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Structural Stability and Chaotic Solutions of Perturbed Benjamin-Ono Equations

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ABSTRACT

A method for proving chaos in partial differential equations is discussed and applied to the Benjamin-Ono equation subject to perturbations. The perturbations are of two types: one that corresponds to viscous dissipation, the so-called Burger's term, and one that involves the Hilbert transform and has been used to model Landau damping. The method proves chaos in the PDE by proving temporal chaos in its pole solutions. The spatial structure of the pole solutions remains intact, but their positions are chaotic in time. Melnikov's method is invoked to show this temporal chaos. It is discovered that the pole behavior is very sensitive to the Burger's perturbation, but is quite insensitive to the perturbation involving the Hilbert transform.

1. Introduction

It is by now well-known that the generic finite degree-of-freedom Hamiltonian system exhibits a variety of chaotic phenomena. However, the extension to partial differential equations of ideas such as Arnold diffusion or Smale horseshoes is a developing area of research. Since baisc models of physics, notably those that describe continuous media, are typically infinite dimensional, it is of interest and importance to investigate the mathematical relevance of nonlinear dynamics concepts. For example, turbulent flows may possess long-lived coherent structures that move and interact chaotically. We discuss a technique for proving chaos in the motion of such coherent nonlinear structures and apply it to the pertubed Benjamin-Ono (BO) equation, a model that possesses many features that are characteristic of turbulent motion.

The technique requires the existence of pole solutions; i.e., exact solutions that are a superposition of rational functions, with poles whose postions in the complex plane vary with time [see (3.1)]. Generally the pole dynamics is governed by a set of ordinary differential equations; thus when a partial differential equation possesses pole solutions the effective number of degrees of freedom is reduced from infinite to finite. If one can show that the pole ODE's exhibit chaos, then the "parent" partial differential equation exhibits a kind of temporal chaos where the spatial structure of a given pole does not change but its position is a random function of time. In practice we begin with an integrable system (in the sense of inverse scattering) with known pole solutions and then add on special perturbations that preserve the spatial structure of the pole solutions. In this way one obtains Hamiltonian equations for the pole dynamics that are composed of an integrable part plus a perturbation. Indeed such systems are generically chaotic and in some cases this can be shown by the method of Melnikov. [See e.g. Guckenheimer and Holmes ('83).]

Mathematically speaking this procedure amounts to showing that the perturbed partial differential equation has invariant manifolds, on which the infinite system reduces to a finite system of ODE's with pole solutions that are structurally unstable. In order to use the Melnikov method we must prove that the unperturbed pole solutions correspond to homoclinic orbits asymptotic to hyperbolic points at infinity, and that the perturbations "break" these orbits. Presumably, the same is true for the soliton solutions.

At first one might suspect that the procedure described above would only work in highly specialized situations, but in fact there exist a variety of systems with pole preserving perturbations. Previously Birnir ('86) has effected this procedure for perturbations of the Korteweg-de Vries (KdV) equation, while here we demonstrate it for the perturbed BO equation given by

$$(1) \qquad u_t = 2uu_x + H(u_{xx}) + \upsilon u_{xx} + \beta H(u_x) \ ,$$

where H is the Hilbert transform and the terms with the coefficients υ and β are the pertubations.

The unperturbed BO equation $(\upsilon = \beta = 0)$ was proposed by Benjamin ('67) and derived from the Navier-Stokes equation by Ono ('75). It describes long wavelength internal waves that propagate in a stratified fluid (e.g. a layer of cold current in the deep ocean). It was proven to be integrable by various authors: Joseph ('77), Meiss and Peirera ('78), Case ('78), Chen, Lee and Peirera ('79), Satsuma and Ishimori ('79), Nakamura ('79) and Ablowitz and Fokas ('81). Integrability was first shown numerically and then proven analytically, a development that was a repeat of the history of the KdV equation and also mathematically similar.

The perturbed BO equation has two additional physical effects. The term υu_{xx} corresponds to viscous dissipation. Physically one expects the viscosity υ to be greater than zero, thereby avoiding backward diffusion. We will set $\upsilon = \epsilon\eta$, explicitly displaying the small parameter ϵ, which we assume satisfies $1 \gg \epsilon > 0$, and consider two cases: (i) where η is a nonnegative constant and (ii) where η is an explicit oscillatory function of time. The latter case with η nonnegative conceivably could arise from fluid parameter fluctuations. The second term, $\beta H(u_x)$, appeared in a perturbed KdV equation that was derived by Ott and Sudan ('69) for modelling electron Landau damping of nonlinear ion acoustic waves. If β is a constant and positive then this term corresponds to Landau damping, while if β is negative it corresponds to growth. We set $\beta = \delta\mu$, with the small parameter δ satisfing $1 \gg \delta > 0$, and consider two cases: (i) where β is constant and (ii) where β has explicit oscillatory time dependence. The latter could serve as a phenomenological model of nonlinear electron Landau damping [see Meiss and Morrison ('83)].

Lee and Chen ('82) have considered Eq. (1), with υ and β constant ($\upsilon > 0$ and $\beta < 0$), as a nonlinear model of plasma turbulence. For this case they argue that the model possesses many features typical of plasma systems. In particular it has the usual quadratic nonlinearity and a linear dispersion relation that is dispersive, marginally stable for $k = 0$, unstable for small k, and damped for large k. Lee and Chen and Meiss ('80) showed that Eq. (1), even with time dependent υ and β, has pole solutions. Their discovery allows us to proceed with the technique described above.

The question of structural stability is determined by the phase space geometry of the pole solutions of the unperturbed BO equation. Investigating this geometry requires work, but fortunately the main tools were developed by Birnir (3, '86), where analogous results for the perturbed KdV equation were obtained. However, the Benjamin-Ono theory is quite different from that of KdV. First, the pole solutions of KdV are singular, with pole-like singularities, whereas the pole solutions of BO are smooth and physically relevant. Secondly, the BO invariant manifolds are unconstrained whereas the poles of the solutions of most other integrable equations, including KdV, are constrained to lie on a subset of the phase space. The third most important difference is that the fixed point at infinity of the BO solutions is elliptic whereas that of the KdV is hyperbolic. This makes it more difficult to apply the Melnikov method to BO than KdV.

There are also similarities between the pole solutions of KdV and BO. Both have a degenerate fixed point at infinity and the phase space of both equations can be completed by embedding them as constraints in the phase space of an N-body problem. For either equation, perturbations that are (i) composed of the Hilbert transform operator acting on lower order gradients and (ii) vector fields in the hierarchy associated with integrable PDE's, result in invariant manifolds; i.e. do not destroy the pole solutions. Perturbations of type (i) and (ii) are different in nature: the former produces chaotic pole solutions while the latter produces regular pole solutions, for <u>both</u> BO and KdV.

This paper is organized into nine sections. In Sec. 2, we list the Hamiltonians, gradients and vector fields for the infinite dimensional hierarchy of Benjamin-One equations. We then discuss the pole solutions in Sec. 3, and describe their shapes and symmetries. Section 4 contains the system of ODE's that determines the invariant manifold of the perturbed BO equation; here it is seen that these equations embed as constraints in an N-body system. We show in Sec. 5 that the unperturbed N-body system has a degenerate fixed point at infinity, which we designularize by a McGehee transformation [see Birnir (3, '86)], to produce the space S^{4n}, the 4n sphere, with a center (elliptic fixed point) on top. Indeed, the unperturbed BO solutions are neither homoclinic nor heteroclinic orbits, but periodic in the independent variable that desingularizes the fixed point. This is disastrous for the Melnikov method which requires a hyperbolic fixed point. Nevertheless, we show that with $\varepsilon = 0$ and $\delta \neq 0$; i.e. with viscous dissipation but without the "Landau" term, the center miraculously turns into a hyperbolic fixed point, whose stable and unstable manifolds are the perturbed BO solutions. Next, in Sec. 6 we give the stable and unstable manifold theorem for the Poincaré maps that arise in the case where the parameters are driven periodically. For η a periodic function one can get a fixed point whose stability changes with time. However, if we avoid the finitely many Poincaré maps that have centers, i.e. where $\eta(t) = 0$, then the stable and unstable manifold theorem applies. This technical difficulty is avoided if we restrict ourselves to the physically relevant case of a nonnegative diffusion coefficient, where η is periodic in time yet nonnegative. In Sec. 7 we compute the Melnikov functions, which are then used to prove the main results in Sec. 8, that the solutions are regular in the limit that δ vanishes, for $\varepsilon \neq 0$, and chaotic when $\delta \neq 0$ and η is a periodic function. Moreover, when $\delta \neq 0$, η is a constant, and $\varepsilon \neq 0$, then the driven "good" perturbation $H(u_x)$ can couple with the "bad" perturbation to produce chaos. Finally, we discuss the perturbation $H(u)$ and compare with the corresponding results for KdV.

2. Hamiltonians, Gradients and Vector Fields.

The Benjamin-Ono equations is a completely integrable Hamiltonian system. This is manifested in the existence of infinitely many conserved quantities, Hamiltonians H_k, the first few of which are listed in Table 1. The functional derivatives of the Hamiltonians produce gradients δH_k; for example, for H_2 we have

$$\dot{H}_2 = \int_{-\infty}^{\infty} \{(1/2)\dot{u}H(u_x)+(1/2)uH(\dot{u})+u^2\dot{u}\}dx$$

$$= \int_{-\infty}^{\infty} \{[H(u_x)+u^2]\dot{u}\}dx = \int_{-\infty}^{\infty} \delta H_2\dot{u}dx,$$

where we have used the skewness of the Hilbert transform and integration by parts.

The Hamiltonians are in involution which means that their Poisson brackets vanish

$$\{H_k,H_j\} = \int_{-\infty}^{\infty} \delta H_k D_x \delta H_j dx = 0, \qquad \text{for all } k \text{ and } j,$$

where $D_x = d/dx$ is the Poisson structure. This implies that the <u>vector fields</u> $X_k(u)$ which are defined by Hamilton's equations,

$$u_t = D_x\delta H_k(u) = X_k(u), \quad (k = 1,....),$$

commute. In other words, there exists a weak symplectic form

$$\Omega(v,w) = 1/2 \int_{-\infty}^{\infty} \int_{-\infty}^{x} \{v(y)w(x)-v(x)w(y)\}dydx$$

defined on the tangent bundle of smooth functions on (v,w) R^1, which vanish at infinity. This symplectic form vanishes on any two vector fields that are generated from the Hamiltonians; i.e.

$$\Omega(X_k(u),X_j(u)) = \int_{-\infty}^{\infty} \delta H_k D_x \delta H_j dx = 0, \qquad \text{for all } k \text{ and } j.$$

The first four gradients and vector fields are listed in Table 1.

The level sets

$$H_k = h_k, \qquad (k = 1,\ldots),$$

are infinte dimensional, possibly degenerate, tori whose tangent bundle is spanned by the vector fields X_k. These tori foliate phase space. Finite dimensional tori are also dense in phase space and we restrict our attention to those below.

Table 1

Order k	Hamiltonians H_k
0	$\int_{-\infty}^{\infty} u\,dx$
1	$1/2 \int_{-\infty}^{\infty} u^2 dx$
2	$\int_{-\infty}^{\infty} \{1/3u^3 + 1/2uHu_x\} dx$
3	$\int_{-\infty}^{\infty} \{1/4u^4 + 3/4u^2Hu_x + 1/2u^2_x\} dx$
4	$\int_{-\infty}^{\infty} \{1/5u^5 + 3/2uu^2_x + 1/2u_xHu_{xx} + 1/2u^2Huu_x + 2/3u^3Hu_x + 1/2u(Hu_x)^2\} dx$

Order k	Gradients δH_k	Vector Fields X_k
0	1	0
1	u	u_x
2	$Hu_x + u^2$	$Hu_{xx} + 2uu_x$
3	$-u_{xx} + 3uHu_x + u^3$	$-u_{xxx} + 3u_xHu_x + 3uHu_{xx} + 3uu_x$
4	$-Hu_{xxx} + uHuu_x - uu_xHu$ $+2u^2Hu_x - 2Hu^2u_x + 1/2(Hu_x)^2$ $+H(u_xHu_{xx}) + H(uHu_{xx})$ $-3/2u_x{}^2 - 3uu_x + u^4$	$-Hu_{xxxx} + u_xHuu_x + uHu_x{}^2 + uHuu_{xx}$ $-u_x{}^2Hu - uu_{xx}Hu - uu_xHu_x + 4uu_xHu_x$ $+2u^2Hu_{xx} - 4Huu_x{}^2 - 2Hu^2u_{xx} + Hu_xHu_{xx}$ $+H(u_{xx}Hu_x) + 2H(u_xHu_{xx}) + H(uHu_{xxx})$ $-3u_xu_{xx} - 3u_x{}^2 - 3uu_{xx} + 4u_xu^3$

3. Rational Pole Solutions.

Matsuno ('79) computed rational (pole) solutions of the Benjamin-Ono equation by the method developed by Hirota ('76) to change integrable non-linear PDE's into bilinear equations. Let

$$u(x,t) = iD_x \ln\{f(x,t)/\bar{f}(x,t)\},$$

where the analytic continuation of f off the real x-axis has zeroes in the lower half plan. Then substitute u into the BO equation to get the equation

$$(iD_t - D_x^2)f'\bar{f} = 0.$$

where $D_t = d/dt' - d/dt$, $D_x = d/dx' - d/dx$ and after performing the differentiation you set $t = t'$ and $x = x'$.

Now if we look for algebraic solutions to the bilinear equation, the result is the following formula

$$f(x,t) = \sum_{m_k=0,1} \prod_{k=1} (x-c_k t)^{m_k} i^{|1-m|} \prod_{j>k} 4^{(1-m_k)(1-m_j)} \tag{1}$$
$$\frac{1}{c_k^{(1-m_k)} c_j^{(1-m_j)}} \left[\frac{(c_k+c_j)}{(c_k-c_j)}\right]^{2(1-m_k)(1-m_j)},$$

where the sum is over all vectors m, of length n, with entries 0 or 1, the c_k's are arbitrary parameters and $|m| = \Sigma m_k$. The first few of the algebraic solutions are listed in Table 2 and their shape and time evolution is shown in Figure 1.

The zeroes of the second algebraic solution are

$$x_1(t) = \tfrac{1}{2}(c_1+c_2)t - \frac{i(c_1+c_2)}{2\, c_1 c_2}$$
$$- \tfrac{1}{2}(c_1-c_2)\sqrt{(t-2i/c_1c_2)^2 + 16/(c_1-c_2)^4}$$

$$x_2(t) = \frac{1}{2}(c_1+c_2)t - \frac{i(c_1+c_2)}{2\ c_1c_2}$$

$$+ \frac{1}{2}(c_1-c_2)\sqrt{(t-2i/c_1c_2)^2+16/(c_1-c_2)^4}\ ,$$

where $f_2(x,t) = (x - x_1(t))(x - x_2(t))$. This suggests that we look for solutions of the form

$$f_{4n} = \prod_{k=1}^{n} \prod_{j=0}^{3} [(x-at-ib)-g_k\Phi^j\sqrt{(t-ie_k)^2+f_k^2}]$$

$$= \prod_{k=1}^{n} [(x-at-ib)^4-g_k^4((t-ie_k)^2+f_k^2)^2],$$

Φ being a fourth root of unity, where a, b, g_k, e_k and f_k are constants. If the number of poles in not a product of 4, f_n can be determined from

$$f_{4n+1} = f_{4n}f_1, \quad f_{4n+2} = f_{4n}f_2, \quad f_{4n+3} = f_{4n}f_2f_1,$$

with the f's from Table 2. Apart from the linear factors $at+ib$, the zeroes move in groups of 4, except for the possible leftover triplet, in the complex plane. Each group expands as fourth roots of unity. These functions will be used below along with their t derivaties

$$y_{4k-j}(t) = g_k\Phi^j\sqrt{(t-ie_k)^2+f_k^2} \tag{2}$$

$$z_{4k-j}(t) = g_k\Phi^j(t-ie_k)/\sqrt{(t-ie_k)^2+f_k^2},$$

$$(k = 1,\ldots), \quad (j = 0,..,4),$$

to compute the Melnikov function. Notice that the full solution of the PDE also requires the complex conjugates of these functions, and that the zeros of f and $\overline{f}$ become the poles of the BO solutions.

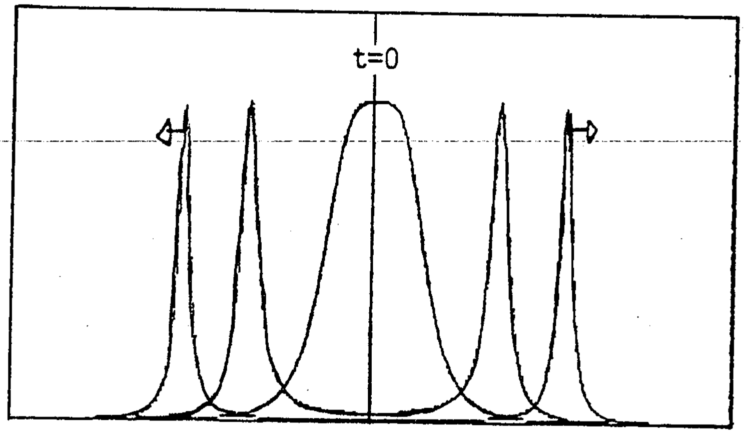
Remark 1: The 4 symmetry implies that the 8n poles in the complex plane combine to make 2n humps on the lines, see Figure 1. The above solutions to BO can be expressed as

$$u(x,t) = 4i\sum_{k=1}^{n}(x-at-ib)^3/[(x-at-ib)^4-g_k^4((t-ie_k)^2+f_k^2)^2]$$

$$-4i\sum_{k=1}^{n}(x-at+ib)^3/[(x-at+ib)^4-g_k^4((t+ie_k)^2+f_k^2)^2],$$

where use has been made of the 4-symmetry.

Figure 1.



t=3

t=7

$u(x_1t) = iD_x \ell n\{f_1/\bar{f}_1\}$

The first pole solution

Table 2.

$$f_1 = i(x-c_1t)+1/c_1$$

$$f_2 = (i(x-c_1t)+1/c_1)(i(x-c_2t)+1/c_2)+4/(c_1-c_2)^2$$

$$f_3 + (i(x-c_1t)+1/c_1)(i(x-c_2t)+1/c_2)(i(x-c_3t)+1/c_3)$$
$$+4[(i(x-c_3t)+1/c_3)/(c_2-c_1)^2+(i(x-c_2t)+1/c_2)/(c_3-c_1)^2$$
$$+(i(x-c_1t)+1/c_1)/(c_2-c_3)^2]+16[1/(c_1-c_2)^2(c_2-c_3)^2$$
$$+1/(c_3-c_1)^2(c_1-c_2)^2+1/(c_2-c_3)^2(c_3-c_1)^2]$$

4. Invariant Manifolds.

An alternate way of writing the rational solutions of BO is to write them explicitly as pole solutions

$$u(x,t) = i \sum_{k=1} [\alpha(x - x_k)^{-1} - \bar{\alpha}(x - \bar{x}_k)^{-1}].$$

These functions are meromorphic functions of x and their time dependence is through the poles, which move in the complex plane. When u is substituted into BO, the PDE reduces to a finite system of nonlinear ODE's. The surprising result that was, as noted above, discovered by Meiss ('80) and Chen and Lee ('82) is that the perturbed BO equation also has finite dimensional invariant manifolds.

<u>Theorem 1</u>: The perturbed Benjamin-Ono equation

$$u_t = H(u_{xx}) + 2uu_x + \delta\eta(t)u_{xx} + \varepsilon\mu(t)H(u_x)$$

has finite dimensional invariant manifolds, which are described <u>exactly</u> by the nonlinear system of ODE's

$$\dot{x}_k = \sum_{j\neq k}^{n} (i-\delta\eta)(x_k - x_j)^{-1} - \sum_{j=1}^{n} (i-\delta\eta)(x_k - \bar{x}_j)^{-1} + i\mu(t)$$

$$\dot{\bar{x}}_k = \sum_{j\neq k}^{n} (i+\delta\eta)(\bar{x}_k - \bar{x}_j)^{-1} - \sum_{j=1}^{n} (i+\delta\eta)(\bar{x}_k - \bar{x}_j)^{-1} - i\mu(t)$$

$$(k+1,\ldots,n).$$

<u>Proof</u>: The ODE's determine the motion of the poles of u in the complex plane. We substitute the pole solutions, with $\alpha = 1+i\delta\eta$, into the PDE, then for fixed t,

$$f(x,\cdot) = u_t - H(u_{xx}) - 2uu_x - \delta\eta(\cdot)u_{xx} - \varepsilon\mu(\cdot)H(u_x)$$

is a function on the Riemann sphere. It has no poles if the ODE's are satisfied, moreover it vanishes at infinity. Therefore it must vanish identically, but this holds for any t, so u is a solution of the PDE if and only if the ODE's are satisfied. Birnir (3, '86) contains more details. QED.

Remark 2: The fact that u has first order poles means that there are no locus conditions restricting the motion of the poles, on the copy of R^{4n}, in C^{4n}, where the poles move because of the conjugation symmetry. This has important consequences for the perturbation $H(u)$ of the BO equation, which will be discussed Section 8.

The existence of invariant manifolds reduces the analysis of perturbations of PDE's into questions about the structural stability of a finite dimensional phase flow. We can apply the qualitative theory of dynamical systems to answer these questions and we hope that the unperturbed BO flow of rational solutions is a homoclinic orbit of a fixed point at infinity. However, to show that this orbit is structurally unstable, we must examine it within a system containing orbits that are not homoclinic. This is accomplished by embedding the perturbed BO system in an N-body system.

Lemma 1: The perturbed Benjamin-Ono system of ODE's embeds as constraints in the perturbed N-body system:

$$\ddot{x}_k = 8(i+\delta\eta(t))^2 \sum_{j\neq k} (x_k-x_j)^{-3} - 16i\delta\eta(t) \sum_{j=1}^{n} (x_k-\bar{x}_j)^{-3} +$$

$$-i\delta\dot{\eta}(t)\left[\sum_{j\neq k}^{n} (x_k-x_j)^{-1} + \sum_{j=1}^{n} (x_k-\bar{x}_j)^{-1}\right] + i\varepsilon\dot{\mu}(t)$$

$$\ddot{\bar{x}}_k = 8(i-\delta\eta(t))^2 \sum_{j\neq k}^{n} (\bar{x}_k-\bar{x}_j)^{-3} + 16i\delta\eta(t) \sum_{j=1}^{n} (\bar{x}_k-x_j)^{-3} +$$

$$+i\delta\dot{\eta}(t)[\sum_{j\neq k}^{n} (\bar{x}_k-\bar{x}_j)^{-1} + \sum_{j=1}^{n} (\bar{x}_k-x_j)^{-1}] - i\varepsilon\dot{\mu}(t)$$

$$(k = 1,\ldots n).$$

<u>Proof</u>: Differentiate the perturbed BO system with respect to t and simplify using the identity

$$(x_k-x_j)^{-2}(x_k-x_m)^{-1} + (x_k-x_j)^{-1}(x_k-x_m)^{-2}$$
$$= (x_k-x_m)^{-1}[(x_k-x_j)^{-2} - (x_k-x_m)^{-2}]. \qquad \text{QED.}$$

<u>Remark 3</u>: The above N-body system admits, but is not restricted to, the cases of physical interest discussed in the Introduction. In particular for the case when η is a nonnegative constant, the system simplifies. If $\eta > 0$ and we drive the system with $\mu(t)$ periodic and nonpositive, then we have an interesting damped-driven system.

5. The Desingularized Phase Space.

In this section we investigate the structure of the phase space of the unperturbed N-body equations. The form of the algebraic solutions, in Section 3, suggests that we look for solutions of the form

$$x_k(t) = at + ib + y_k(t).$$

In these coordinates the unperturbed N-body system becomes

$$\dot{y}_k = z_k$$

$$\dot{z}_k = 8\sum_{j\neq k} (y_k-y_j)^{-3}$$

$$\dot{\bar{y}}_k = \bar{z}_k \qquad\qquad (k = 1,\dots,n).$$

$$\dot{\bar{z}}_k = 8\sum_{j\neq k} (y_k-y_j)^{-3}$$

This is a dynamical system on a copy of R^{4n} in C^{4n}, because of the complex symmetry. To make life easier we make a further simplification

$$y_k(t) = d_k y(t), \qquad\qquad (k = 1,..,n).$$

This amounts to a restriction of the parameters in Section 3, $e_k = e$ and $f_k = f$, however the same analysis goes through in the general case, although with more involved computations. Now the N-body system simplifies and becomes

$$\dot{y}_k = z_k$$

$$\dot{z}_k = 16y_k^{-3}$$

$$\dot{\bar{y}}_k = \bar{z}_k$$

$$\dot{\bar{z}}_k = 16\bar{y}_k^{-3}$$

where the conditions

$$d_k = 1/2 \sum_{j \neq k} (d_k - d_j)^{-3}, \qquad (k = 1,..,n),$$

require the coefficients d_k to be constant multiples of the fourth roots of unity. We have written the full 4n dimensional system, instead of its generic 4 dimensional system, because the perturbation will destroy that symmetry and make the system 4n dimensional again.

Lemma 2: The phase space of the N-body problem is the 4n dimensional sphere S^{4n}. Infinity, $z_k, \bar{z}_k = 0$, $y_k, \bar{y}_k = \infty$, is a degenerate fixed point.

Proof: Consider the kth equation

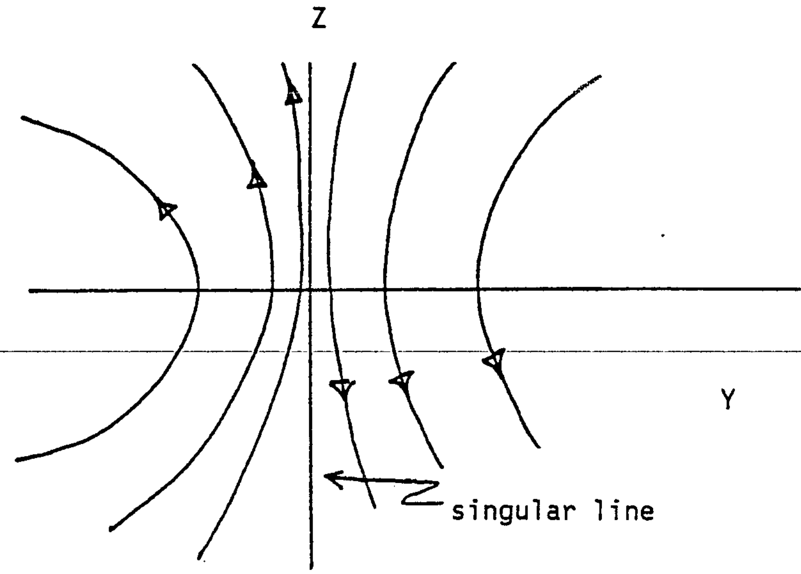
$$\left.\begin{aligned} \dot{y}_k &= z_k \\ \dot{z}_k &= 16 y_k^{-3} \end{aligned}\right\} = f(y_k, z_k).$$

We form a sphere by identifying the lines $y_k = \pm\infty$ and shrinking the lines $z_k = \pm\infty$ to a point. Then $y_k = 0$, is a fixed point. It is badly degenerate

$$\det Df_k = 16 y_k^{-3} = 0,$$

at $y_k = \infty$. The flow in the y_k, z_k plane is shown in Figure 2. QED.

Figure 2.



The flow in the y-z plane.

The degeneracy of the fixed point is a serious problem, since the Melnikov method requires a nondegenerate hyperbolic point. However, the situation is saved by a technique invented by McGehee ('73) to solve the Sitnikov problem and adapted to the KdV equation by Birnir (3,'86).

Lemma 3: A McGehee transformation gives a nondegenerate center at infinity.

Proof: We change variables to local coordinates at infinity and reparametrize time,

$$q_k = y_k^{-1} \quad \text{and} \quad \frac{dt}{dr} = y^2,$$

then

$$\left.\begin{aligned} \frac{dq_k}{dr} &= -(y^2/y_k^2)\dot{y}_k = -\dot{y}_k/d_k^2 = p_k \\ \frac{dp_k}{dr} &= -(y^2/d_k^2)\dot{z}_k = -16y_k^{-1} = -16q_k \end{aligned}\right\} = g(q_k,p_k).$$

Now

$$Dg_k = \begin{bmatrix} 0 & 1 \\ -16 & 0 \end{bmatrix};$$

with eigenvalues $\tau_k = \pm 4i$. Similarly $\bar{\tau}_k = \mp 4i$. This works for any k.

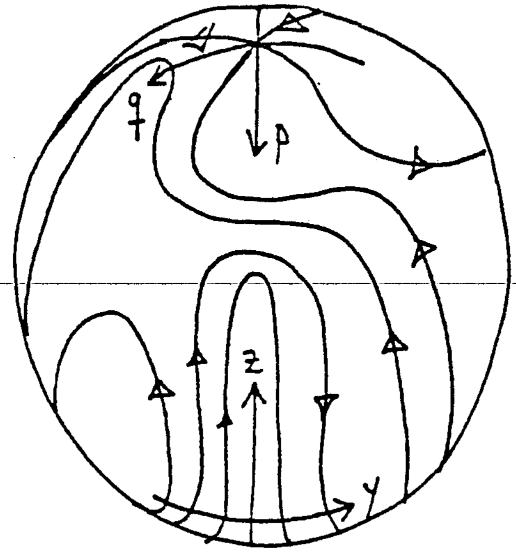
QED.

<u>Proposition 1</u>: The phase space of the desingularized unperturbed N-body flow is the 4n dimensional sphere S^{4n}, lying by the conjugation symmetry in S^{8n}. The z axis is singular and infinity is a 2n fold <u>center</u>.

<u>Proof</u>: We take an atlas consisting of two charts, one S^{4n} punctured at infinity with coordinates (x,y) and the other S^{4n} punctured at the origin with coordinates (q,p). These coordinate pairs are smooth functions of each other on the overlap of the charts and we can continue orbits smoothly from one onto the other. Invoking Lemmas 2 and 3 finishes the proof. The flow on the kth sphere is shown in Figure 3.

QED.

Figure 3.



The desingularized flow in S^1

Proposition 1 is very disappointing. We want to show that perturbed BO has chaotic solution but for that we need a hyperbolic fixed point not a center. However, the N-body flow on S^{4n} is still structurally unstable and there exist arbitrarily small perturbations that change the center, at infinity, into a hyperbolic fixed point.

<u>Proposition 2</u>: The perturbed N-body flow, $\varepsilon = 0$, $\delta \neq 0$, $\eta = 1$, on S^{4n}, has a 2n fold hyperbolic fixed point at infinity. Moreover, the corresponding perturbed Benjamin-Ono solutions are the stable and unstable manifolds of that fixed point.

<u>Proof</u>: The McGehee transformation of the perturbed N-body equations produces the nonlinear system at infinity

$$\frac{dq_k}{dr} = p_k$$

$$\frac{dp_k}{dr} = -16(1+i\delta)^2 q_k + 16i\delta \sum_{j=1}^{n} (2ibq_k\bar{q}_j+\bar{q}_j-q_k)^{-3}\bar{q}_j^3 q_k$$

$$(k = 1,..,n),$$

along with its complex conjugate. The derivatives of the nonlinear terms vanish at infinity so the linearized vector field becomes

$$Dg_k = \begin{bmatrix} 0 & 1 \\ -16(1+i\delta)^2 & 0 \end{bmatrix} ,$$

with eigenvalues $\tau = \pm 4(i-\delta)$, similary $D\bar{g}_k$ has eigenvalues $\tau = \mp 4(i-\delta)$. It remains to prove that the perturbed Benjamin-Ono solutions are exponentially contracting or expanding at infinity. But BO corresponds to the constraints

$$p_k = a + 2(-i+\delta)1_k q_k + (i+\delta) \sum_{j=1} (2ib\bar{q}_k q_j+\bar{q}_j-q_k)^{-3}\bar{q}_j q_k ,$$

$$(k = 1,..,n),$$

where the constants

$$(-i+\delta)1_k = (-i+\delta) \sum_{j\neq k} d_k(d_k-d_j)^{-1} ,$$

have a positive real part.

Now $p_k = dq_k/dr$, so the constraints are first order equations for the q's. We rewrite the constraints as integral equations to see that the solutions are exponentially expanding. The $\bar{q}$'s are similar. QED.

<u>Remark 4</u>: The unperturbed Benjamin-Ono solutions form only one of the orbits, of the N-body flow around the center at infinity, which are periodic in the desingularization parameter. But the perturbations pick

precisely this orbit, out of the continuum of periodic orbits, to make the stable and unstable manifolds of the hyperbolic point at infinity.

6. Persistence of Stable and Unstable Manifolds

In this section we state the technical lemmas that allow us to apply the Melnikov method. We use the stable and unstable manifold theorem for hyperbolic fixed points, first with η a constant (=1) and then with η a periodic function.

Consider the perturbed system in McGehee's coordinates

$$
(1)\quad
\begin{aligned}
q'_k &= p_k \\
p'_k &= -16(1-\delta 1_k\eta'+i\delta\eta)q_k + 16i\delta\eta \sum_{j=1} (2ibq_k\bar{q}_j+\bar{q}_j-q_k)^{-3}\bar{q}_j^3 q_k \\
&\quad + \delta\eta' \sum_{j=1} (2ibq_k\bar{q}_j+\bar{q}_j-q_k)^{-1}\bar{q}_j q_k - i\varepsilon\mu'. \\
\bar{q}'_k &= \bar{p}_k \\
\bar{p}'_k &= -16(1-\delta\bar{1}_k\eta'-i\delta\eta)\bar{q}_k - 16i\delta\eta \sum_{j=1} (-2ib\bar{q}_k q_j+q_j-\bar{q}_k)^{-3} q_j^3\bar{q}_k \\
&\quad + \delta\eta' \sum_{j=1} (-2ibq_k q_j+q_j-\bar{q}_k)^{-1} q_j\bar{q}_k + i\varepsilon\mu'. \\
&\qquad (k = 1,..,n),
\end{aligned}
$$

where ' denotes d/dr, and the 1's are the constants in Proposition 2.

<u>Lemma 4</u>: Suppose $\delta \neq 0$ and $\eta = 1$, then the perturbed system $\varepsilon \neq 0$ has a hyperbolic periodic orbit, whose stable and unstable manifolds are C^∞ close to those of the unperturbed system, $\varepsilon = 0$. Let $x = (q,p)$, then the perturbations expansions

$$x^s(r,r_0,\delta) = x_0^s(r-r_0,\delta) + \varepsilon x_1^s(r,r_0,\delta) + O(\varepsilon^2),\ r \in [r_0,\infty)$$
$$x^u(r,r_0,\delta) = x_0^u(r-r_0,\delta) + \varepsilon x_1^u(r,r_0,\delta) + O(\varepsilon^2),\ r \in (-\infty,r_0]$$

converge uniformly in the indicated time intervals.

Phase space S^{4n} of the perturbed N-body system with $\varepsilon = 0$, $\eta = 1$, and $\delta \neq 0$, consists of two charts as discussed in the proof of Theorem 2. On the chart convering infinity, r parameterizes orbits, whereas t parameterizes the orbits on the chart covering the origin.

The stable and unstable manifolds pass smoothly from one chart to the other. We can choose whether we use t or r to compute the Melnikov function, but t is more natural because its Poincaré map is based on the driving period. Consequently, we restate Lemma 4 in the y and z coordinates, and use those to compute the Melnikov functions.

Lemma 5: With $\delta \neq 0$ and $\eta = 1$, the stable and unstable manifolds of the hyperbolic periodic orbit, of the perturbed system, $\varepsilon \neq 0$, are C^∞ close to those of the unperturbed system, $\varepsilon = 0$. Let $x = (y,z)$, then the perturbation expansions

$$x^s(t,t_0,\delta) = x_0^s(t-t_0,\delta) + \varepsilon x_1^s(t,t_0,\delta) + O(\varepsilon^2),\ t \in [t_0,\infty)$$

$$x^u(t,t_0,\delta) = x_0^u(t-t_0,\delta) + \varepsilon x_1^u(t,t_0,\delta) + O(\varepsilon^2),\ t \in (-\infty,t_0]$$

converge uniformly in the indicated time intervals.

Now consider the perturbed system

$$\begin{aligned}
\dot{y}_k &= z_k \\
\dot{z}_k &= 8(1+i\delta\eta(t))^2 \sum_{j\neq k}^{n} (y_k-y_j)^{-3} - 16i\delta\eta(t) \sum_{j=1}^{n} (2ib+y_k-\bar{y}_j)^{-3} \\
&\quad -\delta\dot{\eta}(t)[\sum_{j\neq k}^{n} (y_k-y_j)^{-1} + \sum_{j=1}^{n} (2ib+y_k-\bar{y}_j)^{-1}] + i\varepsilon\dot{\mu}(t)
\end{aligned} \tag{2}$$

$$\begin{aligned}
\dot{\bar{y}}_k &= z_k \\
\dot{\bar{z}}_k &= 8(1-i\delta\eta(t))^2 \sum_{j\neq k}^{n} (\bar{y}_k-\bar{y}_j)^{-3} + 16i\delta\eta(t) \sum_{j=1}^{n} (-2ib+\bar{y}_k-y_j)^{-3} \\
&\quad -\delta\dot{\eta}(t)[\sum_{j\neq k}^{n} (\bar{y}_k-\bar{y}_j)^{-1} + \sum_{j=1}^{n} (-2ib+\bar{y}_k-y_j)^{-1}] - i\varepsilon\dot{\mu}(t).
\end{aligned}$$

We want to set $\varepsilon = 0$ and say that the perturbed solution $x_\delta(t) = (y_\delta,z_\delta)$ is close to the unpperturbed solutions $x_0(t)$ that we know explicitly.

This would enable us to use x_0 to compute the Melnikov function. However, when $\delta = 0$, the fixed point becomes a center and the stable and unstable manifolds disappear. We use a simple trick to get around this difficulty. Instead of the above system consider the system

$$\dot{y}_k = z_k$$

$$\dot{z}_k = 8(1+i\sigma\eta(t))^2 \sum_{j\neq k}^{n} (y_k-y_j)^{-3} - 16i\delta\eta(t) \sum_{j=1}^{n} (2ib+y_k-\bar{y}_j)^{-3}$$

$$-\delta\dot{\eta}(t)[\sum_{j\neq k}^{n} (y_k-y_j)^{-1} + \sum_{j=1}^{n} (2ib+y_k-\bar{y}_j)^{-1}] + i\varepsilon\dot{\mu}(t)$$

(3)

$$\dot{\bar{y}}_k = \bar{z}_k$$

$$\dot{\bar{z}}_k = 8(1-i\sigma\eta(t))^2 \sum_{j\neq k}^{n} (\bar{y}_k-\bar{y}_j)^{-3} + 16i\delta\eta(t) \sum_{j=1}^{n} (-2ib+\bar{y}_k-y_j)^{-3}$$

$$-\delta\dot{\eta}(t)[\sum_{j\neq k}^{n} (\bar{y}_k-\bar{y}_j)^{-1} + \sum_{j=1}^{n} (-2ib+\bar{y}_k-y_j)^{-1}] - i\varepsilon\dot{\mu}(t).$$

$$(k = 1,..,n),$$

where we have substituted the fixed parameter σ in for δ in the first component of the derivatives of z_k and $\bar{z}_k$.

Now the stable and unstable manifold theorem applies:

<u>Lemma 6</u>: When $\varepsilon = 0$, $\sigma \neq 0$ is fixed and $\eta = 1$, the perturbed system $\delta \neq 0$, has a hyperbolic periodic orbit whose stable and unstable manifolds are C^∞ close to their unperturbed counterparts. Let $x = (y,z)$, then the perturbation expansions

$$x^s(t,t_0,\delta) = x_0^s(t-t_0,\delta) + \varepsilon x_1^s(t,t_0,\delta) + O(\varepsilon^2),\ t \in [t_0,\infty)$$
$$x^u(t,t_0,\delta) = x_0^u(t-t_0,\delta) + \varepsilon x_1^u(t,t_0,\delta) + O(\varepsilon^2),\ t \in (-\infty,t_0]$$

converge uniformly in their time intervals.

Proof: The existence of the stable and unstable manifolds in the (q,p) coordinates implies their existence in the (y,z) coordinates, see McGehee ('73) or Birnir (3,'86).

Finally we are interested in what happens when the diffusion term u_{xx} is driven or when $\eta(t)$ is a periodic function. This situation is more complicated, because the hyperbolic periodic orbit changes its stability during a (driving) period. There are finitely many points t_k where $\eta(t_k) = 0$ and the Poincaré map has a hyperbolic fixed point, except at the times t_k, when its eigenvalues move through the unit circle, see Figure 4. However, we will be able to apply the Melnikov method by avoiding the exceptional transversals:

$$\Sigma_k = \{(x,t) \in S^{4n} \times R \quad t=t_k, \ 0 < k < N+1\}.$$

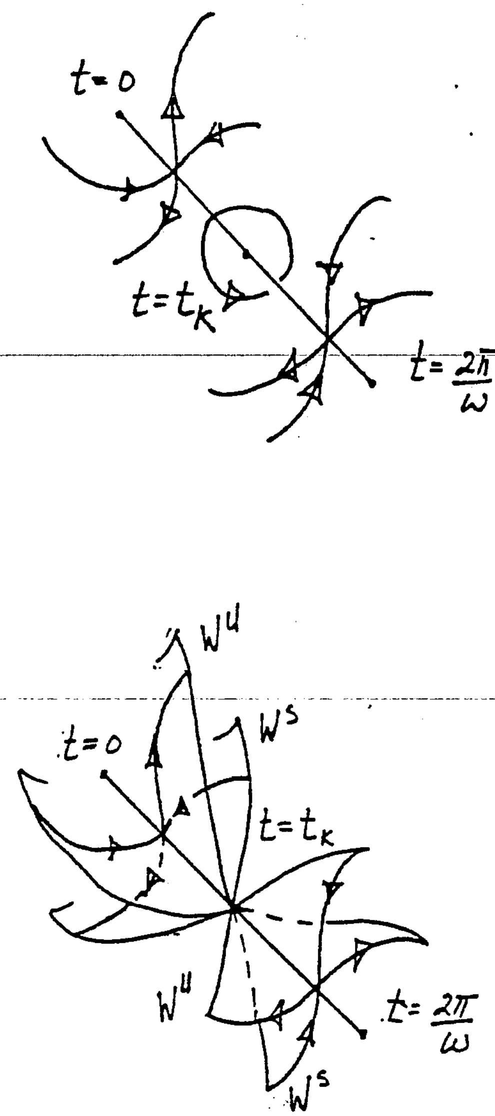
Lemma 7: With $\varepsilon \neq 0$, $\sigma \neq 0$ fixed and $\eta(t)$ a periodic function, the perturbed system, $\delta \neq 0$, has a periodic orbit that is hyperbolic during a period except at finitely many time values t_k, $0 < k < N+1$, where the corresponding Poincaré map has a center. Apart from these exceptional times, the stable and unstable manifolds are C^∞ close to those of the unperturbed, $\delta = 0$, system. The perturbations expansions

$$x^s(t,t_0) = x_0^s(t-t_0) + \delta x_1^s(t,t_0) + O(\delta^2), \ t \in [t_0,\infty)$$
$$x^u(t,t_0) = x_0^u(t-t_0) + \delta x_1^u(t,t_0) + O(\delta^2), \ t \in (-\infty,t_0]$$

converge uniformly in their time intervals.

Now the trick is to set $\sigma = \delta$, for fixed $\delta \neq 0$, then the above systems are identical to the perturbed Benjamin-Ono systems.

Figure 4.



The stable and unstable manifolds

of the fixed point of the Poincaré map.

<u>Remark 5</u>: Recall from Remark 3 that the physically most interesting case is η nonnegative and μ nonpositive. Lemma 6 applies to the case namely η = constant, and μ periodic. In particular, μ could be a negative constant plus an oscillatory function of a smaller amplitude. In Lemma 7, if η is a similar but positive function, then there are no exceptional time values and the eigenvalues of the Poincaré map never cross the unit circle.

7. The Melnikov Method.

The Melnikov function $M(t_0)$ measures the distance between the stable and unstable manifolds of the hyperbolic fixed point of the Poincaré map. If $M(t_0)$ has a simple zero then these intersect transversely, see Guckenheimer and Holmes ('83), and this results in chaotic solutions of the original system.

First we show that the Melnikov function associated with the perturbed BO equation (setting $\eta = 1$)

$$u_t = H(u_{xx}) + 2uu_x + \delta u_{xx} + \varepsilon\mu(t)H(u_x)$$

vanishes identically to all algebraic orders in ε and zeroth order in δ.

<u>Lemma 8</u>: The Melnikov function of the perturbed system 6.2 is identically zero to the zero order in δ and all algebraic orders in ε.

<u>Proof</u>: We compute the Melnikov function for the system 6.2,

$$M(t_0) = \int_{-\infty}^{\infty} f(x_0(t),\delta)\wedge g(t+t_0)dt,$$

where we have written 6.2 in the form

$$\dot{x} = f(x,\delta) + \varepsilon g(t),$$

with $g(t) = (0,0,I,-I)^t\dot{\mu}(t)$, and $x = (y,\bar{y},z,\bar{z})^t$, t denotes transpose. Substituting in from 6.2 gives

$$M(t_0) = i\sum_{k=1}^{n}\int_{-\infty}^{\infty}[z_k(t,\delta)-\bar{z}_k(t,\delta)]\dot{\mu}(t+t_0)dt,$$

$$= i\sum_{k=1}^{n}\int_{-\infty}^{\infty}[z_k(t,0)-\bar{z}_k(t,0)]\dot{\mu}(t+t_0)dt + O(\delta).$$

Recall the formulas for z and $\bar{z}$ from Section 2,

$$z_k(t,0) = d_k(t-ie)/((t-ie)^2+f^2)^{1/2}$$
$$\bar{z}_k(t,0) = \bar{d}_k(t+ie)/((t+ie)^2+f^2)^{1/2}.$$

Therefore, $z_k = d_k$, $\bar{z}_k = \bar{d}_k$, at $t = \pm\infty$ and since $\bar{d}_{k+2} = d_k$, the d's being fourth roots of unity, by integration by parts

$$M(t_0) = i\sum_{k=1}^{n} \int_{-\infty}^{\infty} [\dot{z}_k(t)) - \dot{\bar{z}}_k(t,\delta)]\mu(t+t_0)dt + O(\delta)$$
$$= i\sum_{k=1}^{n} \int_{-\infty}^{\infty} [y_k^{-3}(t) - \bar{y}_k^{-3}(t)]\dot{\mu}(t+t_0)dt + O(\delta).$$

But

$$y_k(t,0) = d_k/((t-ie)^2+f^2)^{1/2}$$
$$\bar{y}_k(t,0) = \bar{d}_k/((t+ie)^2+f^2)^{1/2}$$

so

$$M(t_0) = i\sum_{k=1}^{n} d_k \int_{-\infty}^{\infty} ((t-ie)^2+f^2)^{-3/2}\dot{\mu}(t+t_0)dt$$
$$- i\sum_{k=1}^{n} \bar{d}_k \int_{-\infty}^{\infty} ((t+tie)^2+f^2)^{-3/2}\dot{\mu}(t+t_0)dt + O(\delta)$$
$$= O(\delta),$$

because $\sum_{k=1}^{n} d_k, \sum_{k=1}^{n} \bar{d}_k = 0$, the d's being roots of unity. To see that the higher order (in ε) Melnikov functions are also $O(\delta)$, it is easiest to consider the perturbed system 6.1 in McGehee's coordinates. The system is linear up to order δ and the ε perturbation term does not depend on q and p. This makes all the higher order Melnikov functions vanish to the zero order in δ.

A small driven diffusion perturbation produces a very different result. The perturbed equation

$$u_t = H(u_{xx}) + 2uu_x + \delta\eta(t)u_{xx}$$

corresponds to the perturbed system 6.2, with $\varepsilon = 0$.

Lemma 9: Let $\eta = a\cos(\Omega t)+b\sin(\Omega t)$ be a simple periodic function, $\sigma = \delta$ small, and consider an unexceptional Poincaré map, see Lemma 7. Then the corresponding Melnikov function of the system 6.3 has a simple zero.

Proof: The Melnikov function is

$$M(t_0) = \int_{-\infty}^{\infty} f(x_0(t),\delta) \quad g(x_0(t),t+t_0)dt,$$

with f as above, but now the perturbation vector depends on x:

$$g = (0,0,$$

$$-16i\eta(t)\sum_{j=1}(2ib+y_k-\bar{y}_j)^{-3}-\dot{\eta}(t)[\sum_{j\neq k}(y_k-y_j)^{-1}+\sum_{j=1}(2ib+y_k-\bar{y}_j)^{-1}],$$

$$16i\eta(t)\sum_{j=1}(-2ib+\bar{y}_k-y_j)^{-3}-\dot{\eta}(t)[\sum_{j\neq k}(\bar{y}_k-\bar{y}_j)^{-1}+\sum_{j=1}(2ib+\bar{y}_k-y_j)^{-1}]).$$

We get

$$M(t_0) = -\sum_{k=1}^{n}\int_{-\infty}^{\infty}\{z_k\sum_{j\neq k}^{n}(y_k-y_j)^{-1}+\bar{z}_k\sum_{j\neq k}^{n}(\bar{y}_k-\bar{y}_j)^{-1}\}\dot{\eta}(t+t_0)dt$$

$$-\ 16i\sum_{k=1}^{n}\int_{-\infty}^{\infty}\{(z_k+\bar{z}_j)\sum_{j\neq k}^{n}(2ib+y_k-\bar{y}_j)^{-3}\}\eta(t+t_0)dt$$

$$-\sum_{k=1}^{n}\int_{-\infty}^{\infty}\{(z_k-\bar{z}_j)\sum_{j\neq k}^{n}(2ib+y_k-\bar{y}_j)^{-1}\}\dot{\eta}(t+t_0)dt.$$

Now we substitute the above solutions into these integrals. The first one is easy to do, whereas the last two can be evaluated numerically to show that they do not vanish, except for exceptional parameter (e,f) values. However, we can also get rid of the last two integrals by a simple rescaling of the poles. Recall that the constants d are constant multiplies of roots of unity. We set that constant equal to δ. Then the last two integrals are $O(\delta)$ and

$$M(t_0) = -n(n+1)\int_{-\infty}^{\infty}\{((t-ie)^2+f^2)^{-1}+((t+ie)^2+f^2)^{-1}\}\cdot\{a\cos(\Omega(t+t_0))+b\sin(\Omega(t+t_0))\}dt$$

$$= (\pi/f)\sinh(\Omega e)\cosh(\Omega f)\{a\cos(\Omega t_0)+b\sin(\Omega t_0)\}.$$

This function has a simple zero in t_0, unless $a = b = 0$.

In the next section these Melnikov functions will enable us to prove the main results.

<u>Remark 6</u>: Note that the computations above involve only the time derivatives of η and μ, and are thus unchanged if constants are added. Therefore, Lemmas 8 and 9 hold when η and μ are definite (positive or negative oscillatory functions.

8. Chaotic Solutions.

The moral of the above story is that the global time behavior of, the structurally unstable, pole solutions of the Benjamin-Ono equation is very sensitive to the perturbation u_{xx}, but quite insensitive to the perturbation $H(u_x)$. The time evolution of the solutions in presence of the former perturbation, slightly driven, is chaotic, whereas it is indistinguishable from the integrable case for the latter perturbation, up to exponentially small terms. We state these results in two theorems.

Theorem 2: In the limit as δ tends to zero, the time evolution of the pole solutions to the perturbed Benjamin-Ono equation

$$u_t = H(u_{xx}) + 2uu_x + \delta u_{xx} + \varepsilon\mu(t)H(u_x)$$

is the same as the solutions of the unperturbed equations, up to any algebraic order in ε.

Proof: We must show that the stable and unstable manifolds of the hyperbolic fixed point of the Poincaré map on S^{4n}, from Proposition 2, intersect non-transversely up to exponentially small terms as δ vanishes. But this is the content of Lemma 8, in Section 7. QED

The effect of driving the diffusion term is more spectacular. The solutions remain regular in space but their time behavior is chaotic.

Theorem 5: There exists an invariant hyperbolic set in S^{4n} on which the Poincaré map of the perturbed Benjamin-Ono flow

$$u_t = H(u_{xx}) + 2uu_x + \delta\eta(t)u_{xx}$$

is topologically conjugate to shift on finitely many symbols.

Proof: First we prove that the stable and unstable manifolds of the hyperbolic fixed point of the Poincaré map on S^{4n}, from Proposition 2, intersect transversely. We pick a transversal

$$\Sigma = \{(y,z,t) \varepsilon S^{4n} \times R \quad t = t_0 \neq t_k, 0 < k < N+1\},$$

where the t_k are the exceptional values from Lemma 7, Section 6. Then the corresponding Poincaré map has a hyperbolic fixed point. Now by Lemma 9, in Section 7, the Melnikov function has a simple zero for appropriately chosen driving function η. Consequently, the stable and unstable manifolds intersect transversely.

We want to apply the Smale-Birkoff homoclinic theorem but recall that by Lemma 1, in Section 2, the Benjamin-Ono solutions form a submanifold in the phase space of the N-body equations and we must adapt the Smale-Birkoff theorem to submanifolds. This was done by Birnir (3,'86), Section 9:

The submanifold lemma: The Smale horseshoe can be constructed so that the stable and unstable manifolds, of the fixed point of the Poincaré map, are dense in the horseshoe.

By the submanifold lemma, the perturbed N-body map restricted to the stable and unstable manifolds, which is just the perturbed Benjamin-Ono map, is topologically conjugate to shift on finitely many symbols. QED.

The perturbation breaks the 4 symmetry of the poles, see Remark 1, so more that 2n poles will appear.

Corollary 1: The number of poles (humps) observed in the perturbed flow will vary in the range [2n,8n].

The proof is similar to the analogous statement for KdV, see Birnir (3,'86), Proposition 4, Section 9.

Corollary 2: There exist no analytic integrals of the perturbed Benjamin-Ono equation, with η periodic.

This is a simple consequence of the presence of the horseshoe, see Birnir (3,'86), Section 8, or Moser (1973).

The question still remains what happens when both the above perturbations are present, i.e. δ is not infinitesimal, but the diffusion is not driven, so that η = constant (=1).

Corollary 3: The pole solutions of the perturbed Benjamin-Ono equation

$$u_t = H(u_{xx}) + 2uu_x + \delta u_{xx} + \varepsilon\mu(t)H(u_x)$$

are chaotic in time for δ/ε sufficiently large.

Proof: Here the $O(\delta)$ and higher order terms generated by δu_{xx} come into play and couple with $\mu(t)H(u_x)$. The details are similar to the proof of Theorem 5. QED.

Remark 7: Corollary 3 shows that in the presence of a bad term u_{xx} a driven good term $H(u_x)$ can cause chaotic time behavior. This is in fact what happens in the physically interesting case, when μ is a negative oscillatory function. Recall Remarks 3, 5 and 6.

We conclude with a discussion of a classification of perturbations into those who cause regular and those that cause chaotic time behavior. So far, $H(u_x)$ falls into the first catagory whereas u_{xx} belongs to the second. We want to restrict ourselves to the perturbations whose degree in u and its derivatives is less than that of the unperturbed vector field. If we count each power of u as 1 and the derivative also as 1, the degree of the Benjamin-Ono vector field X_2 is 3. The higher and lower order vector fields in Table 1 are homogeneous. It is reasonable to conjecture that perturbations close to the tangent space of the Benjamin-Ono tori will be regular whereas those in the normal space (spanned by the gradients in Table 1,), will be irregular, and chaotic if driven. Our results for both BO and KdV, see Birnir (3,'86), support this. However, most perturbations will excite all the infinitely many nonlinear modes and the solutions do not stay finite dimensional. For example, the perturbation u will not have invariant manifolds. The conjecture was that the Hilbert transform of either lower order gradients or vector fields will give invariant manifolds, because of the skew action of the Hilbert transform. This is true for KdV, with the Hilbert transform of lower order vector fields producing regular, and the Hilbert transform of lower order gradients chaotic, solutions. For BO, $H(X_1) = H(u_x)$ produces regular solutions but $H(\delta H_1) = H(u)$ <u>does not have</u> an invariant manifold. The reason is the fact discussed in Remark 2, Section 4, that the system of ODE's in Theorem 1, is not accompanied by a locus condition. However, we can produce invariant manifolds for $H(u)$ by adding a $O(\delta^2)$ perturbation, namely, the perturbed BO equation

$$u_t = H(u_{xx}) + 2uu_x + \delta\eta(t)u_{xx} - \frac{\delta\dot{\eta}}{1+\delta^2\eta^2} H(u) + \frac{\delta^2\eta\dot{\eta}}{1+\delta^2\eta^2} u$$

has the invariant manifold $S^{2n} \times R$, and on it an invariant hyperbolic set, where it Poincaré map is topologically conjugate to a shift on finitely many symbols. The proof of this statement is analogous to that of Theorem 3. We summarize these results in Table 3.

Table 3.

Perturbation	u_x	u	$H(u_x)$	$H(u)$	u_{xx}
KdV	integrable	no invar. manifold	regular	chaotic	?
BO	integrable	no invar. manifold	regular	no invar. manifold (but almost chaotic)	chaotic

References

Ablowitz, M. J. and J. Satsuma: J. Math. Phys. 19, p. 2180, 1978.

Ablowitz, M. J. and Fokas, A. S.: A direct linearization associated with the Benjamin-Ono equation, AIP Conference Proceedings 88, 1981, pp. 229-236.

Benjamin, T. B.: J. Fluid Mech. 29, p. 559, 1967.

Birnir, B: Complex Hill's equation and the complex KdV equations. Comm. Pure and Appl. Math., Vol. 39, 1986, pp. 1-49.

Birnir, B: Singularities of the complex KdV flows, Comm. Pure and Appl Math., Vol. 41, 1986, pp.

Birnir, B: The finite time blow-up of the complex KdV equation and existence after blow-up. To appear in Siam Journal of Applied Math..

Birnir, B: Chaotic solutions of KdV I: Rational Solutions. Physica 19D, pp. 238-254, 1986.

Chen, H. H., Y. C. Lee and N. R. Pereira: Algebraic internal wave solitons and the integrable Colegero-Moser-Sutherland N-body problem. Physics of Fluids, 22, p. 187, 1979.

Guckenheimer, J. and P. Holmes: Nonlinear Oscillations, Dynamical System and Bifurcations of Vector Fields. Springer Verlag, New York (1983).

Hirota, R.: Direct Methods of Finding Exact Solutions of Nonlinear Evolution Equations, in Bäcklund Transformations, ed. R. M Miura, Lect. Notes in Math. Vol. 515, Springer: New York, 1976.

Holmes, P. and J. E. Marsden. A partial differential equation with infinitely many periodic orbits: chaotic oscillations of a forced beam. Arch. for Rat. Mech. and Anal., vol. 76 (2), pp. 135-166, 1981.

Howard, L. N. and N. Kopell: Bifurcations and trajectories joining critical points. Adv. Math. 18, pp 498-513, 1976.

Jones, C. K. R. T. and P. Bates: Invariant Manifolds for Semilinear Partial Differential Equations. Preprint 1985.

Joseph, R.I.: J. Math. Phys. 18, p. 2251, 1977.

Lee, Y.C. and H. H. Chen: Nonlinear dynamical models of plasma turbulence. Physica Scripta, T2/1, pp. 41-47, 1982.

Matsuno, Y.: J. Phys. A: Math. Gen 12, p. 619, 1979.

McGehee, R.: A stable manifold theorem for degenerate fixed points with application to celestial mechanics. J. Diff. Eqs. 14. pp. 70-88, 1973.

Meiss, J. and N. Pereira: Physics of Fluids 2, p. 700, 1978.

Meiss, J. D.: Rational Solutions to partial differential equations, Geophysical Fluid Dynamics, WHOI 80-53 (1980), 155-164 (Woods Hole Oceanographic Institute).

Meiss, J. D. and P. J. Morrison: Nonlinear electron Landau damping of ion-acoustic solitons, Physics of Fluids 26, p. 983-989, 1983.

Moser, J.: Stable and Random Motion in Dynamical Systems with Special Emphasis on Celestial Mechanics. Ann. of Math. Studies, no. 77. Princeton Univ. Press, Princeton 1973.

Nakamura, A: J. Phys. Soc. Japan, Vol. 47, No. 4, pp. 1335-1340, 1979.

Ono, H.: J. Phys. Soc. Jpn. 39, p. 1082, 1975.

Ott, E. and R. Sudan: Nonlinear Theory of Ion Acoustic Waves with Landau damping, Physics of Fluids 12, pp. 2388-2394, 1969.

Sastuma, J., and Y. Ishimori: J. Phys. Soc. Japan 46, p. 681, 1979.

Slemrod, M. and J. E. Marsden: Temporal and spatial chaos in a Van der Walls fluid due to periodic thermal fluctuations. Institute for Mathematics and Its Applications, preprint series no. 33, Univ. of Minnesota, 1983.

Smale, S.: Differentiable dynamical systems. Bull. AMS 73, pp. 747-817, 1967.