Variational Principle and Stability of Nonmonotonic Vlasov-Poisson Equilibria

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Dedicated to Professor Dieter Pfirsch on his 60th Birthday

The stability of nonmonotonic equilibria of the Vlasov-Poisson equation is assessed by using nonlinear constants of motion. The constants of motion make up the free energy of the system, which upon variation yields nonmonotonic equilibria. Such equilibria have not previously been obtainable from a variation principle, but here this is accomplished by the inclusion of a passively advected tracer field. Definiteness of the second variation of the free energy gives a sufficient condition for stability in agreement with Gardner's theorem [5]. Previously, we have argued that indefiniteness implies either spectral instability or negative energy modes, which are generically unstable when one adds dissipation or nonlinearity [6]. Such is the case for the nonmonotonic equilibria considered.

1. Introduction

Besides the ubiquitous linear or spectral approach to plasma stability, use has also been made of nonlinear constants of motion to obtain sufficient conditions for stability [1–5]. This latter approach, which we shall refer to as nonlinear stability, yields stronger stability conditions; in fact, there exist systems that are unstable in spite of spectral stability analyses indicating the contrary. (See [6] for examples.)

The connection between a sufficient nonlinear stability condition, similar to those mentioned above, and the Hamiltonian structure of the two-dimensional Euler fluid equations was given in [7]. In [8] this work was extended (including plasma examples) and the terminology Energy-Casimir method was introduced. (Consult this reference for a review of the literature.)

Recently in [6], the question of why nonlinear stability conditions are typically only sufficient for stability was addressed. Here the "free energy principle" was introuduced, which states that when the free energy functional (energy plus Casimir invariant) is indefinite there is either spectral instability or the system has a negative energy mode. Systems with negative energy modes have stable spectra, but are generically unstable when one adds dissipation or nonlinearity.

In this paper we use variational arguments to treat the nonlinear stability of Vlasov-Poisson equilibria with nonmonotonic distribution functions. This is done by retaining Lagrangian variable information, via a tracer field, in an otherwise strictly Eulerian description. This artifice was used by Lin in obtaining variational principles for the dynamical equations of ideal fluids [9]. It has also been used recently in the context of nonlinear stability for fluids [10, 11].

Our principle result is that Vlasov-Poisson equilibria with nonmonotonic distributions are either spectrally unstable or there exist negative energy modes. For the case of stationary Maxwellian ions with drifting Maxwellian electrons, the free energy principle predicts instability for all finite drift speeds, while spectral stability predicts a drift velocity threshold. This is consistent with the work of [12], where instability below threshold has been observed in numerical computations [13]. The free energy principle predicts the same kind of behavior for equilibria with infinitely massive background ions and a multibumped electron distribution.

In Sects. 2–4 we review the noncanonical Hamiltonian structure for the Vlasov-Poisson equation, present and discuss the meaning of its Casimir invariants, and describe the variational principle for monotonic equilibria, from which, in Sect. 5 we

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show stability. (For background material see [14].) This variational principle may be interpreted as the method of the most probable distribution, as in statistical mechanics, but we give an alternate interpretation in (a discursive) Sect. 6 that is more general and better adapted to dynamics. Originally the idea of this variational principle is contained in [1–4], but when we understand the meaning of the Casimir invariants we see that its content is equivalent to Gardner's theorem [5]. In Sect. 7 we give the bracket structure with the inclusion of the tracer field mentioned above. In Sect. 8 we describe the new kind of Casimir invariant that results from this introduction, and in Sect. 8 we present for the first time the variational principle for nonmonotonic equilibria and use it to assess stability. In Sect. 10 we discuss the example with a multibumped electron distribution. Finally in Sect. 11, we summarize.

2. Hamiltonian Structure of the Vlasov-Poisson Equation

The Vlasov-Poisson equation is

$$\frac{\partial f(z,t)}{\partial t} = -v\frac{\partial f}{\partial x} + \frac{e}{m}\frac{\partial \varphi(x;f)}{\partial x}\frac{\partial f}{\partial v}, \tag{1}$$

where, as usual, $f(z,t)$ is the phase space density at the phase space point $z=(x,v)$ for a charged species of particles with charge e and mass m. Now we consider only a single species, but later we will generalize. The electrostatic potential φ is to be viewed as a functional of f determined via Poisson's equation $\varphi_{xx} = -e\int f\,\mathrm{d}z$; thus

$$\varphi(x;f) = e\int V(z,\tilde{z})\,f(\tilde{z})\,\mathrm{d}\tilde{z},$$

where $V(x,\tilde{x})$ is the single particle potential (assumed spatially invariant). The Hamiltonian for this system is the energy functional

$$H[f] = \int T(z)\,f(z)\,\mathrm{d}z + \tfrac{1}{2}e^2\iint V(z,\tilde{z})\,f(z)\,f(\tilde{z})\,\mathrm{d}\tilde{z}\,\mathrm{d}z, \tag{2}$$

where $T(z)=mv^2/2$ is the particle kinetic energy. This system possesses the following noncanonical Poisson bracket [15]:

$$\{F,G\} = \frac{1}{m}\int f(z)\left[\frac{\delta F}{\delta f},\frac{\delta G}{\delta f}\right]\mathrm{d}z \tag{3}$$

where the inner bracket $[\,,\,]$ is defined by $[k,h] = k_x h_v - k_v h_x$. Note that $\delta H/\delta f = T + e\varphi \equiv \mathscr{E}$, where $\mathscr{E}$ is the total particle energy. Evidently,

$$\partial f/\partial t = \{f,H\} = -[f,\mathscr{E}], \tag{4}$$

where $-[f,\mathscr{E}]$ is equivalent to the right hand side of (1).

In obtaining (4) it was necessary to integrate by parts and neglect the surface contribution. For any phase space functions g, h and k we have the identity

$$\int_{\mathrm{D}} g\,[h,k]\,\mathrm{d}z = -\int_{\mathrm{D}} h\,[g,k]\,\mathrm{d}z + \int_{\mathrm{C}} g\,h\,\hat{\boldsymbol{t}}\cdot\nabla k\,\mathrm{d}s, \tag{5}$$

where D is an arbitrary domain in the x-v plane that is bounded by the curve C, and $\hat{\boldsymbol{t}}$ is a unit vector tangent to C. If all phase functions satisfy the boundary condition of constancy on C, then boundary terms like those of (5) can be systematically neglected. In particular when D is the entire x-v plane, phase functions must vanish sufficiently rapidly as $v\to\pm\infty$ and $x\to\pm\infty$. Another case of interest is when D is a semi-infinite strip of finite extent in the x-direction. Again we require phase functions to vanish sufficiently rapidly as $v\to\pm\infty$, but assume periodicity in the x coordinate. For all of these above cases the boundary terms vanish; thus we will freely use the identity

$$\int_{\mathrm{D}} g\,[h,k]\,\mathrm{d}z = -\int_{\mathrm{D}} h\,[g,k]\,\mathrm{d}z. \tag{6}$$

In the future we will not bother with the subscript D on our integrals.

3. Casimir Invariant and Interpretation

The bracket for this system [Eq. (3)] possesses the general Casimir invariant given by

$$C[f] = \int \mathscr{C}(f)\,\mathrm{d}z, \tag{7}$$

where $\mathscr{C}$ is an arbitrary function. A Casimir invariant is a constant of motion that is built into the noncanonical Poisson bracket. It commutes with all Hamiltonians; i.e. $\{C,F\}=0$ for any functional F. That C is a Casimir invariant is easily shown by making use of $\delta C/\delta f = \partial\mathscr{C}/\partial f$, Eq. (6), and the fact that $[\partial\mathscr{C}/\partial f, f] = 0$.

In order to unterstand the meaning of this constant let us ask the following question: Given that C

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is constant, what information about f is obtained? Alternatively, one may wonder what constraints are placed on f by the specification of C. Actually C is an infinity of constants since the function $\mathscr{C}$ is arbitrary. For example, one could consider the infinite sequence of constsants defined by $C^n[f] = \int f^n \mathrm{d}z$, but we will subsequently consider a different set of constants. One would expect that the specification of C for all $\mathscr{C}$ would say a great deal about f, maybe even determine it uniquely, as is the case for systems that are integrable by the inverse scattering method. This is not the case, but we will exploit the arbitrariness of $\mathscr{C}$ in order to see what C does imply.

To begin with we divide our domain D into cells of equal phase space areas, Δz. We suppose that the cell size is sufficiently small so that f can be assumed to have a constant value in each cell. Specification of the field f amounts to specifying a number f_i for each cell. The Casimir C then becomes

$$C = \sum_i \mathscr{C}(f_i)\,\Delta z\,,$$

where the index i ranges over all the cells. Now we choose for the function $\mathscr{C}$ the characteristic function $\tilde{\mathscr{C}}$ defined by

$$\tilde{\mathscr{C}}(f;\tilde{f}) = \begin{cases} 1 & \text{if} \quad f = \tilde{f} \\ 0 & \text{if} \quad f \neq \tilde{f}. \end{cases} \tag{9}$$

With this choice we call the Casimir $\tilde{C}$ and observe that $\tilde{C}/\Delta z$ is equal to the number of cells for which the value of f is equal to $\tilde{f}$. Since $\tilde{f}$ is arbitrary we can use the Casimir C to determine the number of cells with any value of f. Note that this Casimir does *not* tell us where these cells are located, only that there are so many with such-and-such a value of f. This is because $\tilde{C}/\Delta z$ does not depend upon the index i, which lets us know (phase) spatially where a given cell is located. The failure of C to determine spatial correlation arises because these constants are strictly local in nature; i.e. unlike the infinite hierarchy of constants possessed by the Korteweg-de Vries and other integrable equations, the infinity of constants C does not contain derivatives.

Since f corresponds to a phase space density (number of particles per unit phase space volume) there is a physical interpretation of the Casimir invariants C. (Note that this interpretation also applies to the case where f is a point vortex density.) It is apparent that the quantities f_i correspond to the number of particles in the i-th cell. Thus specification of the Casimir C places a constraint on the placement of particles in the cells; i.e. specifying $\mathscr{C}(f)$ tells us the number of cells with a given number of particles. This constant clearly arises because of Liouville's theorem for particle dynamics.

4. Variational Principle for Monotonic Equilibria

As alluded to in Sect. 1, equilibria are stationary points of the free energy $F = H + C$, for which in the case of the Vlasov-Poisson equation H is given by (2) and C by (7). Upon varying this quantity we obtain

$$\begin{aligned}\delta F[f;\delta f] &= \int (\mathscr{E} + \partial\mathscr{C}/\partial f)\,\delta f\,\mathrm{d}z \\ &= \int (m v^2/2 + e\,\varphi + \partial\mathscr{C}/\partial f)\,\delta f\,\mathrm{d}z\,.\end{aligned} \tag{10}$$

Thus equilibria (f_e) are given by

$$\mathscr{E} + \partial\mathscr{C}(f)/\partial f = 0\,. \tag{11}$$

There are two things to notice about these equilibria that are obtained as extremals of F: firstly, in order to solve for $f_e(\mathscr{E})$ the quantity $\partial\mathscr{C}/\partial f$ must be monotonic and therefore its inverse must also be monotonic. This gives

$$f_e = f_e(\mathscr{E})\,, \tag{12}$$

where $f_e(\mathscr{E})$ is a monotonic function of $\mathscr{E}$, and we see that there is a one-to-one correspondence between an equilibrium and a choice for the function $\mathscr{C}$. Secondly, since f_e is a function of $\mathscr{E}$, only velocity symmetric distributions $[f_e(v) = f_e(-v)]$ are obtained. Evidently, extremals of F only make up a subclass of equilibria of the Vlasov-Poisson equation, since this system is known to possess nonmonotonic and velocity asymmetric (for untrapped particles) equilibria. (Below we will rectify this deficiency.)

5. Stability of Monotonic Equilibria

For the above restricted class of equilibria we can obtain a criterion for stability by taking the second

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variation of F; viz.

$$\delta^2 F[f;\delta f] = e^2 \iint \delta f(z)\, V(z,\tilde{z})\, \delta f(\tilde{z})\, \mathrm{d}z\, \mathrm{d}\tilde{z} + \int (\delta f)^2\, \partial^2 \mathscr{C}/\partial f^2\, \mathrm{d}z. \quad (13)$$

The time rate of change of this quantity can be identified with the input power due to an external current source: $\mathscr{P} = -\int J_{\mathrm{ext}}\, \delta E\, \mathrm{d}x = \partial(\delta^2 F/2)/\partial t$. This identification is easily made by making use of the linearized Vlasov equation and the Maxwell-Ampére law with the inclusion of the external current.

Observe that the first term of (13) is positive definite (it corresponds to the second variation of the electrostatic energy which goes as the square of the electric field), while the second term will be positive definite provided $\partial^2 \mathscr{C}/\partial f^2 > 0$. For stability this must be true over the entire domain of integration when f is set equal to f_{e}, since we can make $\delta^2 F$ negative by choosing δf such that the first term of (13) vanishes and such that δf is localized where $\partial^2 \mathscr{C}/\partial f^2 < 0$. This statement translates into a statement about f_{e}: upon differentiating (11) with respect to $\mathscr{E}$ we obtain $\partial f_{\mathrm{e}}/\partial \mathscr{E} = -[\partial^2 \mathscr{C}(f_{\mathrm{e}})/\partial f^2]^{-1}$. Therefore we have stability if f_{e} is *any* monotonic decreasing function of the energy. The case where $\delta^2 F$ is indefinite is treated in Section 9.

6. Reciprocal of the Method of the Most Probable Distribution

The above variation principle for dynamical equilibria of the Vlasov-Poisson equation can be compared to the statistical mechanical method of the most probable distribution for thermodynamic equilibrium. Again suppose that we have divided our phase space into cells, of size Δz, where each cell is labelled by the index i. The index i represents the phase space coordinates of a given cell. Also suppose that there is an energy ε_i associated with the i-th cell and let f_i represent the number of particles in this cell. The probability of finding a system composed of N particles in the state where there are f_1 particles in cell 1, f_2 particles in cell 2, etc. is given by $P = \mathrm{const} \times n!/(f_1!\, f_2! \ldots)$. In the method of the most probable distribution it is the natural log of this P, the entropy, that is maximized subject to the constraints of constant particle number and energy:

$$N = \sum_i f_i, \qquad E = \sum_i f_i\, \varepsilon_i. \quad (14)$$

This is achieved by utilizing Lagrange multipliers, and results in the Maxwell-Boltzmann law. We will see that this case corresponds to the choice: $\mathscr{C}(f) = \lambda_1 f + \lambda_2 f \ln(f)$, where $\lambda_{1,2}$ are constants. [Using Sterling's approximation $\ln(P) \approx \sum -f_i \ln(f_i)$.]

Extremals obtained by varying a quantity subject to a constraint are the same as those of the reciprocal problem where the roles of the quantity and constraint are reversed (except for singular cases that are not of interest here). This is apparent if one recalls that the Lagrange multiplier method is based upon the idea that extrema occur at points where the surfaces defined by the quantity and constraint are tangent. Since for standard Hamiltonian systems dynamical equilibria are stationary points of the energy, and for noncanonical Hamiltonian systems equilibria are stationary points of the free energy, it is more natural to invert the variational principle of the preceding paragraph: we vary the Hamiltonian (the energy) subject to the constraint of constant entropy (as well as particle number). Note that the entropy is a Casimir and now it serves merely as a constraint. (For the function $\mathscr{C}$ to represent the usual entropy, an extensive quantity, it must satisfy the derivation property $\mathscr{C}(fg) = f\mathscr{C}(g) + g\mathscr{C}(f)$. The interpretation of entropy as a constraint is a third interpretation of entropy, differing from the notion of accessible phase space that generally appears in statistical mechanics and the information theoretic interpretation [16].) It is now apparent how one can generalize the method of the most probable distribution by considering other constraints (Casimir invariants) besides that corresponding to the usual entropy.

Consider now an energy minimization principle. Suppose we have N particles that are to be placed in cells (labelled by i). Associated with each cell is an energy ε_i. We wish to minimize the total energy

$$E = \sum_i f_i\, \varepsilon_i \quad (15)$$

subject to the constraint that there are $\tilde{f}_1$ particles in m_1 cells, $\tilde{f}_2$ particles in m_2 cells ... or in general $\tilde{f}_j$ particles in m_j cells.

If there were no constraints the state of minimum energy would be to put all the particles in the cell

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with the smallest ε_j, call it ε^*. The energy would then be $E_{\min} = N\varepsilon^*$. Since there is no bound on the cell energy value, we know $E_{\max} = \infty$. Thus we have a well-posed minimization principle, with the obvious answer, which is to put most the particles in the cells with the least energy values while obeying the constraints. This yields a distribution that is monotonically decreasing in energy. In essence what we have done is to restate Gardner's theorem [5].

What we would like to do now is state this principle in mathematical terms as a variational principle. Since the solution of this problem is to find f_i, we would like to build an action for this quantity: $A = H + C$, where $H(f_i)$ is identical to the energy given by (15) and C represents the constraints. The particle number constraint is easily stated as in (14)

$$\sum_i f_i - N = 0 \tag{16}$$

while the constraint that there exist m_j cells with $\tilde{f}_j$ particles can be written as

$$\sum_i I(f_i - \tilde{f}_j) - m_j = 0, \tag{17}$$

where the function I is given by

$$I(x) = \begin{cases} 1 & \text{if} \quad x = 0, \\ 0 & \text{if} \quad x \neq 0. \end{cases} \tag{18}$$

Thus our problem is to minimize the following:

$$A(f_i) = \sum_i f_i \varepsilon_i + \lambda_N \Big(\sum_i f_i - N \Big) + \sum_j \lambda_j \Big[\sum_i I(f_i - \tilde{f}_j) - m_j \Big], \tag{19}$$

where the f_i are integers and the problem only makes sense of $\sum m_j = N$. Here the λ's, the Lagrange multipliers, are determined by the specification of the constraint values N and the m_j. In general these relationships are complicated and particular to the problem at hand, but we note that the Lagrange multipliers have the physical interpretation of being the rate of change of the total energy evaluated on the extremal f_i's with respect to the constraint values m_j; i.e. $\partial E(f_i)/\partial m_j = \lambda_j$.

Since in (19) f_i only takes on integer values we cannot use calculus, but we are really interested in the continuum limit anyway. Thus we let $f_i \to f(x, v)$, where (x, v) is any point in D. The action becomes a functional of the function f. It is clear that the first two terms of (19) become functionals of f such that their functional derivative is $\mathscr{E} + \lambda_N$, while the continuum limit of the last term is more subtle. In (19) the function I is only defined on the integers. We will replace this discrete function by a tent function extension or linear interpolation, which is represented by

$$\begin{aligned} I(f - \tilde{f}) &\to T_{\Delta f}(f - \tilde{f}) \\ &= \frac{1}{\Delta f}[H(f - \tilde{f} + \Delta f)[f + \Delta f - \tilde{f}] \\ &\quad + 2H(f - \tilde{f})[\tilde{f} - f] + H(f - \tilde{f} - \Delta f)[f - \Delta f - \tilde{f}], \end{aligned} \tag{20}$$

where H, the Heavyside function, is defined by

$$H(x) = \begin{cases} 1 & \text{if} \quad x > 0, \\ 0 & \text{if} \quad x < 0. \end{cases} \tag{21}$$

In the continuum limit the last term of (19) becomes

$$\sum_j \lambda_j [\int T_1(f - \tilde{f}_j)\, \mathrm{d}z - m_j]. \tag{22}$$

Observe that (22) involves the integration over the tent function expansion of a function of f

$$\mathscr{S}(f) = \sum_j \lambda_j T_1(f - \tilde{f}_j). \tag{23}$$

Note that the Lagrange multipliers, λ_j, are the values that the function $\mathscr{S}$ takes on at each vertex; in principle they are determined by the numbers m_j, which physically correspond to the number of particles in each cell.

Thus we see that the continuum limit of our energy minimization principle is the variational principle given above, for which we had stability if f was monotonic.

7. Hamiltonian Structure with Tracer Field

Now we generalize the above to yield a principle for general nonmonotonic equilibria. This requires the inclusion of Lagranian variable information, which is accomplished by envisioning the dumping of dye into the phase space fluid. The intensity of the dye is measured by a tracer field, $g(x, v)$, that advects with the phase velocity $(v, \pm E)$. Although we can do general nonmonotonic equilibria with any number of species, we will first consider the case where there is a respondable ion background

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with a monotonic distribution through which streams a distribution of electrons. The equations of motion in this case are

$$\frac{\partial f_\alpha(z,t)}{\partial t} - v\frac{\partial f_\alpha}{\partial x} - \frac{e\,\alpha}{m\,\alpha}E\frac{\partial f_\alpha}{\partial v}, \tag{24}$$

$$\frac{\partial g(z,t)}{\partial t} = -v\frac{\partial g}{\partial x} + \frac{e}{m_e}E\frac{\partial g}{\partial v}, \tag{25}$$

where α indicates species, e meaning electrons and i meaning ions; e.g., m_e is the electron mass and $e_e = -e$, e being the absolute value of the electron charge. In this case the tracer field advects with the electrons.

The noncanonical Poisson bracket for the addition of a passive advective scalar field to an existing system takes a general form given for our example by the following:

$$\{F,G\} = \sum_\alpha \int \frac{1}{m_\alpha} f_\alpha [F_\alpha, G_\alpha]\,\mathrm{d}z + \int g\,[[F_e, G_g] + [F_g, G_e]]\,\mathrm{d}z, \tag{26}$$

where we have introduced the notation: $F_\alpha \equiv \delta F/\delta f_\alpha$ and $F_g \equiv \delta F/\delta g$. Utilizing the Hamiltonian

$$H = \sum_\alpha \int T_\alpha(z)\, f_\alpha(z)\,\mathrm{d}z + \frac{e^2}{2}\iint V(z,\tilde{z}) \cdot [f_e(z) - f_i(z)][f_e(\tilde{z}) - f_i(\tilde{z})]\,\mathrm{d}\tilde{z}\,\mathrm{d}z. \tag{27}$$

Equations (23) and (24) can be written in the form

$$\frac{\partial f_\alpha}{\partial t} = \{f_\alpha, H\} = -[f_\alpha, \mathscr{E}_\alpha], \quad \frac{\partial g}{\partial t} = \{g, H\} = -[g, \mathscr{E}_e], \tag{28}$$

where $\mathscr{E}_\alpha = (m_\alpha v^2)/2 + e_\alpha \varphi$.

8. Casimir Invariants with Tracer Field

The Casimir invariants for the bracket of (26) are

$$C_1 = \int \mathscr{C}(g)\,\mathrm{d}z, \quad C_2 = \int \mathscr{T}(g)\, f_e\,\mathrm{d}z, \quad C_3 = \int \mathscr{U}(f_i)\,\mathrm{d}z, \tag{29}$$

where $\mathscr{C}$, $\mathscr{T}$, and $\mathscr{U}$ are arbitrary functions. The Casimir invariants C_1 and C_3 have the same interpretation as that given above for the single species Casimir invariant except they involve the tracer field g and the ion phase space density f_i.

Now we interpret the new Casimir, C_2, that involves the two fields f_e and g. This Casimir is a measure of how "mixed up" the two fields are. Consider

$$C_2 = \int \mathscr{T}(g)\, f_e\,\mathrm{d}z, \tag{30}$$

where now $\mathscr{T}(g)$ is an arbitrary function. Again dividing the domain into cells and letting $\mathscr{T}(g)$ be the characteristic function of (9), we obtain

$$\tilde{C}_2 = \sum_i f_i\,\Delta z. \tag{31}$$

Thus with this Casimir we are able to determine the sum of the values of f_e on those cells with a particular value of the field g. We are also in a position to obtain the average value of the field f_e on the cells with a given value of g; this quantity is given by $\tilde{C}_2/\tilde{C}_1$. In fact it is apparent that we can obtain the average of f_e subject to a probability density given by $\mathscr{T}(g)$,

$$\langle f_e \rangle_{\mathscr{T}} = \frac{\int f_e\,\mathscr{T}(g)\,\mathrm{d}z}{\int \mathscr{T}(g)\,\mathrm{d}z} = C_2/C_1. \tag{32}$$

Thus the variable g plays a role that is analogous to that of the energy for usual averages in statistical mechanics. We can also use the Casimir invariants C_1 and C_2 to obtain an average of g with f_e as the density,

$$\langle g \rangle = \frac{\int f_e\, g\,\mathrm{d}z}{\int f_e\,\mathrm{d}z}. \tag{33}$$

A sort of cross correlation for the two fields is given by the following:

$$R(f_e, g) = \frac{\int f_e\, g\,\mathrm{d}z}{(\int f_e\,\mathrm{d}z)(\int g\,\mathrm{d}z)}. \tag{34}$$

Since $R(f_e, g)$ is a Casimir this "correlation" is conserved by the dynamics. Observe that $R(g,g) = \langle g^2 \rangle / \langle g \rangle^2$, where the average is with respect to uniform density.

There is another (equivalent) interpretation of the Casimir invariants C_1 and C_2 that is enlightening. Suppose that g corresponds physically to a density, then the quantities g_i corresponds to the number of particles (the number of dyeons) in the i-th cell. We can view the Casimir C_1 as being a constraint on the placement of dyeons in the cells; for example, specifying $\mathscr{C}(g)$ tells us the number of cells with a given number of dyeons. Now in our example both

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fields correspond to densities, the field f_e, of course, corresponding to the electron density. Now C_1 tells us the number of cells that contain a particular number of dyeons, and C_2 tells us the total (or average) number of electrons placed in these cells.

In closing this section we note that the equations of reduced magnetohydrodynamics possess Casimir invariants analogous to C_1 and C_2, where the role of g is played by the parallel vector potential and that of f_e by the scalar vorticity [17]. In this case C_1 and C_2 correspond respectively to the magnetic and cross helicities which are conserved on any given flux surface.

9. Nonmonotonic Equilibria and Stability

Now we return to the variational principle for equilibria; i.e. we extremize the free energy functional

$$F = H + \sum_{i=1}^{3} C_i . \tag{35}$$

Varying with respect to f_i, f_e and g respectively gives

$$\frac{\delta F}{\delta f_i} = \mathscr{E}_i + \mathscr{W}'(f_i) = 0, \tag{36a}$$

$$\frac{\delta F}{\delta f_e} = \mathscr{E}_e + \mathscr{T}(g) = 0, \tag{36b}$$

$$\frac{\delta F}{\delta g} = \mathscr{C}'(g) + f_e \mathscr{T}'(g) = 0. \tag{36c}$$

Assuming that $\mathscr{W}'$ has an inverse yields the following arbitrary monotonic ion distribution function:

$$f_i = f_i(\mathscr{E}_i). \tag{37}$$

The choice of the Casimir C_1 determines this distribution. Now suppose that $\mathscr{T}$ is invertible but otherwise arbitrary. Equation (36b) then implies

$$g = g(\mathscr{E}_e). \tag{38}$$

Solving (36c) for f_e and then making use of (38) yields the following:

$$f_e = -\frac{\mathscr{C}'(g)}{\mathscr{T}'(g)} = \mathscr{V}(\mathscr{E}_e). \tag{39}$$

Since the function $\mathscr{C}'(g)$ is arbitrary, $\mathscr{V}(\mathscr{E}_e)$ is an arbitrary function of $\mathscr{E}_e$: it need not be monotonic. The monotonicity of the tracer field, as a function of $\mathscr{E}_e$, serves to uniquely label the phase space fluid elements and thus acts as an intermediary that allows the distinction between elements (particles) with the same values of f_e and different values of $\mathscr{E}_e$. In this way equilibria with nonmonotonic distributions are obtained as stationary points of the free energy given by (35).

This generalization is important since it is already known by Gardner's theorem that monotonic equlibria are nonlinearly stable [5]. Moreover, it would be desirable to obtain equilibria with spatial variation; i.e., Bernstein-Greene-Kruskal [18] type equilibria from a variational principle, and it is known that such equilibria are not possible with monotonic distribution functions [19].

Turning now to the question of stability we take the second variation of the free energy functional given by (35), complete the square and evaluate it on an equilibrium. Using g, f_e, and f_i to indicate equilibrium quantities, we obtain

$$\delta^2 F[f_\alpha, g; \delta f_\alpha, \delta g] = \int (\delta E)^2 \,\mathrm{d}x + \int \mathscr{W}''(f_i)(\delta f_i)^2 \,\mathrm{d}z + \int \left[(\mathscr{C}''(g) + f_e \mathscr{T}''(g)) \left(\delta g + \frac{\delta f_e \mathscr{T}'(g)}{\mathscr{C}'' + f_e \mathscr{T}''} \right)^2 - \frac{(\mathscr{T}'(g)\,\delta f_e)^2}{\mathscr{C}'' + f_e \mathscr{T}''} \right] \mathrm{d}z . \tag{40}$$

In this expression the first term, that involving δE, stands for the second variation of the second term of (27). Equation (40), as it stands, is always indefinite since $\mathscr{C}'' + f_e \mathscr{T}''$ cannot be both positive and negative. But, since the dye is constrained to move with the electron phase space fluid, δf_e and δg are not independent. Specifically, since we are not interested in dye instabilities we require that these quantities be connected by the "neighboring equilibrium" relation

$$\delta g\,(\mathscr{C}'' + f_e \mathscr{T}'') = -\mathscr{T}'\,\delta f_e \tag{41}$$

or equivalently

$$\delta g = \delta f_e \frac{(\partial g/\partial \mathscr{E}_e)}{(\partial f_e/\partial \mathscr{E}_e)} . \tag{42}$$

Inserting (41) or (42) into (40) and making use of the equilibrium relations, Eqs. (36), yields

$$\delta^2 F[f_\alpha; \delta f_\alpha] = \int (\delta E)^2 \,\mathrm{d}x - \int ((\delta f_i)^2/(\partial f_i/\partial \mathscr{E}_i) + (\delta f_e)^2/(\partial f_e/\partial \mathscr{E}_e)) \,\mathrm{d}z . \tag{43}$$

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Thus we see from (43) that $\delta^2 F$ is definite provided the equilibrium distribution functions are monotonically decreasing functions. The introduction of the tracer field has allowed us to use variational calculus by enlarging the class of extremal points to include nonmonotonic distribution functions.

The class of equilibrium of the Vlasov-Poisson system is even larger than that above. In particular energy contours that do not have turning points need not have the same number of particles. Thus for fixed x a distribution function is not required to be velocity symmetric. Also for spatially homogeneous equilibria f_e can be an arbitrary function of velocity, provided the ion distribution function cancels out the electric field. Consider first the second case. Since $\varphi = 0$, we have $\mathscr{E}_e = m_e v^2/2$ and so the equilibrium relation (36b) becomes

$$m_e v^2/2 + \mathscr{T}(g) = 0. \tag{44}$$

If we pick $\mathscr{T}(g) = -m_e g^2/2$, then (44) has the solutions $v = \pm g$. Choosing the upper sign and then inserting $g = v$ into (36c) we find that f_e equal to an arbitrary function of v is extremal.

As an example, suppose that the equilibrium of interest is specified by

$$f_i = [2\pi v_i^2]^{-1/2} \exp[-v^2/2v_i^2] \quad \text{and}$$

$$f_e = [2\pi v_e^2]^{-1/2} \exp(-(v - v_D)^2/2v_e^2],$$

where v_i and v_e are the electron and ion thermal speeds and v_D is the drift of the electrons with respect to the ion distribution. The relevant free energy in this case is given by

$$\delta^2 F[f_\alpha; \delta f_\alpha] = \int (\delta E)^2 \, dx \tag{45}$$

$$- \int (m_i v (\delta f_i)^2/(\partial f_i/\partial v) + m_e v (\delta f_e)^2/(\partial f_e/\partial v)) \, dz.$$

If we choose δf_i and δf_e such that there is no perturbed charge density, then the nonnegative electrostatic energy density term vanishes. This will happen if we suppose both δf_i and δf_e depend only on v and that their velocity integrated difference vanishes. The second term of (45) is positive definite. Thus if we choose $\delta f_e = 0$, then $\delta^2 F > 0$. Conversely, if we choose $\delta f_i = 0$ and assume $\int \delta f_e \, dv = 0$, then we obtain

$$\delta^2 F = \sqrt{2\pi}\, m_e v_e^3 \int \left[\frac{v}{(v - v_D)} \right] \cdot \exp[(v - v_D)^2/2v_e^2] (\delta f_e)^2 \, dv, \tag{46}$$

where the domain of integration in (46) is $-\infty < v < \infty$. Choose δf_e as follows:

$$\delta f_e = v_e^{-2} (v - v_D) \exp[-(v - v_D)^2/4v_e^2] \, h(v), \tag{47}$$

where $h(v)$ vanishes outside of $0 < v < v_D$, but is otherwise arbitrary. We have concentrated δf_e in the region where $v > 0$ and $\partial f_e/\partial v < 0$, and we have chosen δf_e so that the free energy $\delta^2 F$ is finite. Inserting (47) into (46) yields

$$\delta^2 F = m_e \sqrt{2\pi}\, v_e^{-1} \int_0^{v_D} v (v - v_D) \, h^2(v) \, dv. \tag{48}$$

Regardless of $h(v)$ the right hand side of (48) is negative. Thus $\delta^2 F$ is indefinite for any finite v_D, whereas spectral stability analysis yields a threshold, say v^*, which is a necessary and sufficient condition for exponentially growing modes. In the region $0 < v_D < v^*$, this system possesses negative energy modes. In the numerical experiments of [13] growth was observed below the v^* threshold.

In order to neglect the electrostatic energy term of $\delta^2 F$ the following must be true:

$$\int \delta f_e \, dv = v_e^{-2} \int_0^{v_D} (v - v_D) \cdot \exp[-(v - v_D)^2/4v_e^2] \, h(v) \, dv = 0. \tag{49}$$

Since $h(v)$ is arbitrary, the second equality of (49) can surely be made to hold, and simultaneously preserve the constraints to first order.

10. Multibumped Electron Distribution

Now suppose that the ions constitute a uniform infinitely massive background charge and the equilibrium electron distribution is double bumped with maxima occurring at $v = 0$ and $v = v_D$ and with a minimum at v_0. The free energy is now the same as that of (45) with δf_i set equal to zero. As in Sect. 9 we can choose δf_e such that the electrostatic term vanishes, leaving

$$\delta^2 F = -\int m_e v (\delta f_e)^2/(\partial f_e/\partial v) \, dv. \tag{50}$$

If we concentrate δf_e in the region where $\partial f_e/\partial v > 0$; then $\delta^2 F$ is negative and we have indefiniteness. For this type of equilibrium one would thus expect nonlinear instability below the spectral threshold.

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11. Summary

We have reviewed the Hamiltonian structure of the Vlasov-Poisson equation and the related variation principle for monotonic equilibria. We have discussed the meaning of the Casimir invariants and the variational principle for which they are constants. It was shown how to obtain nonmonotonic equilibria from a variational principle by including a tracer field. Nonmonotonic equilibria were seen to have an indefinite free energy, a threshold that differs from that for spectral stability. Explicitly, the cases of Maxwellian electrons drifting through ions and of a multibumped electron distribution were treated.

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