

$$k_2=\frac{32\pi}{\sqrt{2}}\int_1^{\infty}dW\int_0^{2\pi}d\alpha\frac{\cos\alpha}{\sqrt{W-\cos\alpha}}$$
$$\times\left(\int_0^{2\pi}d\alpha\sqrt{W-\cos\alpha}\int_0^{2\pi}\frac{d\alpha}{\sqrt{W-\cos\alpha}}\right)^{-1}$$
$$\simeq 6.3. \tag{75}$$

Note that roughly half of this integral comes from distances in χ' which are 2/3 of an island width away from the island separatrix. Thus, it is not highly sensitive to the region near the separatrix.

V. CONCLUSIONS AND FURTHER APPLICATIONS

We have derived a the Grad–Shafranov equation, Eq. (57), to describe MHD equilibria in the vicinity of thin islands in tokamaks. The resistive evolution of the island width is given by Eq. (71). Note that the latter indeed resembles Eq. (7) since D_I is small. Thus the qualitative discussion of Sec. II is pertinent. In particular, we have shown that finite pressure effects, while affecting initial island growth, become irrelevant for island widths exceeding the Δx_c of Eq. (1).

The average curvature in tokamaks is usually favorable. In other configurations, such as reversed field pinches, the curvature is unfavorable. Resistive interchanges are likely to be unstable for such cases, and we believe that Eq. (71) describes the coherent evolution of these instabilities in the nonlinear phase. If Δ'_* is stabilizing, Eq. (1) gives the saturated island width for these modes. Of course, the analysis given here does not describe any further evolution if two islands overlap.

APPENDIX A: PRESSURE PROFILE CALCULATION

We assume that there is a diffusion process operating in the equilibrium, and pressure sources exist in the plasma interior. In steady state, the pressure gradient in the island region is found by the condition that the flux be a constant. With the island growing, the condition of constant flux still determines the pressure gradient if the diffusion coefficient D is sufficiently large that the local pressure equilibrates rapidly compared to the island growth rate, i.e., if $D/\Delta x^2 > \gamma$. From Eq. (71), $\gamma \sim \eta\Delta'/\Delta x$ so that the criterion is $D > \eta\Delta'\Delta x$. Even for the case $\Delta'\Delta x \sim 1$, this is satisfied if the pressure diffusion coefficient exceeds the classical magnetic diffusion coefficient.

We furthermore assume that D is constant and unchanged by the island. The flux Γ is

$$\Gamma = D\int d\mathbf{s}\cdot\nabla p = D\int \frac{d\alpha\, d\zeta}{J}\nabla\alpha\times\nabla\zeta\cdot\psi_h\frac{\partial\bar{p}}{\partial\psi_h}.$$

Thus, the constant flux condition in the island region is

$$\frac{\partial\bar{p}}{\partial\psi}\int d\alpha\,\chi = C$$

for some constant C. Therefore,

$$\frac{\partial\bar{p}}{\partial\psi_h} = \frac{C}{\int_0^{2\pi} d\alpha\,\sqrt{\psi_h - A_0\cos\alpha}}.$$

Equation (74) results by choosing the constant C to make the value of $\partial\bar{p}/\partial\chi$ far from the island agree with the equilibrium value.

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