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NONLINEAR TOROIDAL PLASMA DYNAMICS BY REDUCED FLUID MODELS

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Abstract

NONLINEAR TOROIDAL PLASMA DYNAMICS BY REDUCED FLUID MODELS.

Fluid models are presented which generalize reduced MHD by allowing for compressibility, Finite Larmor Radius, and long mean free path in toroidal geometry. The Hamiltonian structure of the models leads to a generalized energy principle for determining linear and nonlinear MHD stability of equilibria with flows and Finite Larmor Radius effects. Neoclassical effects from the long mean free path lead to new dissipative terms giving rotation damping, bootstrap currents and the Ware pinch. Rotation damping from non-ambipolar transport in stellarators can lead to self-consistent plasma currents which strongly reduce or "heal" steady-state magnetic stochasticity (e.g., from coil errors or Pfirsch–Schlüter currents). The bootstrap current in tokamaks causes the growth of nonlinear magnetic islands in the Rutherford regime. Thus, otherwise stable moderate mode number islands can potentially overlap, with serious detriment to confinement.

INTRODUCTION

We present models that generalize reduced MHD by allowing for compressibility, Finite Larmor Radius, and long mean free path. The Hamiltonian structure of the models leads to a significantly generalized energy principle. Applications include linear and nonlinear instabilities and results regarding self-consistent magnetic stochasticity.

1. MORE REALISTIC FLUID MODELS

A. Conventional high β reduced MHD[1] contains three fields: the poloidal flux function ψ, a velocity stream function ϕ, and pressure p. To allow for compressible flows, we add parallel velocity. Previous Alfvénic normalizations are used[1]. FLR terms enter proportional to a parameter δ, defined so that the normalized diamagnetic velocity $(cT/neB)\hat{z} \times \nabla n/\epsilon v_A$ is equal to $\delta\hat{z} \times \nabla p$. We also define $[Q_1, Q_2] = \hat{z} \cdot \nabla Q_1 \times \nabla Q_2$, $\nabla_{\parallel} Q = \frac{\partial Q}{\partial z} - [\psi, Q]$, $F = \phi + k\delta p$, $W = \nabla_{\perp}^2 F$, $J = \nabla_{\perp}^2 \psi$, with $k = T_i/T_e$. The conserved energy is

$$H = \frac{1}{2}\int |\nabla\psi|^2 + |\nabla F|^2 + \left(\frac{1+k}{2}\right)p^2 + v^2 \qquad (1)$$

We therefore write the four-field model with the terms needed for energy conservation on the left, and higher order FLR terms needed to be Hamiltonian on the right.

$$\begin{aligned}\dot{W} + [F, W] - \delta k \nabla \cdot [p, \nabla F] + \nabla_{\parallel} J + (1+k)[h,p] \\ = (k\delta\beta/2)\big\{ -2\delta(1+k)\nabla_{\perp}^2 [h,p] - \nabla_{\perp}^2 \nabla_{\parallel}(v + 2\delta J) \\ + k\delta \nabla^2 [p + 2\beta h, W]\big\}\end{aligned} \tag{2}$$

$$\dot{\psi} + \nabla_{\parallel} F - \delta(1+k)\nabla_{\parallel} p + k\delta^2 \beta \nabla_{\parallel} W = 0 \tag{3}$$

$$\dot{p} + [F,p] + \beta\nabla_{\parallel}(v + 2\delta J) - 2\beta\,[h, F - (1+k)\delta p] = k\delta^2\beta[p + 2\beta h, W] \tag{4}$$

$$\begin{aligned}\dot{v} + [F - kp, v] + \frac{1}{2}(1+k)\nabla_{\parallel} p = -k\delta\left[v, 2\beta h + k\delta^2\beta\nabla_{\perp}^2 p - 2\delta\beta W\right] \\ + \frac{1}{2} k\delta\beta\nabla_{\parallel} W\end{aligned} \tag{5}$$

where h gives the effects of curvature; h is the normalized distance in the direction outward from the torus.

B. In modern tokamaks the collisional mean free path exceeds the device size. Nonetheless, fluid equations can be rigorously derived to describe the region near a rational surface. The calculation uses kinetic theory together with a systematic two-scale expansion in the parallel gradients, and is similar in spirit to previous MHD analysis by Glasser et al. and Kotschenreuther et al. For simplicity we use an aspect ratio expansion and plateau collisionality. For maximum generality, we include FLR effects, parallel compressibility, resistive flux diffusion in the inner layer, semicollisionality, rotation damping of parallel velocity, nonlinear convection and nonlinear modifications of the equilibrium field. A maximal ordering to systematically include all the above, in which all quantities are ordered in terms of ϵ, is: $\Delta r \sim \rho_p$, $\rho_p/r \sim \epsilon^2$, $\omega \sim \epsilon^2 v_i/R$, $\beta \sim \epsilon^2$, $B_p/B \sim \epsilon$, $\nu_e R/v_e \sim \epsilon$, $\sqrt{m_e/m_i} \sim \epsilon$, where Δr, r, v_i, and ρ_p are the layer width, minor radius, ion thermal velocity and poloidal gyroradius.

The final fluid equations have non-dissipative and dissipative terms usually found in slab models, but also additional dissipative terms giving rotation damping, bootstrap currents, and the Ware pinch; similar effects were previously found by Callen and Shaing[2]. The dissipation-free terms are, in fact, the same as those on the left sides of Eqs. (2)–(5), but with $h = 0$ (also, the terms on the right do not occur—they are higher order). The new neoclassical terms in Eqs. (2) and (5) are

$$\dot{W} + \ldots = -\nu \frac{\partial}{\partial r}\left(\frac{\partial F}{\partial r} + \Theta v\right) \tag{6}$$

$$\dot{v} + \ldots = -\Theta\nu\left(\frac{\partial F}{\partial r} + \Theta' v\right) \tag{7}$$

For an axisymmetric tokamak with nearly circular surfaces, $\nu = \sqrt{\pi\beta_i q}$, $\theta = \theta'$ = the ratio of poloidal to toroidal field. We have general expressions for non-circular non-symmetric geometries, which are omitted due to space.

The vorticity is damped at a rate ν, and the parallel velocity is damped at a rate $\theta\theta'\nu \sim \epsilon^2\nu$. For moderate β, the rotation damping term is much larger than the usual inertial term in Eq. (2). For axisymmetry the damping terms tend to equilibrate both F and v to satisfy $v = -\theta^{-1}\partial F/\partial r$, as in equilibrium neoclassical theory. The toroidal angular momentum is conserved by the damping terms for axisymmetry. In a stellarator, $|\theta'| \gg |\theta|$. Then the damping terms tend to make both F and v vanish. This also is the condition in equilibrium neoclassical theory, where the ions are electrostatically confined.

The Ohms' Law and pressure evolution, Eqs. (3) and (4), have neoclassical dissipative terms

$$\dot{\psi} + \ldots = \eta\left[J + \alpha\frac{\partial}{\partial r}(1+k)p - \frac{\alpha}{\delta}\left(\frac{\partial F}{\partial r} + \Theta v\right)\right] \tag{8}$$

$$\dot{p} + \ldots = 2\beta\alpha\eta\frac{\partial}{\partial r}\left[J + \alpha\frac{\partial}{\partial r}(1+k)p - \frac{\alpha}{\delta}\left(\frac{\partial F}{\partial r} + \Theta v\right)\right] \tag{9}$$

For equilibrated ion flows, these give a bootstrap current and Ware pinch in agreement with equilibrium neoclassical theory. For plateau collisionality, $\alpha \sim \epsilon v_e/R\nu_e$.

2. ENERGY PRINCIPLES FOR NONLINEAR STABILITY

Here we present generalizations of the ideal MHD energy principle, δW, for fluid and kinetic models, that can handle equilibrium flow and FLR effects.

Consider a noncanonical Hamiltonian system with field variable $f(z,t)$. Functionals called Casimirs exist for such systems, which are constants of the motion. (For example, in the Vlasov equation with distribution function f, $\int dx\, dv G(f)$ is constant for any function G.) Denote such a Casimir by $C[f]$, and denote the Hamiltonian by $H[f]$. It has previously been shown[3,4] that for each Casimir C, an equilibrium can be found by solving $\frac{\delta H[f]}{\delta f} = \frac{\delta C[f]}{\delta f}$, where $\delta/\delta f$ means functional differentiation with respect to f. Consider perturbations δf away from f_0. We have

$$C[f_0 + \delta f] - C[f_0] = \int dz \frac{\delta C}{\delta f}\delta f + \frac{1}{2}\int dz \frac{\delta^2 C}{\delta f^2}\delta f^2 + \text{Higher order terms} \tag{10}$$

$$H\left[f_0+\delta f\right]-H[f_0]=\int dz\frac{\delta H}{\delta f}\delta f$$
$$+\frac{1}{2}\int dz\frac{\delta^2 H}{\delta f^2}\delta f^2+\text{Higher order terms} \tag{11}$$

Now we can obtain the change in energy from a perturbation δf. Since the system is constrained by $C(f)=$ constant, the only *accessible* perturbations are those where the left side of Eq. (10) vanishes. Then by the equilibrium condition we have that the change in H at constant Casimir C, $\Delta H\big|_c$, is

$$\Delta H\big|_c=\frac{1}{2}\int dz\frac{\delta^2(H-C)}{\delta f^2}\delta f^2+\text{Higher order terms}$$

and we define $F=H-C$ to be the free energy for accessible perturbations. As expected, the perturbed energy is second order in δf. Also note that the energy released or absorbed to create an allowable perturbation δf depends on the equilibrium through C.

If $\delta^2 F$ is positive definite, then all accessible small perturbations δf increase H. Since H is conserved, no small perturbation can grow indefinitely, and we have both linear and nonlinear stability for small δf. (Technically this only implies formal stability, but the step from formal to nonlinear stability is usually straightforward[3,4].)

Note that we have not considered the linearized equations of motion to come to this conclusion. The linearized equations depend on both H and the Poisson bracket. The latter can be quite complicated for noncanonical systems.

If $\delta^2 F$ is indefinite, then either there is a linear instability or a linearly stable negative energy wave (or direction in function space). It is not possible to determine which without considering the Poisson bracket. Nonetheless, note that a linearly stable negative energy wave can lead to serious nonlinear instabilities. For example, explosive instabilities are possible in Hamiltonian systems with three interacting waves with resonant frequencies. More complicated nonlinear phenomena are also possible in continuous media; these are currently under investigation. The important point is that when $\delta^2 F$ is indefinite, an instability of some kind is likely.

Negative energy waves can also be found in simple cases by applying the Brillouin-Laue condition $\omega\partial\epsilon/\partial\omega<0$. We can show that this condition must agree with $\delta^2 F$. However, ϵ can only be easily defined in uniform homogeneous media. Dielectric response theory in non-homogeneous media is very lengthy or intractable. The free energy functional $\delta^2 F$ is far more practical in such cases.

Thus, we believe that $\delta^2 F$ is a better tool than linear spectral theory to find the total instability potential of many configurations.

As an example we consider reduced low β MHD with poloidal flow. Suppose we wish to consider stability of cylindrically symmetric equilibria to perturbations with a given helicity. Let χ be the relevant helical flux, $\chi = \psi - r^2/2q_0$. The Casimirs are $\int H(\psi)dx$ and $\int UG(\psi)$, for arbitrary functions H and G. The free energy F is

$$F = \int \left|\nabla_\perp \left(\chi + \frac{r^2}{2q_0}\right)\right|^2 + |\nabla\phi|^2 + H(\chi) + UG(\chi)dx \qquad (12)$$

Equilibria are given by $\nabla_\perp^2 \left(\chi + r^2/\partial q_0\right) = H'(\chi) + \nabla_\perp^2 GG'$, $\phi = G(\chi)$. To examine stability, take the second variation and rearrange terms to get

$$\delta^2 F = \int \Big(|\nabla\delta\phi - \nabla G'\delta\chi|^2 + |\nabla\delta\chi|^2 \left(1 - G'^2\right) + \delta\chi^2 \left[G''\nabla^2 G + H'' + G'\nabla\cdot G''\nabla\chi\right] \Big) dx \qquad (13)$$

An instability or negative energy wave exists if $\delta^2 F$ can be made negative. To minimize $\delta^2 F$, choose $\delta\phi$ to make the first term vanish. The minimizing $\delta\chi$ can be found by standard techniques, e.g., solve the relevant Eulers equation, or for an approximate answer, insert trial functions and vary their parameters, etc.

For equilibria without flow, $G = 0$, the Eulers equation is equivalent to the small aspect ratio δw result for ideal kinks; the second term gives the stabilizing influence of line bending, and the H'' term gives the destabilizing influence of current gradients. We see that the nonlinear stability with flow can be found by a similar Eulers equation; no qualitatively new features arise in the mathematical analysis given Eq. (13).

3. APPLICATIONS

A. Large self-consistent plasma currents can arise in magnetic stochasticity and islands when the neoclassical transport is not intrinsically ambipolar. Here we consider only steady-state stochasticity (from non-axisymmetric equilibrium Pfirsch-Schlüter currents or coils).

Let us estimate the size of the self-consistent parallel currents and fields from radial electron diffusion in stochasticity. Resonant magnetic perturbations $\delta B_r = \delta B_{\text{ind}} + \delta B_{\text{ext}}$ arise on an integrable background field B, due to both an external agent (i.e. field coils or Pfirsch-Schlüter currents) and the induced plasma response. The flux average radial electron current from parallel motion in the stochasticity is $\langle j_r^e \rangle = \langle \delta j_\| \delta B_r / B \rangle$. Breaking this into the various helicities, which we label by perpendicular wavenumber k, $\langle j_r^e \rangle = \sum_k \left\langle \delta j_\|^{-k} \delta B_r^k / B \right\rangle$. The motion of electrons in a stochastic field is only correlated with the resonant harmonic near its respective rational surface; call the distance Δx_c. Thus, at a given radius, only terms from nearly rational surfaces contribute. Denote the typical spacing between the rational surfaces by λ; then $\Delta x_c/\lambda$ harmonics contribute at a given point, and if all resonant harmonics have roughly the same amplitude we have $\langle j_r^e \rangle \sim (\Delta x_c/\lambda)\, \delta B^k / B \delta j_{\parallel}^{-k}$. The induced magnetic perturbation can be simply computed using the "constant ψ" approximation, which here implies $(\Delta'/k)\, \delta B_{\text{ind}}^k = (4\pi/c) \int dx \delta j_{\parallel}^k \sim (4\pi/c) \delta j_{\parallel}^k \Delta x_c$. Self-consistency is important when $\delta B_{\text{ind}} \gtrsim \delta B_r$, which gives the criterion

$$\left(\frac{\delta B^k}{B}\right)^2 < \frac{4\pi}{c} \frac{\lambda \langle j_r^e \rangle}{B} \frac{k_\perp}{\Delta'} \tag{14}$$

A neoclassical ion current arises because the magnetic stochasticity changes the radial electric field. Equal radial electron and ion current occurs. For typical reactor parameters and ion transport levels consistent with the Lawson criterion, self-consistency is important for $\delta B/B \sim 10^{-2}$, which can cause very large overlapping islands.

More detailed analysis of both the single helicity and multihelicity cases reveals that the self-consistent currents cause the resonant perturbation to be shielded out, and thus islands and stochasticity are "healed" in steady state if the inequality in (13) is satisfied.

B. The bootstrap current term in Ohm's Law can cause magnetic islands to grow in the Rutherford regime. For sufficiently large islands, the vorticity equation implies $\nabla_{\parallel} J = 0$, and J is a flux function, so as in previous analysis the current can be found by taking the flux average of Ohm's Law. Also, the pressure becomes a flux function because of parallel convection by sound waves. The pressure profile is found as in Appendix A of Ref. 6.

We obtain the following evolution equation for Δx, the island width

$$\frac{K_1}{\eta} \frac{d\Delta x}{dt} = \Delta' + k_2 \frac{S' \alpha}{\Delta x} \frac{dp}{dx} \tag{15}$$

where $K_1 \simeq 1.6$. $k_2 \simeq 6$, and $S' = q^2/r(dq/dr)$. For small islands the bootstrap current term dominates and the island grows. For $\Delta' < 0$, the steady-state island width is given by $\Delta x = \frac{K_2 s' \alpha (dP/dx)}{\Delta'}$.

This is similar to previous results for unstable interchanges[5,6]. Thus, as in that case, island overlap always occurs if islands up to sufficiently high mode numbers are considered[6]. For typical parameters and $\beta \sim 1\%$, island overlap occurs for islands with $m \sim 5 - 10$.

One must keep in mind that the above calculation is only valid for single helicity islands; the dynamics may change considerably as overlap is approached. If islands significantly overlap, the stochastic diffusion can be crudely estimated from the quasilinear formula; for $\beta \gtrsim 1\%$, the confinement deterioration is significant for current and future devices and rapidly worsens with increasing β.

C. We find that the shear Alfvén resonant mode is destabilized by gradients in the equilibrium ω_*-profile. Significantly, its growth rate dominates the tearing mode for either large ω_* or low shear and depends only weakly on the resistivity, $\gamma \propto \eta^{1/4}$. Thus, it should predominate in reactors. The mode's stability is independent of Δ'.

D. Numerical studies of the $m = 1$ tearing mode show that it continues exponential growth well into the nonlinear regime. We have an analytic theory explaining this. Space does not allow a full discussion of this, but the primary reason for the rapid growth of the $m = 1$ mode is that its island size depends linearly on the magnetic flux perturbation.

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