A generalized reduced fluid model with finite ion-gyroradius effects

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Reduced fluid models have become important tools for studying the nonlinear dynamics of plasma in a large aspect-ratio tokamak. A self-consistent nonlinear reduced fluid model, with finite ion-gyroradius effects is presented. The model is distinctive in being correct to $O((\rho_i/a)^2)$ and in satisfying an exact, relatively simple energy conservation law.

I. INTRODUCTION

The term "reduced fluid model" refers to a set of simplified fluid equations that describes the nonlinear dynamics of large aspect ratio tokamak plasmas. The simplification is based on the following assumptions:

$$(1)\ \frac{\text{poloidal magnetic field}}{\text{toroidal magnetic field}} \sim O(\epsilon),$$

$$(2)\ \frac{\text{compressional Alfvén time}}{\text{time scale of interest}} \sim O(\epsilon),$$

$$(3)\ \frac{\text{transverse scale length}}{\text{parallel scale length}} \sim O(\epsilon),$$

where the ordering parameter $\epsilon \equiv a/R_0 \ll 1$ is the inverse aspect ratio; a is the perpendicular length scale, and R_0 is the toroidal length scale (or say, the major radius of the magnetic axis). The first assumption limits the plasma safety factor to be of order unity, the second assumption eliminates compressional Alfvén dynamics, and the third assumption is appropriate to flute-like perturbations. Consequently, $O(\epsilon)$ terms in the moment equations describe the compressional Alfvén equilibration, $O(\epsilon^2)$ terms describe the shear-Alfvén dynamics, and $O(\epsilon^3)$ terms are dropped. Early reduced fluid models were described in the work of Kadomtsev and Pogutse[1] and Rosenbluth *et al.*[2] Strauss[3,4] generalized this work and produced what is now called "reduced magnetohydrodynamics" (RMHD). These simple models qualitatively describe important tokamak phenomena such as island formation, magnetic stochasticity, and plasma disruption, in reasonable agreement with the actual tokamak experiments.[5] For this reason, RMHD has become a principle tool in understanding the nonlinear processes in tokamaks and has attracted a great deal of attention from the tokamak research community in the last decade. We also note that similar work has been done for mirror machines in the "long-thin limit" by Newcomb.[6] However, RMHD excludes many potentially important kinetic and nonideal effects, because of its MHD origin. Even in its ideal context, RMHD omits, for instance, poloidal curvature and density gradient terms. Thus there have been some recent efforts to understand and extend reduced fluid systems.[7–10] For a review we direct the reader to Ref. 11.

The point is that instabilities of interest are often so slow that the omission of the diamagnetic drift and comparable terms arising from viscosity is not appropriate. A recent model called the four-field model[7] keeps electron beta terms and is valid for a characteristic time as slow as the diamagnetic drift frequency ω_*. This fluid model possesses desirable features, such as a physically realistic long mean-free-path electron response (that usually requires a kinetic treatment); also, unlike conventional RMHD, the threshold for tearing instability is given by the appropriate finite critical value of Δ'.[7] However, the four-field model's treatment of finite gyroradius terms is less complete and less self-consistent than that of the present model. As in the four-field model, we consider slow time scale shear-Alfvén dynamics, with constant temperature; we also allow for comparable electron and ion temperatures, which is realistic in most modern tokamaks. We derive a fully self-consistent system that includes cross-field viscosity terms, as well as electron diffusive terms provided by the Spitzer resistivity,[12] and ion viscous terms caused by ion–ion collisions. The latter terms have been widely used as a damping mechanism of plasma momentum in computational works. We assume the collision frequency to be much larger than the particle transit frequency in order to avoid particle trapping effects. The inclusion of compressibility and viscosity couples the parallel flow to the usual fields of RMHD. The magnetic field curvature terms included in our reduced equations are consistent with those obtained from drift kinetic theory. Of course, important kinetic effects such as Landau damping and magnetic trapping are beyond the scope of the present work. Also, we have excluded potentially important effects because of the variation of temperature.[13] A reduced fluid model that includes the electron temperature as a dynamical variable, but is restricted to zero ion temperature, was derived by Drake and Antonsen.[8] Unlike previous works, we do not scale the ion-gyroradius ρ_i or the plasma beta β in terms of ϵ; rather, we treat them as independent small parameters. We thus have a system that is self-consistent and possesses a conceptually simple energy conservation law. We remark that energy conservation laws are necessary for the description of reduced fluid models as a noncanonical Hamiltonian field theory, a formalism that has been useful for obtaining additional constants of motion and nonlinear stability criteria for a reduced system.[9] Also, energy constants can be used as a computational diagnostic. Even though we do omit $O((\rho_i/a)^3)$ terms for simplicity, we remark that the model retains significant finite Larmor radius (FLR) physics even when $\rho_i^2 \nabla^2 \sim 1$.

Finally, although the model can be used for the calcula-

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tion of equilibrium, stability, and nonlinear dynamics of tokamak plasmas, here we restrict our attention to showing how we can reproduce the usual toroidal transport. Other applications are left to future research.

II. FLUID EQUATIONS

A. Derivation

We start with the exact moment equations

$$\frac{\partial}{\partial t} n_\alpha + \nabla\cdot n_\alpha \mathbf{V}_\alpha = 0, \tag{1}$$

$$n_\alpha m_\alpha \left(\frac{\partial}{\partial t} + \mathbf{V}_\alpha\cdot\nabla\right)\mathbf{V}_\alpha + \nabla\cdot\mathsf{P}_\alpha = e_\alpha n_\alpha \left(\mathbf{E} + \frac{\mathbf{V}_\alpha}{c}\times\mathbf{B}\right) + \mathbf{F}_\alpha, \tag{2}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{V}_\alpha\cdot\nabla\right)\mathsf{P}_\alpha + \nabla\cdot\mathsf{q}_\alpha + [(\mathsf{P}_\alpha\cdot\nabla\mathbf{V}_\alpha) + \mathrm{Tr}] + \mathsf{P}_\alpha \nabla\cdot\mathbf{V}_\alpha = \left(\frac{m_i e_\alpha}{m_\alpha e_i}\right)\Omega(\mathsf{P}_\alpha\times\mathbf{b} + \mathrm{Tr}) + \mathsf{C}_\alpha, \tag{3}$$

where α ($=i,e$) is the species label, Ω is ion gyrofrequency, and the moment tensors P_α, q_α, and C_α are defined by

$$\mathsf{P}_\alpha \equiv \int d\mathbf{v}\, m_\alpha(\mathbf{v}-\mathbf{V}_\alpha)(\mathbf{v}-\mathbf{V}_\alpha)f_\alpha,$$

$$\mathsf{q}_\alpha \equiv \int d\mathbf{v}\, m_\alpha(\mathbf{v}-\mathbf{V}_\alpha)(\mathbf{v}-\mathbf{V}_\alpha)(\mathbf{v}-\mathbf{V}_\alpha)f_\alpha,$$

$$\mathsf{C}_\alpha \equiv \int d\mathbf{v}\, m_\alpha(\mathbf{v}-\mathbf{V}_\alpha)(\mathbf{v}-\mathbf{V}_\alpha)C_\alpha(f_\alpha),$$

where $C_\alpha(f)$ is the collision operator, n_α, m_α, and $\mathbf{V}_\alpha$ are density, mass, and velocity, respectively. As usual, $\mathbf{B}$ and $\mathbf{E}$ are the magnetic and electric fields, $\mathbf{b}$ is the unit vector along the magnetic field line, and I is the unit tensor. Also we use Tr to denote the transpose of the preceding tensor. For isothermal systems, it is adequate to write the friction force $\mathbf{F}_\alpha$ in the Spitzer–Härm[12] form

$$\mathbf{F}_\alpha \equiv \int d\mathbf{v}\, m_\alpha(\mathbf{v}-\mathbf{V}_\alpha)C_\alpha(f_\alpha) \simeq -n_\alpha e_\alpha \eta_s \mathbf{J},$$

where η_s is the Spitzer–Härm resistivity, and $\mathbf{J}$ is the plasma current. Then, we assume quasineutrality, sum Eq. (2) over species, and obtain

$$nm_i\left(\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla\right)\mathbf{V} + \nabla\cdot(\mathsf{P}_i + \mathsf{P}_e) = \frac{\mathbf{J}}{c}\times\mathbf{B}, \tag{4}$$

where we have neglected terms of $O(m_e/m_i)$. Similarly upon subtraction Eq. (2) leads to the usual Ohm's law

$$\mathbf{E} + (\mathbf{V}_e/c)\times\mathbf{B} = \eta_s \mathbf{J} - \nabla nT_e/ne, \tag{5}$$

where $n = n_e = n_i$, $\mathbf{V}\sim\mathbf{V}_i = \mathbf{J}/ne - \mathbf{V}_e$. We can therefore compute $\mathbf{V}_{e\perp}$, to all orders in ϵ, from Ohm's law. Therefore, Eqs. (1) and (5) describe mainly the electron dynamics, while Eq. (4) describes mainly the ion dynamics.

The ion gyromotion comes into our fluid system through the pressure tensor P_i which can in general be expressed as

$$\mathsf{P}_i = P_\perp(\mathsf{I} - \mathbf{bb}) + P_\parallel \mathbf{bb} + \hat{\mathsf{P}}_i \equiv \mathsf{P}^{\mathrm{CGL}} + \hat{\mathsf{P}}_i,$$

where

$$P_\perp \equiv \int d\mathbf{v}\, \frac{m_i v_\perp^2}{2} f_i, \quad P_\parallel \equiv \int d\mathbf{v}\, m_i v_\parallel^2 f_i,$$

define the well-known Chew–Goldberger–Low[14] stress tensor $\mathsf{P}^{\mathrm{CGL}}$, and $\hat{\mathsf{P}}_i$ is called the cross-field viscosity tensor. For convenience, we hereafter use ^ to denote the non-CGL (see Appendix A) portion of an arbitrary tensor; namely,

$$\hat{\mathsf{A}} = \mathsf{A} - \mathsf{A}^{\mathrm{CGL}} = \mathsf{A} - \left[\mathbf{bb}(\mathsf{A}:\mathbf{bb}) + (\mathsf{I} - \mathbf{bb})\left(\frac{\mathsf{I} - \mathbf{bb}}{2}:\mathsf{A}\right)\right]. \tag{6}$$

A method, presented in Appendix A, leads to the following cross-field viscosity tensor:

$$\hat{\mathsf{P}}_i \simeq \hat{\mathsf{P}}^g + \hat{\mathsf{P}}^c,$$

where $\hat{\mathsf{P}}^g$, the gyroviscosity tensor, is

$$\hat{\mathsf{P}}^g \simeq (1/4\Omega)\{[\mathbf{b}\times\hat{\mathsf{S}}^g\cdot(\mathsf{I} + 3\mathbf{bb})] + \mathrm{Tr}\}, \tag{7}$$

and $\hat{\mathsf{P}}^c$, the collisional viscosity tensor, is

$$\hat{\mathsf{P}}^c \simeq -\frac{3\nu_i}{10\Omega}\frac{nT_i}{\Omega}[\hat{\mathsf{W}} + 3(\hat{\mathsf{W}}\cdot\mathbf{bb} - \mathrm{Tr})].$$

Here $\hat{\mathsf{S}}^g$ is given in Eq. (A5), $\mathsf{W} = \nabla\mathbf{V} + \mathrm{Tr}$, and ν_i is the ion–ion collision frequency. It is important to note that the $O(\epsilon)$ terms of Eq. (7) give exactly Braginskii's[15] cross-field viscosity and the $O(\epsilon^2)$ terms give higher-order finite Larmor radius corrections.

Before proceeding with the reduction process, we first briefly review the normalized geometry that is based on the large aspect ratio orderings described in Sec. I. For details, we refer the reader to Ref. 10. The dimensionless coordinates (h, y, z, τ) are defined by

$$h = \frac{R - R_0}{a}, \quad y = \frac{Z}{a}, \quad z = -\zeta, \quad \tau = \frac{\epsilon v_A t}{a};$$

where (R, ζ, Z) are the usual cylindrical coordinates centered on the tokamak symmetry axis, the Alfvén speed $v_A \equiv B_0^2/4\pi m_i n_0$, and B_0, n_0 are the volume-averaged vacuum field and plasma density, respectively. Therefore, the dimensionless gradient can be written as

$$\nabla = \nabla_\perp + \hat{z}\frac{\epsilon}{1+\epsilon h}\frac{\partial}{\partial z}$$

and the reduced geometry can be generally described by the metric coefficients

$$g^{ij} \equiv \nabla x^i\cdot\nabla x^j = \begin{cases} \left(\dfrac{\epsilon}{1+\epsilon h}\right)^2, & i = j = z, \\ \delta_{ij}, & \text{otherwise}. \end{cases}$$

For an isothermal system, the normalized electron pressure can be defined by $p \equiv (\beta/\epsilon)(n/n_0 - 1)$, where the constant $\beta \equiv 8\pi n_0 T_e/B_0^2$ is the electron beta. The magnetic field can thus be written as

$$\mathbf{B} = [B_0/(1+\epsilon h)]\hat{z} + \nabla\times\mathbf{A} = B_0\{[1 + \epsilon(B_z - h)]\hat{z} - \epsilon\hat{z}\times\nabla_\perp \psi\} + O(\epsilon^2), \tag{8}$$

where $\mathbf{A}$ is the vector potential resulting from plasma current,

$$B_z \equiv \hat{z}\cdot(\nabla\times\mathbf{A})/B_0\epsilon \tag{9}$$

is the normalized diamagnetic correction to the toroidal magnetic field, and

$$\psi \equiv \frac{\mathbf{A}\cdot\hat{z}}{aB_0\epsilon} = \frac{\Psi_p}{2\pi B_0 a^2}\,[1+O(\epsilon)]$$

is the normalized poloidal magnetic flux, where Ψ_p is the usual poloidal magnetic flux. Also note here that the lowest-order field line curvature is

$$\boldsymbol{\kappa} = \mathbf{b}\cdot\nabla\mathbf{b} = (1/R_0)\nabla_\perp\, h\,[1+O(\epsilon)].$$

From Ohm's law and the toroidal component of Faraday's law, one finds $\nabla\cdot\mathbf{V} = O(\epsilon^2)$. The smallness of the compressibility is physically justified by the inability of plasma to compress the toroidal field. Note that the same conclusion can be made from the continuity equation and the assumption of $n = n_0 + O(\epsilon)$. We therefore write $\mathbf{V}$ as

$$\mathbf{V} = \epsilon v_A(\hat{z}\times\nabla_\perp F + v\hat{z}) + O(\epsilon^2), \tag{10}$$

where v is the normalized parallel flow and F is the normalized stream function. By using this form for $\mathbf{V}$ and the $O(\epsilon)$ contribution of $\nabla\cdot\mathsf{P}_i$,

$$\nabla_\perp P_\perp + \nabla\cdot\widehat{\mathsf{P}}_0 = -en\nabla_\perp\phi + \Omega\mathbf{V}\times\mathbf{b}, \tag{11}$$

Eq. (2), for $\alpha = i$, is reduced to

$$(1 + a_i^2\nabla_\perp^2)F = \varphi + \delta p\frac{T_i}{T_e} + \delta\beta\frac{T_i}{T_e}\frac{P_\perp - nT_i}{\epsilon n_0 T_i}, \tag{12}$$

where the constant $\delta \equiv v_A/2\Omega a$ is a measure of FLR effects, ϕ is the electrostatic potential, and $\varphi \equiv (c/\epsilon B_0 a v_A)\,\phi$ is the normalized electrostatic potential. It is also worth mentioning here that $a_i^2 \equiv \delta^2\beta\, T_i/T_e$ and the operator $a_i^2\nabla_\perp^2$ corresponds to $\rho_i^2\nabla_\perp^2$, a well-known FLR operator. The remaining unnormalized variable is $P_\perp - nT_i$ in the last term of Eq. (12), which usually vanishes when ion gyromotion and trapping effects are not considered. However, even in the absence of particle trapping effects, when the ion gyromotion becomes important, we find (see Appendix B) that

$$(P_\perp - nT_i)/\epsilon n_0 T_i = 2\delta\nabla_\perp^2 F - h. \tag{13}$$

Note that in deriving this result, we have also neglected $O(a_i^3)$ terms. Equation (12) thus becomes

$$(1 - a_i^2\nabla_\perp^2)F = \varphi + \delta(T_i/T_e)(p - \beta h). \tag{14}$$

Similarly, the $O(\epsilon)$ terms of Eq. (4) give the reduced pressure balance law

$$B_z = -\frac{1 + T_i/T_e}{2}p - \frac{\delta\beta T_i}{2T_e}\nabla_\perp^2 F + \frac{\beta T_i}{2T_e}h, \tag{15}$$

which describes compressional Alfvén equilibration; again, the second term of the right-hand side is an FLR correction.

In the reduced equations, we keep $O(\epsilon^2)$ terms and drop $O(\epsilon^3)$ terms. Consider first Eqs. (1) and (2) with $\alpha = e$. From Eq. (1), Faraday's law, and $\hat{z}\cdot\nabla\times$ [Eq. (5)], we obtain

$$\begin{aligned}\frac{\partial}{\partial\tau}p &+ [F + 2\delta B_z, p]\\ &= -\beta\{[F + 2\delta B_z, 2h] + \nabla_\parallel(v + 2\delta J)\}\\ &\quad + \beta\left(\frac{\partial}{\partial\tau}B_z + [F, B_z]\right) - \eta\beta\nabla_\perp^2 B_z,\end{aligned} \tag{16}$$

where the operator $\partial/\partial\tau + [F + 2\delta B_z,\ ]$ is simply a reduced form of $(\partial/\partial t + \mathbf{V}_e\cdot\nabla)$, and the right-hand side is because of electron compressibility. Similarly, the parallel component of Ohm's law gives

$$\frac{\partial}{\partial\tau}\psi + \nabla_\parallel F = -2\delta\nabla_\parallel B_z + \eta J, \tag{17}$$

where

$$J \equiv \nabla_\perp^2\psi \tag{18}$$

is the normalized parallel current, the constant

$$\eta \equiv c^2\eta_s/4\pi a v_A\epsilon \tag{19}$$

is the normalized resistivity,

$$\nabla_\parallel \equiv \mathbf{b}\cdot\nabla = \frac{\partial}{\partial z} - [\psi,\ ] \tag{20}$$

is the nonlinear parallel gradient, and

$$[f,g] \equiv \hat{z}\cdot\nabla_\perp f\times\nabla_\perp g \tag{21}$$

defines the bracket. We also note that if we allow for anisotropic resistivity,[15] then

$$\mathbf{F}_\alpha = n_\alpha e_\alpha(\eta_\perp\mathbf{J}_\perp + \eta_\parallel\mathbf{J}_\parallel)$$

and our model is modified by simply replacing η by a normalized $\eta_\perp$ in Eq. (16) and by a normalized $\eta_\parallel$ in Eq. (17).

The other two equations to be derived are the reduced parallel momentum equation and parallel vorticity equation. Taking $\mathbf{b}\cdot$Eq. (4) and $\mathbf{b}\cdot\nabla\times$Eq. (4) yields, respectively,

$$\begin{aligned}&nm_i\left(\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla\right)(\mathbf{V}\cdot\mathbf{b}) + \mathbf{b}\cdot(\nabla\cdot\widehat{\mathsf{P}}_i) + \mathbf{b}\cdot\nabla(P_\parallel + nT_e\\ &\quad = O(\epsilon^3),\\ &nm_i\left(\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla\right)(\mathbf{b}\cdot\nabla\times\mathbf{V}) + \mathbf{b}\cdot\boldsymbol{\kappa}\times\nabla(P_\perp - P_\parallel)\\ &\quad + \mathbf{b}\cdot\nabla\times(\nabla\cdot\widehat{\mathsf{P}}_i)\\ &\quad = -\frac{\mathbf{J}}{c}\cdot\nabla B + \frac{\mathbf{B}}{c}\cdot\nabla(\mathbf{J}\cdot\mathbf{b}) - \frac{B}{c}\boldsymbol{\kappa}\cdot\mathbf{J} + O(\epsilon^3),\end{aligned}$$

where $\boldsymbol{\kappa}$ is the field line curvature. The difficult terms to evaluate are $\mathbf{b}\cdot\nabla\cdot\widehat{\mathsf{P}}$ and $\mathbf{b}\cdot\nabla\times(\nabla\cdot\widehat{\mathsf{P}})$, since $\widehat{\mathsf{P}}$ is a complicated tensor; however, by using the identities

$$\begin{aligned}\mathbf{b}\cdot\nabla\cdot\widehat{\mathsf{P}} &= -\nabla\mathbf{b}:\widehat{\mathsf{P}} + \nabla\cdot(\mathbf{b}\cdot\widehat{\mathsf{P}})\\ &= -\nabla\mathbf{b}:\widehat{\mathsf{P}}_0 + \nabla\cdot\left(\frac{1}{\Omega}\mathbf{b}\cdot\widehat{\mathsf{S}}\times\mathbf{b}\right) + O(\epsilon^3),\\ \mathbf{b}\cdot\nabla\times(\nabla\cdot\widehat{\mathsf{P}}) &= (\nabla\times\mathbf{b})\cdot\nabla\cdot\widehat{\mathsf{P}} - \nabla_k(\widehat{\mathsf{P}}\cdot\nabla\mathbf{b})_{ij}\,\epsilon_{ijk}\\ &\quad + \tfrac{1}{2}\nabla\nabla:(\widehat{\mathsf{P}}\times\mathbf{b} + \mathrm{Tr})\\ &= (\nabla\times\mathbf{b})\cdot\nabla\cdot\widehat{\mathsf{P}}_0 - \nabla_k(\widehat{\mathsf{P}}_0\cdot\nabla\mathbf{b})_{ij}\,\epsilon_{ijk}\\ &\quad + \frac{1}{2}\nabla\nabla:\left(\frac{1}{\Omega}\widehat{\mathsf{S}}\right) + O(\epsilon^3),\end{aligned}$$

the calculation becomes much easier. Here, $\widehat{\mathsf{S}}$ and $\widehat{\mathsf{P}}_0$ are given by Eqs. (A4) and (A5). Regarding the parallel collisional viscosity, we can simply adopt Braginskii's result and find it yields no contribution to our reduced model. Finally, we obtain

$$\frac{\partial}{\partial\tau}(1-2a_i^2\nabla_\perp^2)v+\left[(1-a_i^2\nabla_\perp^2)F-\delta\frac{T_i}{T_e}[p+\beta(4h-B_z)],v\right]+\frac{1+T_i/T_e}{2}\nabla_\parallel p-\frac{6}{5}\beta\mu\nabla_\perp^2 v$$
$$=\delta\beta(T_i/T_e)\{\nabla_\parallel\nabla_\perp^2 F+[\nabla_\perp F;\nabla_\perp\psi]+2\delta\nabla_\perp\cdot[F,\nabla_\perp v]\}, \tag{22}$$

and

$$\frac{\partial}{\partial\tau}\left(1-\frac{1}{2}a_i^2\nabla_\perp^2\right)\nabla_\perp^2 F+\left[F,\left(1-\frac{1}{2}a_i^2\nabla_\perp^2\right)\nabla_\perp^2 F\right]+\nabla_\parallel J+2[B_z,h]-\frac{3}{10}\beta\mu\nabla_\perp^4 F$$
$$=-\delta\beta\frac{T_i}{T_e}\left\{\nabla_\perp\cdot\left(\left[\nabla_\perp F,\frac{p}{\beta}-B_z+h\right]+[\nabla_\perp\psi,v]\right)+\frac{1}{2}\nabla_\perp^2\left[\nabla_\parallel v-\frac{1}{\beta}\left(\frac{\partial}{\partial\tau}p+[F,p]\right)\right]\right\}, \tag{23a}$$

where the notation

$$[\mathbf{A};\mathbf{B}]=\sum_k[A_k,B_k],$$

and the constant

$$\mu\equiv\nu_i\delta T_i/\epsilon\Omega T_e$$

is the normalized viscosity coefficient. Because of the omission of $O(a_i^3)$ in deriving Eqs. (14) and (15), to assure the system to be self-consistent energy conserving, we sum $-(a_i^2/\delta)\nabla_\perp^2$ [Eq. (B2)] over Eq. (23a) and obtain

$$\frac{\partial}{\partial\tau}\left(1-\frac{5}{2}a_i^2\nabla_\perp^2\right)\nabla_\perp^2 F+\left[F,\left(1-\frac{5}{2}a_i^2\nabla_\perp^2\right)\nabla_\perp^2 F\right]+\nabla_\parallel J+2[B_z,h]-\frac{3}{10}\beta\mu\nabla_\perp^4 F$$
$$=-\delta\beta\frac{T_i}{T_e}\left\{\nabla_\perp\cdot\left[\nabla_\perp F,\frac{p}{\beta}-B_z+3h-4\delta\nabla_\perp^2 F\right]+(\nabla_\perp\cdot[\nabla_\perp\psi,v]+\nabla_\perp^2\nabla_\parallel v)\right.$$
$$\left.+\frac{1}{2}\nabla_\perp^2\left[\nabla_\parallel v+\frac{1}{\beta}\left(\frac{\partial}{\partial\tau}p+[F,p]\right)\right]\right\}. \tag{23b}$$

It is expected that the linear version of Eqs. (22) and (23) can also be derived through the ion gyrokinetic equation to the order of $(\rho_i/a)^2$, but the calculation would be extremely complicated. Here, we only remark that the second term on the left-hand side of Eq. (22),

$$\left[(1-a_i^2\nabla_\perp^2)F-\delta\frac{T_i}{T_e}(p+\beta(4h-B_z)),v\right]=\left[\varphi-\delta\frac{T_i}{T_e}\beta(4h-B_z),v\right],$$

is the reduced form of

$$\int d\mathbf{v}\, v_\parallel \mathbf{v}_d\cdot\nabla f,$$

where $\mathbf{v}_d$ is the particle drift velocity which includes $\mathbf{E}\times\mathbf{B}$, curvature, and gradient B drifts. On the other hand, in a constant magnetic field, i.e., where $\nabla\psi$ and B_z are constant and curvature is neglected, Eqs. (22) and (23) can be derived alternatively by using

$$m_i n\left(\frac{\partial}{\partial t}+\mathbf{V}\cdot\nabla\right)\mathbf{V}_D+\nabla\cdot\hat{\mathrm{P}}_0=-\frac{1}{2}\nabla_\perp\left(\frac{nT_i}{\Omega}\mathbf{b}\cdot\nabla_\perp\times\mathbf{V}_\perp\right)-\mathbf{bb}\cdot\nabla\left(\frac{nT_i}{\Omega}\mathbf{b}\cdot\nabla_\perp\times\mathbf{V}_\perp\right)$$
$$+\mathbf{bb}\cdot\left(\nabla_\perp V_\parallel\times\nabla_\perp\frac{nT_i}{\Omega}\right)-\frac{nT_i}{2\Omega}\mathbf{b}\times\nabla_\perp(\nabla_\perp\cdot\mathbf{V}_\perp)+O(\epsilon^3), \tag{24}$$

where $\mathbf{V}_D=(1/m_in\Omega)\mathbf{b}\times\nabla nT_i$ is the diamagnetic velocity. The first two terms on the right-hand side of Eq. (24), involving, respectively, the perpendicular and parallel gradients of the parallel vorticity, display anisotropy in that the coefficients differ by a factor of two. This factor can be related to the dimensionality of the system. To our knowledge this anisotropy has not previously been noticed. We also note that the above equation differs from the usual gyroviscosity cancellation[16] because of the last three terms, which are caused by parallel gradients, parallel flow, and compressibility. It also implies that when the compressibility is considered, the usual gyroviscosity cancellation will not be valid.

B. Summary

We have derived a closed reduced-fluid system; namely, the particle conservation law

$$\frac{\partial}{\partial\tau}p+[F+2\delta B_z,p]=-\beta\{[F+2\delta B_z,2h]+\nabla_\parallel(v+2\delta J)\}+\beta\left(\frac{\partial}{\partial\tau}B_z+[F,B_z]\right)-\eta\beta\nabla_\perp^2 B_z, \tag{25}$$

the generalized Ohm's law

$$\frac{\partial}{\partial\tau}\psi+\nabla_\parallel F=-2\delta\nabla_\parallel B_z+\eta J, \tag{26}$$

the parallel acceleration law

$$\frac{\partial}{\partial\tau}(1-2a_i^2\nabla_\perp^2)v+\left[(1-a_i^2\nabla_\perp^2)F-\delta\frac{T_i}{T_e}(p+\beta(4h-B_z)),v\right]+\frac{1+T_i/T_e}{2}\nabla_\parallel p-\frac{6}{5}\beta\mu\nabla_\perp^2 v$$
$$=\delta\beta\frac{T_i}{T_e}\{\nabla_\parallel\nabla_\perp^2 F+[\nabla_\perp F;\nabla_\perp\psi]+2\delta\nabla_\perp\cdot[F,\nabla_\perp v]\},\tag{27}$$

and the shear Alfvén law

$$\frac{\partial}{\partial\tau}\left(1-\frac{5}{2}a_i^2\nabla_\perp^2\right)\nabla_\perp^2 F+\left[F,\left(1-\frac{5}{2}a_i^2\nabla_\perp^2\right)\nabla_\perp^2 F\right]+\nabla_\parallel J+2[B_z,h]-\frac{3}{10}\beta\mu\nabla_\perp^4 F$$
$$=-\delta\beta\frac{T_i}{T_e}\left\{\nabla_\perp\cdot\left[\nabla_\perp F,\frac{p}{\beta}-B_z+3h-4\delta\nabla_\perp^2 F\right]+(\nabla_\perp\cdot[\nabla_\perp\psi,v]+\nabla_\perp^2\nabla_\parallel v)\right.$$
$$\left.+\frac{1}{2}\nabla_\perp^2\left(\nabla_\parallel v+\frac{1}{\beta}\left(\frac{\partial}{\partial\tau}p+[F,p]\right)\right)\right\}.\tag{28}$$

Also, the electrostatic potential and toroidal magnetic field are given by

$$\varphi=(1-a_i^2\nabla_\perp^2)F-\delta(T_i/T_e)(p-\beta h),\tag{29}$$

$$B_z=-\frac{1+T_i/T_e}{2}p-\frac{\delta\beta T_i}{2T_e}\nabla_\perp^2 F+\frac{\beta T_i}{2T_e}h.\tag{30}$$

Although Eqs. (25)–(30) appear to form a system of five fields, only four of these are independent. This is because B_z (or p) can be straightforwardly eliminated through the simple relation of Eq. (30); nevertheless, the resulting set of equations will be much more complicated unless some simplifications, such as dropping $O(\beta^2)$ terms, are made. We also remark that our system is not only exact to $O((\rho_i/a)^2)$, but also a fairly good approximation to a full FLR system for a wide range of $\rho_i^2\nabla^2$. The details of these discussions will be studied in the work in process. Furthermore, the linear version of our model has been used for studying the toroidal drift-tearing modes,[17] in the case where terms with $\beta T_i/T_e$ are omitted; i.e., the FLR effects are omitted while the semicollisional and drift effects are kept. The result includes the ion-sound stabilizing effects to the usual drift-tearing result. By further specifying the scale of the small parameters, e.g., δ or β, our system can be reduced to many well-known reduced fluid models. For instance, it is of interest that a high-β version of RMHD[4] with both ion and electron drift corrections can be derived by simply setting $\beta,\eta,\mu\to 0$, $T_i/T_e\to 1$, and redefining $P\equiv 8\pi nT_e/B_0^2$. We thus obtain a three-field model, given by

$$\frac{\partial}{\partial\tau}p+[F,p]=0,$$
$$\frac{\partial}{\partial\tau}\psi+\nabla_\parallel F-2\delta\nabla_\parallel p=0,\tag{31}$$
$$\frac{\partial}{\partial\tau}\nabla_\perp^2 F+[F,\nabla_\perp^2 F]+\nabla_\parallel J-2[p,h]+\delta\nabla_\perp\cdot[\nabla_\perp F,p]=0.$$

The electrostatic potential, and toroidal magnetic field are, respectively,

$$\varphi=F-\delta p,$$
$$B_z=-p.$$

The parallel flow is decoupled from the other fields and satisfies

$$\frac{\partial}{\partial\tau}v+[F,v]+\nabla_\parallel p+\delta[v,p]=0;\tag{32}$$

also, the compressibility is given by

$$\nabla\cdot\mathbf{V}=\epsilon^2\frac{v_A}{a}\{[F-2\delta p,2h]+\nabla_\parallel(v+2\delta J)\}.\tag{33}$$

This model arises since when β is of order ϵ, compressibility and viscosity are both discarded from the system. Therefore, the equation of adiabatic compression acts the same as the equation of isothermal particle conservation, during the course of shear-Alfvén motion. If we further set $\delta\to 0$, we will get the usual high-β version of RMHD, and further setting $p\to 0$ clearly leads to the low-β version of RMHD.[3]

III. ENERGY CONSERVATION

In a dynamical system, if a quantity Q satisfies

$$\frac{\partial}{\partial t}Q+\nabla\cdot\mathbf{U}=0,$$

where $\langle\ \rangle$ denotes the fixed volume average (all the surface terms are omitted), then $\langle Q\rangle$ is a constant of motion. For a general discussion of the constants of motion in RMHD, we refer the reader to Ref. 9. Here we study the most common constant of motion: the energy.

The energy conservation law for our primitive fluid system, Eqs. (1), (4), and (5), can be determined by calculating $\langle\mathbf{V}\cdot$Eq. (4)$\rangle$, $\langle\mathbf{J}\cdot(\mathbf{V}\times\mathbf{B})\rangle$, and $\langle\mathbf{B}\cdot(\partial/\partial t)\mathbf{B}\rangle$. With the aid of Ohm's law and Maxwell's equations, we derive

$$\frac{\partial}{\partial t}\left\langle\frac{m_i n V^2}{2}+\frac{B^2}{8\pi}\right\rangle$$
$$= -\eta_s\langle|\mathbf{J}|^2\rangle + \langle\nabla\mathbf{V}: (\mathsf{P}_i + \mathsf{P}_e)\rangle$$
$$= \langle -\eta_s|\mathbf{J}|^2 + \nabla\mathbf{V}: \hat{\mathsf{P}}^c\rangle + \langle n(T_e+T_i)(\nabla\cdot\mathbf{V})\rangle$$
$$+ \langle[(P_\perp - nT_i)\,\mathsf{I} + (P_\parallel - P_\perp)\mathbf{bb} + \hat{\mathsf{P}}^g]: \nabla\mathbf{V}\rangle. \tag{34}$$

Recall that the omission of the electron anisotropic stress tensor is because of the smallness of m_e. We note here that the right-hand side of this equation corresponds to the rate of change of the internal energy of the system; therefore, the equation simply represents the conservation of the total of kinetic energy, magnetic energy, and internal energy. From the thermodynamics point of view, the change of the internal energy is caused by the entropy heat production and work done on the system, i.e.,

$$n\,du = nT\,ds + p\frac{dn}{n},$$

where u and s are the internal energy and entropy per unit volume, respectively. The term involving $T\,ds$ corresponds to the collisional terms on the right-hand side of Eq. (34); while the second and the third terms on the right-hand side of Eq. (34) represent the generalized work done because of isotropic stress and anisotropic stress, respectively. Thus, Eq. (34) is equivalent to

$$\frac{\partial}{\partial t}\left\langle\frac{m_i n V^2}{2}+\frac{B^2}{8\pi}+nu\right\rangle = 0.$$

However, except for the entropy production, which must be a positive quantity because of the well-known H theorem, all forms of energy are expected to merge into an energy functional $\langle H\rangle$ such that

$$\frac{\partial}{\partial t}\langle H\rangle = -\left\langle nT\frac{ds}{dt}\right\rangle$$

= a negative quantity usually called the

dissipation of energy.

To absorb the work done into the energy functional, we use Eq. (A3) to derive the equality

$$\langle\nabla\mathbf{V}: \hat{\mathsf{P}}\rangle = -(\Omega/2nT_i)\,\langle\hat{\mathsf{P}}_0: \hat{\mathsf{S}}\rangle,$$

where, again, $\hat{\mathsf{P}}_0$ and $\hat{\mathsf{S}}$ are given by Eqs. (A5) and (A2), respectively. For the present paper, we also adopt the large-aspect-ratio scalings; that is, we use Eqs. (A4) and (B1) to obtain the energy conservation law

$$\frac{\partial}{\partial t}\langle H\rangle = -\eta_s\,\langle|\mathbf{J}|^2\rangle - \frac{3\nu_i}{10\Omega}\frac{nT_i}{\Omega}$$
$$\times\left\langle 4|\nabla V_\parallel|^2 + \sum_{i,j}(\nabla_i V_{\perp j})^2\right\rangle, \tag{35}$$

where the energy functional $\langle H\rangle$ has the form

$$\langle H\rangle = \left\langle\frac{m_i n V^2}{2}+\frac{B^2}{8\pi}+n(T_i+T_e)\left(\ln\frac{n}{n_0}\right)\right.$$
$$\left.+\frac{(P_\perp - nT_i)^2}{2n_0 T_i}\right\rangle + \frac{nT_i}{8\Omega^2}$$
$$\times\left\langle 4|\nabla V_\parallel|^2 + \sum_{i,j}(\nabla_i V_{\perp j})^2\right\rangle.$$

Notice that the stress induced by ion-gyromotion tends to expand the plasma, the same behavior as that indicated by conventional isotropic pressure. Again, the right-hand side of Eq. (35) contains the Ohmic and viscous entropy heat production.

Equation (35) involves the large-aspect-ratio approximation but is not expressed in terms of reduced field variables. We next compute the reduced energy functional in terms of p, ψ, F, and v from the reduced fluid equations (25)–(28). First note that, without surface terms, we have the identity $\langle[f,g]h\rangle = \langle f[g,h]\rangle$. Then, we calculate

$$-\left\langle F\frac{\partial}{\partial\tau}\left(1-\frac{5}{2}a_i^2\nabla_\perp^2\right)\nabla_\perp^2 F\right\rangle$$
$$+\left\langle v\frac{\partial}{\partial\tau}(1-2a_i^2\nabla_\perp^2)\,v\right\rangle$$
$$-\left\langle J\frac{\partial}{\partial\tau}\psi\right\rangle + \left\langle\frac{1+T_i/T_e}{2\beta}p\frac{\partial}{\partial\tau}p\right\rangle$$

and find that

$$\frac{\partial}{\partial\tau}\langle\hat{H}\rangle = -\eta\langle J^2\rangle - \frac{3}{10}\beta\mu\,\langle(\nabla_\perp^2 F)^2 + 4|\nabla_\perp v|^2\rangle, \tag{36}$$

where the reduced energy functional has the form

$$\langle\hat{H}\rangle = \left\langle\frac{v^2+|\nabla_\perp F|^2}{2}+\frac{B_z^2+|\nabla_\perp\psi|^2}{2}+\frac{1+T_i/T_e}{4\beta}p^2\right\rangle$$
$$+a_i^2\left\langle\left(\frac{5}{4}\nabla_\perp^2 F\right)^2+|\nabla_\perp v|^2\right\rangle. \tag{37}$$

By noticing that direct reduction from Eqs. (34) and (35) leads exactly to Eqs. (36) and (37), the self-consistency of our model is further indicated.

On the other hand, to understand the appearance of $\langle 2hp\rangle$ in the energy functional of the high-β version of RMHD,[4] we first note that this system satisfies

$$\left\langle\frac{\partial}{\partial t}\frac{P}{\gamma-1}\right\rangle = \langle\mathbf{V}\cdot\nabla P\rangle,$$

where γ is the ratio of specific heats. This describes $2/(\gamma-1)$ dimensional adiabatic compression. From Eqs. (31)–(33), with $\delta\to 0$, we find that the above time rate of change of the internal energy is reduced to

$$-\epsilon^3(v_A^3/a)m_i n\langle p[F,2h] + p\nabla_\parallel v\rangle.$$

This energy involves v because of parallel compressibility. However, the evolution of the kinetic energy because of parallel flow is

$$\frac{\partial}{\partial\tau}\left\langle\frac{v^2}{2}\right\rangle = -\langle v\nabla_\parallel p\rangle.$$

Thus the term $\langle -2hp\rangle$ which appears in the conserved energy functional of the high-β version of RMHD comes from

$$\left\langle\frac{\partial}{\partial t}\frac{P}{\gamma-1}+\frac{\partial}{\partial t}m_i n\frac{V_\parallel^2}{2}\right\rangle\rightarrow\frac{\partial}{\partial\tau}\langle-2hp\rangle. \tag{38}$$

It also implies that even though the parallel flow is decoupled from the reduced system within the scalings of the high-β version of shear-Alfvén dynamics, it still implicitly exchanges energy with the thermal field.

IV. APPLICATION TO TOROIDAL PARTICLE TRANSPORT

Plasma transport in axisymmetric toroidal devices was first studied by Pfirsch and Schlüter[18] and later on extensively studied in more rigorous ways that combine both kinetic and fluid approaches.[19] In this section we show how we can reproduce, from our model, the isothermal particle transport coefficient in the isothermal Pfirsch–Schlüter regime, where the trapped population is negligibly small.

We first choose the transport ordering in the reduced system; that is,

$$\frac{\partial}{\partial\tau}\sim O(\eta\beta);\quad F,v\sim O(\delta\beta);\quad p\sim O(\beta);\quad J,\psi\sim O(1).$$

Also, axisymmetry gives $\nabla_\parallel=-[\,\psi,\ ]$. Therefore, the equilibrium configuration can be derived from the lowest order (in δ,β) of Eqs. (25)–(28). Note that both the plasma pressure and the electrostatic potential are flux functions

$$p=\bar{p}(\psi)+O(\delta^2\beta^2),$$
$$\varphi=\bar{\varphi}(\psi)+O(\eta\beta),$$

and that the return flows are included in the equilibrium parallel flow

$$v=\bar{v}(\psi)+2h\frac{\partial}{\partial\psi}\left(\bar{\varphi}+\delta\frac{T_i}{T_e}\bar{p}\right)+O(\eta\beta),$$
$$J=\bar{J}(\psi)-h\left(1+\frac{T_i}{T_e}\right)\frac{\partial}{\partial\psi}\bar{p}, \tag{39}$$

where Eq. (39) is also recognized as the reduced Grad–Shafranov equation.

Now, we extract the radial particle transport by flux surface averaging $\langle\ \rangle_\psi$ the equation of particle conservation, just as in the usual neoclassical transport theory. Before doing so, we note two important identities relating to flux surface averaging in reduced form:

$$\langle\nabla_\parallel f\rangle_\psi=0,$$
$$\langle[F,G]\rangle_\psi=\frac{1}{q}\frac{\partial}{\partial\psi}(q\langle G\nabla_\parallel F\rangle_\psi)(1+O(\epsilon)),$$

where q is the plasma safety factor. Then, by calculating $\langle$Eq. (25)$\rangle_\psi$ and $\langle$Eq. (26)$\rangle_\psi$, we have

$$\Gamma^\psi=-\eta\beta\frac{1+T_i/T_e}{2}\left(\frac{\partial}{\partial\psi}\bar{p}\right)\langle|\nabla\psi|^2\rangle_\psi+2\beta\langle h\nabla_\parallel(F+2\delta B_z)\rangle_\psi,$$
$$\langle h\nabla_\parallel(F+2hB_z)\rangle_\psi\sim\eta\langle hJ\rangle_\psi.$$

Finally, the radial flux has the form

$$\Gamma^\psi=-\eta\beta\frac{1+T_i/T_e}{2}\left(\frac{\partial}{\partial\psi}\bar{p}\right)\langle|\nabla\psi|^2\rangle_\psi(1+2q^2),$$

where the last term $2q^2$ is just the usual toroidal enhancement. Of course, ∇T effects[17] are omitted as mentioned before.

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APPENDIX A: CROSS-FIELD VISCOSITY

The cross-field viscosity tensor is usually derived via kinetic theory; here, we consider a method for deriving this tensor from exact moment equations. An earlier, linear application of this method was presented by Lee.[20] The result differs from Braginskii's result by including higher-order FLR corrections. We first define a tensor operator

$$\mathsf{K}(\mathsf{P})\equiv[(\mathsf{P}\times\mathbf{b})+\mathrm{Tr}],$$

such that Eq. (3) can be written as

$$(\mathsf{K}(\mathsf{P})=(1/\Omega)\hat{\mathsf{S}}, \tag{A1}$$

where

$$\mathsf{S}\equiv\left(\frac{\partial}{\partial t}+\mathbf{V}\cdot\nabla\right)\mathsf{P}+\nabla\cdot\mathsf{q}+[(\mathsf{P}\cdot\nabla\mathbf{V})+\mathrm{Tr}]+\mathsf{P}(\nabla\cdot\mathbf{V})-\mathsf{C}. \tag{A2}$$

Recall that of a tensor is defined by

$$\hat{\mathsf{A}}\equiv\mathsf{A}-\mathsf{A}^{\mathrm{CGL}}\equiv\mathsf{A}-\{\mathbf{bb}(\mathsf{A}:\mathbf{bb})+(\mathsf{I}-\mathbf{bb})[(\mathsf{I}-\mathbf{bb})/2:\mathsf{A}]\},$$

for the reason that $\mathsf{K}(\mathsf{A}^{\mathrm{CGL}})=0$. Then, by using the following tensor identity for any symmetric second-order tensor A:

$$\mathbf{b}\times\mathsf{A}\times\mathbf{b}=\mathsf{A}-(\mathbf{bb}\cdot\mathsf{A}+\mathrm{Tr})-(\mathsf{I}-\mathbf{bb})(\mathsf{I}-\mathbf{bb}):\mathsf{A}+\mathbf{bb}(\mathbf{bb}:\mathsf{A}),$$

we find the inverse operator K^{-1},

$$\mathsf{K}^{-1}(\mathsf{A})=\tfrac{1}{4}\{[\mathbf{b}\times\mathsf{A}\cdot(\mathsf{I}+3\mathbf{bb}]+\mathrm{Tr}\}.$$

By simple algebra one can prove that the homogeneous solution of Eq. (A1), i.e., solution of $\mathsf{K}(\mathsf{P})=0$, must be $\mathsf{P}^{\mathrm{CGL}}$. This agrees with the Chew–Goldberger–Low result that the lowest-order (in ρ_i) pressure tensor is $\mathsf{P}^{\mathrm{CGL}}$. We thus obtain

$$\mathsf{P}=\mathsf{K}^{-1}(\hat{\mathsf{S}})+\mathsf{P}^{\mathrm{CGL}}=\frac{1}{4\Omega}[\mathbf{b}\times\hat{\mathsf{S}}\cdot(\mathsf{I}+3\mathbf{bb})+\mathrm{Tr}]+\mathsf{P}^{\mathrm{CGL}}. \tag{A3}$$

To derive $\hat{\mathsf{S}}$ in terms of observable quantities, it is necessary to specify one or more small parameters such as ϵ,β, etc. For the present paper, we adopt the large aspect ratio scalings and find that

$$\hat{\mathsf{S}}=\left(\frac{\partial}{\partial t}+\mathbf{V}\cdot\nabla\right)\hat{\mathsf{P}}_0+[(\hat{\mathsf{P}}_0\cdot\nabla\mathbf{V})+\mathrm{Tr}]+P_\perp\hat{\mathsf{W}}-\hat{\mathsf{C}}+O(\epsilon^3)\equiv\hat{\mathsf{S}}^g-\hat{\mathsf{C}}, \tag{A4}$$

where

$$\hat{\mathsf{P}}_0=\mathsf{K}^{-1}[(nT_i/\Omega)\hat{\mathsf{W}}] \tag{A5}$$

is the $O(\epsilon)$ cross-field viscosity, which is identical to Braginskii's result. Here we have neglected the contribution from q which is $O(\rho_i^3)$. We remark that Newcomb[21] has studied an incompressible, collisionless nonlinear system with FLR corrections in the paraxial limit. He includes terms from q; however, we note that these terms are important only when particle trapping effects are considered.

Consider now the collisional cross-field viscosity originating from $\widehat{\mathsf{C}}$. Braginskii's result is accurate enough (under our orderings on ν_i) for the reduction process, but we present the derivation for instructional purposes. The second-order collisional moment tensor C, after some manipulations, can be written as

$$\mathsf{C} \simeq m_i\, \Delta \int d\mathbf{v}\, f_i(\mathbf{v}) \left(\mathbf{vv} - \frac{v^2}{3}\,\mathsf{I}\right)\Theta(x),$$

where

$$x^2 = m_i v^2/2T_i, \quad \Delta = -\tfrac{8}{2}\sqrt{2P}\,\nu_i,$$

$$\Theta(x) = \left(x^2 - \frac{3}{2}\right)\frac{\mathrm{erf}(x)}{x^5} + \frac{3}{\sqrt{P}}\,\frac{\exp(-x^2)}{x^4},$$

and $\mathrm{erf}(x)$ is the usual error function. Note that we have neglected terms of order m_e/m_i resulting from ion–electron collisions. We first expand the distribution function in spherical-harmonics and Laguerre polynomials. Then we use the identity (under the assumption that zeroth order f_i is a moving Maxwellian) that

$$\mathsf{K}(\mathsf{M}_{2k}) = \frac{1}{\Omega}\, n m_i \left(\frac{T_i}{m_i}\right) k\, \frac{(k+\frac{3}{2})!}{(\frac{5}{2})!}\,\mathsf{W},$$

where

$$\mathsf{M}_{2k+2} = \int d\mathbf{v}\, \frac{m_i \mathbf{vv} v^{2k}}{2^k} f_i,$$

and obtain, in agreement with Ref. 15,

$$\widehat{\mathsf{C}} \simeq -\tfrac{6}{5}\,\nu_i\, \widehat{\mathsf{P}}_0.$$

We remark that the above technique is very useful in deriving the cross-field moment tensor in the magnetized plasma regime, because of the small factor $1/\Omega$ in front of $\widehat{\widehat{\mathsf{S}}}$. This approach has been used to reproduce the usual neoclassical cross-field velocity (unpublished).

APPENDIX B: FLR corrections to $\mathsf{P}^{\mathrm{CGL}}$

When ion gyromotion becomes important, it is expected that the diagonal form of the CGL tensor breaks down. We start by operating on Eq. (A2) with $(\mathsf{I} - \mathbf{bb})$: and obtain

$$\left(\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla\right)(P_\perp - nT_i) + nT_i[(\mathsf{I} - \mathbf{bb}):\nabla\mathbf{V}] = 0, \tag{B1}$$

and its reduced form

$$\frac{\partial}{\partial t}\frac{P_\perp - nT_i}{n_0 T_i \epsilon} + \left[F, \frac{P_\perp - nT_i}{n_0 T_i \epsilon}\right] = \frac{1}{\beta}\left(\frac{\partial}{\partial \tau} p + [F, p]\right) + \nabla_\parallel v + [F, h]. \tag{B2}$$

By using Eq. (16) and comparing Eq. (B2) with $O(a_i^0)$ terms of the vorticity equation, we find that

$$(P_\perp - nT_i)/n_0 T_i \epsilon = 2\delta \nabla_\perp^2 F - h + O(a_i^3). \tag{B3}$$

We note that this result can also be derived from the linear gyrokinetic theory. The point is, the nonlinearity will not appear here because we need $P_\perp$ only to $O(\epsilon)$.

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