Wave energy flow conservation for propagation in inhomogeneous Vlasov–Maxwell equilibria

H. J. Kull,[a] H. L. Berk, and P. J. Morrison
Institute for Fusion Studies and Department of Physics, The University of Texas at Austin, Austin, Texas 78712

(Received 4 February 1988; accepted 20 September 1988)

Wave energy flow conservation is demonstrated for Hermitian differential operators that arise in the Vlasov–Maxwell theory for propagation perpendicular to a magnetic field. The energy flow can be related to the bilinear concomitant, for a solution and its complex conjugate, by using the Lagrange identity of the operator. This bilinear form obeys a conservation law and is shown to describe the usual Wentzel–Kramers–Brillouin (WKB) energy flow for asymptotically homogeneous regions. The additivity and lack of uniqueness of the energy flow expression is discussed for a general superposition of waves with real and complex wavenumbers. Furthermore, a global energy conservation theorem is demonstrated for an inhomogeneity in one dimension and generalized reflection and transmission coefficients are thereby obtained.

I. INTRODUCTION

In the Vlasov–Maxwell theory, wave energy conservation is commonly associated with Hermitian wave operators. Such operators arise when particle resonances are not present or neglected, e.g., for wave propagation perpendicular to the magnetic field. In the present paper we wish to examine the conservation of wave energy flow for Hermitian operators. Such operators satisfy relations known as Lagrange identities. The purpose is to show how the general structure of these identities determines both the conservation law and the asymptotic properties of the energy flow.

Lagrange identities are well known from the theory of ordinary differential equations[1] and in an integrated form are familiar as Green's formula in potential theory.[2] They establish a relation between an operator, its adjoint, and a boundary form, which is commonly called the bilinear concomitant. More generally, Lagrange identities can be derived for vector systems of differential equations of arbitrary order, such as those that occur frequently in the Vlasov–Maxwell theory.[3–4] For these vector systems explicit Lagrange identities can be obtained where the concomitant is, in general, only uniquely determined up to the curl of a bilinear vector field. We note in comparison that, in general, the electromagnetic energy flow in a vacuum can only be identified as the Poynting vector up to a curl, but this is sufficient to identify the total energy flow out of a closed surface.

The concomitant of a Hermitian operator is related to a conserved current of the system, which has previously been used to express energy flow conservation for simple systems.[5,6] Here we wish to emphasize the general structure of these relations and present a discussion independent of the actual form of the operator. An important issue in this context is the identification of the concomitant expression with the physical wave energy flow. We find that this can be achieved asymptotically (i.e., in the boundary regions where the system is spatially homogeneous), and for general vector systems we obtain a straightforward derivation of the Wentzel–Kramers–Brillioin (WKB) energy flow expression from the Lagrange identity. With the relationship between the concomitant and the energy flow one can demonstrate a global energy conservation theorem for these Hermitian systems. For solutions corresponding to general radiative boundary conditions we discuss the uniqueness and the additivity (i.e., sum over normal mode energies) of the asymptotic energy flow and obtain generalized reflection and transmission coefficients that satisfy global energy conservation in one dimension.

While the concomitant conservation follows most immediately from the Lagrange identity, it may be instructive to discuss an alternative derivation from a version of Noether's theorem.[7] For Hermitian operators one can find variational forms that play a role analogous to the action integral in field theories. However, while the most common action integrals depend only on the fields and their first derivatives, the present functionals include derivatives up to an arbitrarily high order. It is this generalization that leads to the concomitant expression. From the symmetries of the variational form one obtains in a straightforward way conservation laws. These are well known for certain wave Lagrangians describing wave propagation for weakly inhomogeneous media.[8] Using the Lagrange identity one can show that energy flow conservation follows generally for arbitrary inhomogeneities from a gauge invariance. Furthermore, translational invariance for time independent and homogeneous media yields energylike and momentum like conserved quantities that can be related to the concomitant expression.

II. EXACT CONSERVATION RELATIONS

In the Vlasov–Maxwell theory, the propagation of waves with frequency ω and electric field $\mathbf{E}(\mathbf{x})e^{-i\omega t} = i(\omega/c)\mathbf{A}e^{-i\omega t}$ (for definiteness the gauge condition with

[a] Permanent address: Institut für Angewandte Physik, Technische Hochschule, 6100 Darmstadt, Federal Republic of Germany.

 0899-8221/89/010055-07$01.90

zero scalar potential is chosen) is governed by vector systems,

$$\mathbf{L}\cdot\mathbf{A} = 0\,, \tag{1}$$

where

$$\mathbf{L}\cdot\mathbf{A} = \nabla\times(\nabla\times\mathbf{A}) - \frac{\omega^2}{c^2}\mathbf{A} + \frac{4\pi i\omega}{c^2}\hat{\sigma}(\omega)\cdot\mathbf{A}\,,$$

and $i\hat{\sigma}(\omega)$ is a nonlocal conductivity tensor that is assumed to be Hermitian for real ω. Hence, in general, $\mathbf{L}$ represents an integral operator, and under not very restrictive approximations can be approximated by differential operators of arbitrarily high order. A specific form for $\mathbf{L}$ from Vlasov–Maxwell theory can be found in Ref. 3. In the following we will assume that the operator $\mathbf{L}$ satisfies a Lagrange identity and from this structure discuss some consequences for the wave energy flow in inhomogeneous media. The Lagrange identity for $\mathbf{L}$ and its adjoint $\mathbf{L}^\dagger$ is a relation of the form

$$\boldsymbol{\phi}^*\cdot(\mathbf{L}\cdot\boldsymbol{\psi}) - (\mathbf{L}^\dagger\cdot\boldsymbol{\phi})^*\cdot\psi = i\nabla\cdot\mathbf{J}(\boldsymbol{\phi},\boldsymbol{\psi}) \tag{2}$$

that holds for *arbitrary* vector fields $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$. The vector $i\mathbf{J}(\phi,\psi)$ is bilinear with respect to its arguments and is known as the concomitant. The star denotes the complex conjugate and the factor i has been inserted for convenience. In the Appendix we derive Lagrange identities for vector systems of differential equations of arbitrary order. The concomitant here is in general only determined up to the curl of a bilinear vector field, $\nabla\times\mathbf{b}(\phi,\psi)$. This ambiguity, however, is unimportant for one-dimensional problems and for global conservation relations as discussed in Sec. III.

From the Lagrange identity (2) there follows by Gauss' theorem the Green's formula

$$\int_V d^3\mathbf{x}[\boldsymbol{\phi}^*\cdot(\mathbf{L}\cdot\boldsymbol{\psi}) - (\mathbf{L}^\dagger\cdot\boldsymbol{\phi})^*\cdot\boldsymbol{\psi}] = i\int_{\partial V} d\boldsymbol{\sigma}\cdot\mathbf{J}(\boldsymbol{\phi},\boldsymbol{\psi})\,, \tag{3}$$

for any volume V with surface ∂V, and $d\boldsymbol{\sigma}$ is the surface element. If the whole plasma is contained in V and $\boldsymbol{\phi},\boldsymbol{\psi}$ are subject to boundary conditions where the surface term vanishes, Eq. (3) expresses the general definition of the adjoint operator $\mathbf{L}^\dagger$.

Let us now choose $\boldsymbol{\psi}$ to be a solution of Eq. (1) and $\boldsymbol{\phi}$ to satisfy the adjoint equation

$$\mathbf{L}^\dagger\cdot\boldsymbol{\phi} = 0\,. \tag{4}$$

Then, according to Eq. (2), $\mathbf{J}(\boldsymbol{\phi},\boldsymbol{\psi})$ represents a conserved current, whose divergence is zero. This relation becomes especially useful for Hermitian operators, where $\mathbf{L} = \mathbf{L}^\dagger$, and $\boldsymbol{\phi} = \boldsymbol{\psi}$ is a solution of the same physical system. The relation

$$\nabla\cdot\mathbf{J}(\boldsymbol{\psi},\boldsymbol{\psi}) = 0 \tag{5}$$

becomes a conservation law for any solution and its complex conjugate. We will see that $\omega\mathbf{J}(\boldsymbol{\psi},\boldsymbol{\psi})$ can be identified asymptotically as a flux, which for the special operator in Eq. (1) is the wave energy flux. For non-Hermitian operators we define the energy flux by an expression of the same form, however, with $\mathbf{L}$ replaced by its Hermitian part $\mathbf{L}_\mathrm{H}$,

$$\nabla\cdot\mathbf{J}_\mathrm{H}(\boldsymbol{\psi},\boldsymbol{\psi}) = -i[\boldsymbol{\psi}^*\cdot(\mathbf{L}_\mathrm{H}\cdot\boldsymbol{\psi}) - (\mathbf{L}_\mathrm{H}\cdot\boldsymbol{\psi})^*\cdot\boldsymbol{\psi}]\,. \tag{6}$$

Writing $\mathbf{L} = \mathbf{L}_\mathrm{H} + i\mathbf{L}_A$ with an anti-Hermitian part $i\mathbf{L}_A$ and choosing $\boldsymbol{\psi}$ again as a solution of Eq. (1), we obtain, from Eq. (6),

$$\nabla\cdot\mathbf{J}_\mathrm{H}(\boldsymbol{\psi},\boldsymbol{\psi}) = -[\boldsymbol{\psi}^*\cdot\mathbf{L}_A\cdot\boldsymbol{\psi} + (\mathbf{L}_A\cdot\boldsymbol{\psi})^*\cdot\boldsymbol{\psi}]\,. \tag{7}$$

The right-hand side no longer vanishes, and it thus describes the dissipation of energy associated with $\mathbf{L}_A$. Henceforth, we only analyze the case where $\mathbf{L}_A = 0$.

III. GLOBAL ENERGY CONSERVATION

We now evaluate the concomitant expression for Hermitian operators asymptotically as the system approaches infinity, where spatial homogeneity is assumed. In such a region the well-known expression for the WKB wave energy flow is obtained.[9] More generally, the uniqueness and the additivity of the asymptotic energy flow for a general superposition of waves, including propagating and evanescent waves, is discussed. For an inhomogeneity in one spatial dimension, a global energy conservation theorem is demonstrated and generalized reflection and transmission coefficients are obtained.

Let us consider an arbitrary inhomogeneous medium in a finite volume that is surrounded by spatially homogeneous regions. Integrating Eq. (5) over a volume whose surface ∂V lies entirely in the homogeneous medium, one obtains by Gauss' theorem a global conservation law for the asymptotic flux,

$$\int_{\partial V} d\boldsymbol{\sigma}\cdot\mathbf{J}(\boldsymbol{\psi},\boldsymbol{\psi}) = 0\,. \tag{8}$$

We now show that $\mathbf{J}(\boldsymbol{\psi},\boldsymbol{\psi})$ represents asymptotically the wave energy flow, so that Eq. (8) becomes a statement of global energy conservation.

Asymptotically, the solutions of Eq. (1) can be taken as a superposition of plane waves,

$$\boldsymbol{\psi} = \sum_n \hat{\boldsymbol{\psi}}_n e^{i\mathbf{k}_n\cdot\mathbf{x}}\,. \tag{9}$$

For a homogeneous medium, the form of Eq. (9) is an exact solution for each mode n (below we frequently suppress the subscript n) and the operator of Eq. (1) becomes an algebraic system,

$$\mathbf{D}(\mathbf{k})\cdot\hat{\boldsymbol{\psi}} = 0\,. \tag{10}$$

Solutions of (10) require that the ith component of $\hat{\boldsymbol{\psi}}$ satisfy

$$\hat{\psi}^i = D^{ij}\hat{\psi}^j/D^{jj} \quad (j \text{ not summed})\,, \tag{11}$$

where D^{ij} is the transpose of the cofactor of D_{ij} (D_{ij} is the index form of the tensor $\mathbf{D}$), and the index j is arbitrary. The solubility condition for nontrivial solutions is

$$\sum_l D_{il}D^{lj} \equiv \Delta(\mathbf{k})\delta_{ij} = 0\,,$$

where

$$\Delta(\mathbf{k}) = \det\mathbf{D}\,.$$

The solubility condition determines the possible values of one component of $\mathbf{k}$ in terms of the other components and the frequency ω.

If L is Hermitian and the components of $\mathbf{k}$ are real, then, as shown in the Appendix, $\mathbf{D}$ is Hermitian ($D_{ir} = D^*_{ir}$) and it follows that $D^{rj} = D^{ir*}$. Thus, for arbitrary real $\mathbf{k}$,

$$\delta_{ij}\Delta(\mathbf{k}) = \sum_r D_{ir}(\mathbf{k}) D^{rj}(\mathbf{k})$$

$$= \sum_r D^{jr*}(\mathbf{k}) D^*_{ri}(\mathbf{k}) = \Delta^*(\mathbf{k})\delta_{ij} \,. \qquad (12)$$

Equation (12) implies $\Delta(\mathbf{k})$ is a real function of real $\mathbf{k}$; hence the zeros of Δ must occur in complex conjugate pairs in one of the $\mathbf{k}$ components if the other two $\mathbf{k}$ components are real.

Now consider a general asymptotic solution

$$\psi = \sum_n C_n \hat{\psi}_n e^{i\mathbf{k}_n\cdot\mathbf{x}} , \qquad (13)$$

where the complex coefficients C_n are inserted to allow for arbitrary normalization of the plane waves $\hat{\psi}_n$ which have corresponding wavenumbers $\mathbf{k}_n$. The asymptotic expression for the bilinear concomitant then takes the form

$$\mathbf{J}(\psi,\psi) = \sum_{n,m} C^*_n C_m \mathbf{J}_{nm} e^{i(\mathbf{k}_m - \mathbf{k}^*_n)\cdot\mathbf{x}} , \qquad (14)$$

with $\mathbf{J}_{nm} \equiv \mathbf{J}(\hat{\psi}_n,\hat{\psi}_m)$, a spatially independent function. In the double sum (14) there occur spatially constant terms when $\mathbf{k}_m = \mathbf{k}^*_n$ and spatially periodic terms when $\mathbf{k}_m \neq \mathbf{k}^*_n$. We denote by n^* the mode with wave vector k^*_n, so $\hat{\psi}_{n^*}$ is the amplitude of this mode. The constant terms in the expansion (14) are represented by $\mathbf{J}_m \equiv \mathbf{J}(\hat{\psi}_{m^*},\hat{\psi}_m)|C_m|^2$. The Lagrange identity given in Eq. (6) determines the form of the vector $\mathbf{J}_n$ in the following way. In Eq. (11) we have written a relationship that determines all the components $\hat{\psi}^i$ in terms of a specific but arbitrary component $\hat{\psi}^r$. Since the problem is homogeneous, solutions are only determined up to an arbitrary multiplicative constant. We thus write

$$\hat{\psi}^i(\mathbf{k}') = \alpha(\mathbf{k}') D^{ir} ,$$

where $\alpha(\mathbf{k})$ is an arbitrary proportionality constant independent of i. Using Eq. (A14) yields

$$\hat{\psi}^{i*}(\mathbf{k}'') = \alpha^*(\mathbf{k}'') D^{ri}(\mathbf{k}''^*) \,.$$

For now $\mathbf{k}'$ and $\mathbf{k}''$ are arbitrary, but ultimately we set $\mathbf{k}' = \mathbf{k}_n$ and $\mathbf{k}'' = \mathbf{k}_{m^*}$, where $\Delta(\mathbf{k}_n) = \Delta(\mathbf{k}_{m^*}) = 0$. (Note $\mathbf{k}^*_{m^*} = \mathbf{k}_m$.) From Eq. (6), if we choose a single coefficient of $\exp[i(\mathbf{k}'_n - \mathbf{k}''^*_{m^*})\cdot\mathbf{x}]$ (where we anticipate setting $\mathbf{k}'_n = \mathbf{k}_n$ and $\mathbf{k}''_{m^*} = \mathbf{k}_{m^*}$, by putting indices on $\mathbf{k}'$ and $\mathbf{k}''$) we have

$$-(\mathbf{k}'_n - \mathbf{k}''^*_{m^*})\cdot\mathbf{J}_{m^*n}$$

$$= \sum_{p,q} [\alpha^*(\mathbf{k}''_{m^*}) D^{rp}(\mathbf{k}''^*_{m^*}) D_{pq}(\mathbf{k}'_n) D^{qr}(\mathbf{k}'_n)\alpha^*(\mathbf{k}'_n)$$

$$- \alpha^*(\mathbf{k}''_{m^*}) D^{rp}(\mathbf{k}''^*_{m^*}) D_{pq}(\mathbf{k}''^*_{m^*}) D^{qr}(\mathbf{k}'_n)\alpha(\mathbf{k}'_n)]$$

$$= \alpha^*(\mathbf{k}''_{m^*}) D^{rr}(\mathbf{k}''^*_{m^*})\Delta(\mathbf{k}'_n)\alpha(\mathbf{k}'_n)$$

$$- \alpha^*(\mathbf{k}''_{m^*})\Delta(\mathbf{k}''^*_{m^*}) D^{rr}(\mathbf{k}'_n)\alpha(\mathbf{k}'_n) \,. \qquad (15)$$

The last equality follows by contraction of the matrix $\mathbf{D}$ with its cofactor transpose. Now if we let $\mathbf{k}'_n \to \mathbf{k}_n$ and $\mathbf{k}''^*_{m^*} \to \mathbf{k}_m$, but with $\mathbf{k}_m \neq \mathbf{k}_n$, the right-hand side of Eq. (15) vanishes because $\Delta(\mathbf{k}_n) = \Delta(\mathbf{k}^*_{m^*}) = \Delta(\mathbf{k}_m) = 0$. Hence $J_{nm^*} = 0$ if $m^* \neq n^*$. For the case $\mathbf{k}'_n \to \mathbf{k}''^*_{m^*} \to \mathbf{k}_n$, we Taylor expand the middle term of Eq. (15) in $\mathbf{k}'_n - \mathbf{k}_n$ and $\mathbf{k}''^*_{m^*} - \mathbf{k}_m$. Use

$$D_{pq}(\mathbf{k}'_n) - D_{pq}(\mathbf{k}''^*_{n^*}) = (\mathbf{k}'_n - \mathbf{k}''^*_{n^*})\cdot\frac{\partial}{\partial\mathbf{k}} D_{pq}(\mathbf{k}_n)$$

and note that the linear expansion of the other terms can be ignored, since they are multiplied by either

$$\sum_r D^{rp}(\mathbf{k}_n) D_{pq}(\mathbf{k}_n) = \Delta(\mathbf{k}_n)\delta_{rq}$$

or

$$\sum_q D_{pq}(\mathbf{k}_n) D^{qr}(\mathbf{k}_n) = \Delta(\mathbf{k}_n)\delta_{pr},$$

both of which vanish. The Taylor expansion, using $k^*_{n^*} = k_n$ and thus $k_{n^*} = k^*_n$, yields

$$-(\mathbf{k}'_n - \mathbf{k}''^*_{n^*})\cdot\Big(\mathbf{J}_{n^*n} + \hat{\psi}^*_{n^*}(\mathbf{k}^*_n)$$

$$\cdot\frac{\partial}{\partial\mathbf{k}}\mathbf{D}(\mathbf{k}_n)\cdot\hat{\psi}_n(\mathbf{k}_n)\Big) = 0 \,.$$

Hence, $\mathbf{J}_n$ is given by

$$\mathbf{J}_n = \mathbf{J}_{n^*n} C_n C_{n^*} = -\hat{\psi}^*_{n^*}(\mathbf{k}^*_n)\cdot\frac{\partial D}{\partial\mathbf{k}}(\mathbf{k}_n)\cdot\hat{\psi}_n(\mathbf{k}_n) C_n C_{n^*} \,. \qquad (16)$$

Some basic properties of the asymptotic wave energy flow follow immediately from Eqs. (13)–(16). First we note that the concomitant (14) can actually be identified asymptotically with the wave energy flow. For a single plane wave with a real wave vector the solution (13) consists of only one term and the concomitant assumes the form given by Eq. (16), in agreement with previous expressions for the wave energy flow in homogeneous media.[8,9] More generally, Eq. (16) applies also to complex wave vectors $\mathbf{k}_m$.

For a general superposition of waves, the sum in Eq. (14) also contains spatially oscillating interference terms $\mathbf{J}_{nm}$ with $\mathbf{k}_m \neq \mathbf{k}^*_n$. According to Eq. (15) these terms are restricted by an orthogonality constraint, $(\mathbf{k}^*_n - \mathbf{k}_m)\cdot\mathbf{J}_{nm} = 0$, but are otherwise not determined by the Lagrange identity. As previously mentioned, the concomitant expression is in general only uniquely defined up to the curl of an arbitrary bilinear vector field $\mathbf{b}(\psi,\psi)$. This quantity has the asymptotic form

$$\nabla\times\mathbf{b}(\psi,\psi) = -\sum_{n,m}(\mathbf{k}^*_n - \mathbf{k}_m)\times\mathbf{b}_{nm}\, C^*_n C_m e^{i(\mathbf{k}_m - \mathbf{k}^*_n)\cdot\mathbf{x}} , \qquad (17)$$

where $\mathbf{b}_{nm}$ is defined as $\mathbf{J}_{nm}$. This expression is zero for $\mathbf{k}_m = \mathbf{k}^*_n$ and otherwise spatially oscillating. In general, the ambiguity of the asymptotic concomitant expression can be removed by taking the spatial average in Eq. (14).

Consider now a global energy conservation theorem for an inhomogeneity in one dimension. For simplicity of notation we denote by k and J the vector components along the inhomogeneity direction and assume the same transverse wave vector for all waves. The possible values k_n of k are given by the roots of the local dispersion relation. In this case the oscillating terms in Eq. (14) yield $(k_n^* - k_m)J_{nm}C_mC_n^* = 0$ and therefore **J** becomes

$$J(\psi,\psi) = \sum_{\substack{n \\ k_n \text{ real}}} |C_n|^2 J_{nn} + \sum_{\substack{n \\ k_n \text{ complex}}} C_n C_{n^*} J_{nn^*} . \quad (18)$$

We now restrict ourselves to scattering problems where for definiteness at $x = -\infty$ an incoming wave i; its amplitude, real k, and $\partial\omega/\partial k > 0$ are given. For a $2N$th-order differential equation this incoming wave induces N reflected (transmitted) waves. Suppose n_r (n_t) of these waves have real k values with $\partial\omega/\partial k < 0$ $(\partial\omega/\partial k > 0)$, and $N - n_r$ $(N - n_t)$ of these waves are evanescent with complex k values and $\mathrm{Im}\, k < 0$ $(\mathrm{Im}\, k > 0)$ as required for spatially decaying solutions. Thus Eq. (18) can be written as

$$J(\psi,\psi) = J_{ii}|C_i|^2 + \sum_{n_r}|C_n|^2 J_{nn} = J_i + \sum_{n_r} J_n , \quad x < 0 , \quad (19a)$$

and

$$J(\psi,\psi) = \sum_{n_t}|C_n|^2 J_{nn} = \sum_{n_t} J_n , \quad x > 0 , \quad (19b)$$

where n_r (n_t) refers to the real k_n reflected (transmitted) solutions. Note that the assumed boundary conditions eliminate either C_n or C_{n^*} in the summation over the complex k solutions; the restricted summation will be indicated by $n^<$ $(n^>)$ when $x < 0$ $(x > 0)$. Since $J(\psi,\psi)$ is conserved, we obtain our fundamental scattering relationship

$$J_i + \sum_{n^<} J_{n_r} = \sum_{n^>} J_{n_t} . \quad (20)$$

We can rewrite Eq. (20) somewhat to emphasize a more physical interpretation. Note that

$$J_{n_r} = \hat{\psi}_n^* \frac{\partial D}{\partial k}(k_n)\hat{\psi}_n$$
$$= -\frac{\Delta_{,k}}{\Delta_{,\omega}} \hat{\psi}_n^* \frac{\partial D}{\partial \omega}(k_n)\hat{\psi}_n = v_{gn}\hat{\psi}_n^* \frac{\partial D}{\partial \omega}(k_n)\hat{\psi}_n ,$$

where v_g, the group velocity, is given by

$$v_g = \frac{\partial\omega}{\partial k} = -\frac{\Delta_{,k}}{\Delta_{,\omega}} .$$

(The above relation follows from $(d/dk)[\hat{\psi}^* D(k,\omega(k))\hat{\psi}] = 0$, and manipulations similar to those used in obtaining Eq. (16).) For the specific operator of Eq. (1) it can be shown[9] that

$$\hat{\psi}_n^* \frac{\partial(\omega D)}{\partial\omega}\hat{\psi}_n = \omega\hat{\psi}_n^* \frac{\partial D}{\partial\omega}\hat{\psi}_n \equiv \omega J^0$$

is exactly the wave energy density of mode n and J^0 is the action. Therefore, $J_n = v_{gn}J^0$ is the action flux. Now dividing by the incident flux $|J_i|$ and defining reflection and transmission coefficients as

$$R_n = -\frac{J_n}{|J_i|} , \quad T_n = \frac{J_n}{|J_i|} ,$$

one obtains

$$\sum_{n_r}\sigma_n R_n + \sum_{n_t}\sigma_n T_n = \sigma_i . \quad (21)$$

Here σ_n is the sign of $\omega\hat{\psi}_n^*(\partial D/\partial\omega)\hat{\psi}_n$, which is negative for negative energy waves.

IV. WAVE ACTION, ENERGY, AND MOMENTUM CONSERVATION

We now discuss a version of Noether's theorem and relations for wave action, energy, and momentum in space- and time-dependent media. For Hermitian operators the wave fields can be derived from a variational principle and Noether's theorem provides a general method to obtain field invariants corresponding to the symmetries of the action function. Since this functional depends on derivatives of the field up to an arbitrarily high order, the conserved quantities can be expressed in compact form by the concomitant. The Lagrangian formalism for wave propagation in slowly varying media is thereby generalized to arbitrary inhomogeneities.

In the following we consider wave propagation in a medium with an arbitrary space and time dependence. We assume that the system can be described by a Hermitian operator **L**, which satisfies a Lagrange identity in four-dimensional space-time x^μ,

$$\mathscr{L}(\psi^*,\psi,x^\mu) - \mathscr{L}(\psi^*,\psi,x^\mu)^* = i\frac{\partial}{\partial x^\mu}J^\mu(\psi,\psi) , \quad (22)$$

with $\mathscr{L}(\psi^*,\psi,x^\mu) = \psi^*\cdot\mathbf{L}\cdot\psi$. We choose the notation A^μ for a four-dimensional vector with time component A^0 and use the repeated index sum notation. The number of components of the fields ψ is unconstrained, but $\mathscr{L}$ is assumed to be a scalar covariant quantity. We define the variational form

$$S = \int d^4x\, \mathscr{L}(\psi^*,\psi,x^\mu) \quad (23)$$

and note that the wave equation (1) follows from the variational principle $\delta S = 0$ with respect to variations of ψ and ψ^*. Let us now assume an infinitesimal point transformation,

$$\tilde{x}^\mu = x^\mu + \delta x^\mu(x^\mu) ,$$
$$\tilde{\psi}(\tilde{x}^\mu) = \psi(x^\mu) + \delta\psi(x^\mu) . \quad (24)$$

The Jacobian of the transformation of x^μ is

$$1 + \frac{\partial}{\partial x^\mu}\delta x^\mu \quad (25)$$

and the change of ψ at x^μ is defined as

$$\overline{\delta\psi} \equiv \tilde{\psi}(x^\mu) - \psi(x^\mu) = \delta\psi(x^\mu) - \delta x^\mu \frac{\partial}{\partial x^\mu}\psi(x^\mu) . \quad (26)$$

With Eqs. (25) and (26), the variation of $S(\psi)$ corresponding to the transformation (24) assumes the form

$$\delta S = \int d^4\tilde{x}\ \mathscr{L}(\tilde{\psi}^*,\tilde{\psi},\tilde{x}^\mu) - \int d^4x\ \mathscr{L}(\psi^*,\psi,x^\mu)$$
$$= \int d^4x\left(\delta\mathscr{L} + \mathscr{L}\frac{\partial}{\partial x^\mu}\delta x^\mu\right)$$
$$= \int d^4x\left(\overline{\delta\boldsymbol{\psi}^*}\cdot\mathbf{L}\cdot\boldsymbol{\psi} + \boldsymbol{\psi}^*\cdot\mathbf{L}\cdot\overline{\delta\boldsymbol{\psi}} + \frac{\partial}{\partial x^\mu}(\mathscr{L}\delta x^\mu)\right), \tag{27}$$

where

$$\delta\mathscr{L} = \mathscr{L}(\tilde{\psi}^*(\tilde{x}(x)), \tilde{\psi}(\tilde{x}(x)), \tilde{x}(x)) - \mathscr{L}(\boldsymbol{\psi}^*(x),\psi(x),x)$$

expanded to first order. We now use the Lagrange identity (22) to obtain

$$\delta S = \int d^4x[\ \overline{\delta\boldsymbol{\psi}^*}(\mathbf{L}\cdot\boldsymbol{\psi}) + \overline{\delta\boldsymbol{\psi}}\cdot(\mathbf{L}\cdot\boldsymbol{\psi})^*]$$
$$+ \int d^4x\frac{\partial}{\partial x^\mu}[iJ^\mu(\boldsymbol{\psi},\overline{\delta\boldsymbol{\psi}}) + \mathscr{L}\delta x^\mu]\,. \tag{28}$$

If $\boldsymbol{\psi}$ is an extremal of the variational form (23) and S is invariant with respect to the transformation for an arbitrary volume (i.e., $\delta S = 0$), then Noether's theorem follows in the form

$$\frac{\partial}{\partial x^\mu}[iJ^\mu(\psi,\overline{\delta\psi}) + \mathscr{L}\delta x^\mu] = 0\,. \tag{29}$$

Let us now consider the particular transformation $\boldsymbol{\psi}\to e^{i\theta}\boldsymbol{\psi}$, corresponding to a uniform phase shift of the fields. The infinitesimal changes here are $\delta x^\mu = 0$ and $\delta\boldsymbol{\psi} = \overline{\delta\boldsymbol{\psi}} = i\delta\theta\boldsymbol{\psi}$. Under this transformation the variational form (23) remains invariant, which is called a gauge invariance of the first kind (or sometimes called global gauge invariance). According to Noether's theorem (29) this symmetry leads to the conservation law

$$\frac{\partial}{\partial x^\mu}J^\mu(\boldsymbol{\psi},\boldsymbol{\psi}) = 0\,. \tag{30}$$

This conservation law possesses an indeterminancy, since any transformation

$$J^0\to J^0 + \nabla\cdot\mathbf{a}\,,\quad \mathbf{J}\to\mathbf{J} - \partial_t\mathbf{a} + \nabla\times\mathbf{b}\,,$$

with vector fields **a** and **b**, leaves Eq. (30) invariant. Nevertheless, asymptotically where we have an exact expansion of plane waves with given frequencies and wave vectors, the ambiguous terms vanish and unique expressions are then obtained.

Integrating Eq. (30) over a volume V that encloses the whole system yields a field invariant

$$\mathscr{I}^0 = \int_V d^3x\,J^0(\boldsymbol{\psi},\boldsymbol{\psi})\,. \tag{31}$$

The asymptotic expression for $J^0(\boldsymbol{\psi},\boldsymbol{\psi})$ can be found from Eq. (22) in analogy with the treatment in Sec. III. For a single wave $\propto e^{-i\omega t}$ we obtain

$$J^0(\boldsymbol{\psi},\boldsymbol{\psi}) = \boldsymbol{\psi}^*\cdot\left(\frac{\partial}{\partial\omega}\mathbf{D}\right)\cdot\boldsymbol{\psi}\,. \tag{32}$$

The expression (32) is commonly called the wave action, which is generalized by Eq. (31) to arbitrary inhomogeneities.

For time-independent homogeneous media Eq. (23) is also invariant with respect to infinitesimal translations δx^μ. Using Eq. (26) with $\delta\psi(x^\mu) = 0$, we find from Eq. (29) the conservation law

$$\frac{\partial}{\partial x^\mu}\left[J^\mu\left(\boldsymbol{\psi},i\frac{\partial}{\partial x^\nu}\boldsymbol{\psi}\right)\right] = 0\,. \tag{33}$$

This defines a stress tensor with components that are an energy

$$\mathscr{W} = \int_V W\,d^3x = \int_V d^3x\,J^0(\boldsymbol{\psi},i\,\partial_t\,\boldsymbol{\psi}) = \omega\mathscr{I}^0 \tag{34}$$

and a momentum flux tensor

$$\mathscr{P} = \int_V d^3x\,\mathbf{J}(\boldsymbol{\psi},i\nabla\boldsymbol{\psi}) = -\mathbf{k}\int_V d^3x\,\boldsymbol{\psi}^*\cdot\frac{\partial\mathbf{D}}{\partial\mathbf{k}}\cdot\boldsymbol{\psi} = \mathbf{k}\mathbf{v}_g\mathscr{I}^0\,, \tag{35}$$

where Eq. (32) has been used.

For inhomogeneous, time-dependent media the change of these quantities is given by

$$\frac{\partial}{\partial x^\mu}J^\mu\left(\boldsymbol{\psi},i\frac{\partial}{\partial x^\nu}\boldsymbol{\psi}\right) = -\boldsymbol{\psi}^*\cdot\left(\frac{\partial}{\partial x^\nu}\mathbf{L}\right)\cdot\boldsymbol{\psi}\,. \tag{36}$$

Expression (36) follows from Eq. (22) and the relation

$$\frac{\partial}{\partial x^\nu}\mathscr{L}(\boldsymbol{\psi}^*,\boldsymbol{\psi},x^\mu) = 0\,, \tag{37}$$

which itself follows since $\mathscr{L}\equiv 0$ when $\mathbf{L}\cdot\boldsymbol{\psi} = \mathbf{L}^\dagger\boldsymbol{\psi}^* = 0$.

To illustrate the general procedure we now consider a specific example describing wave propagation in a cold plasma with no external fields. In our chosen gauge the wave equation for the four-vector potential A^μ has the form

$$\nabla\times(\nabla\times\mathbf{A}) + (1/c^2)\partial_t^2\mathbf{A} + [\omega_p^2(\mathbf{x},t)/c^2]\mathbf{A} = 0\,, \tag{38}$$

where ω_p is the plasma frequency and c is the speed of light. We have allowed for the possibility that the medium is space and time dependent, but require ω_p to approach a constant asymptotically in space and time.

Since the operator of Eq. (38) is self-adjoint the variational form exists with

$$\mathscr{L} = \frac{-1}{16\pi}\mathbf{A}^*\cdot\left(\nabla\times\nabla\times + \frac{1}{c^2}\partial_t^2 + \frac{\omega_p^2}{c^2}\right)\cdot\mathbf{A}\,, \tag{39}$$

and Eqs. (22) and (30) imply

$$i\frac{\partial}{\partial x^\mu}J^\mu = \frac{1}{16\pi}\Bigg[\partial_t\left(-\frac{1}{c^2}(\mathbf{A}^*\cdot\partial_t\mathbf{A} - \mathbf{A}\cdot\partial_t\mathbf{A}^*)\right)$$
$$+ \nabla\cdot[(\nabla\times\mathbf{A}^*)\times\mathbf{A} - (\nabla\times\mathbf{A})\times\mathbf{A}^*]\Bigg] = 0\,. \tag{40}$$

Thus

$$J^0 = (i/16\pi c^2)(\mathbf{A}^*\cdot\partial_t\mathbf{A} - \mathbf{A}\cdot\partial_t\mathbf{A}^*)\,, \tag{41}$$

$$\mathbf{J} = -(i/16\pi)[(\nabla\times\mathbf{A}^*)\times\mathbf{A} - (\nabla\times\mathbf{A})\times\mathbf{A}^*]\,. \tag{42}$$

It is evident from Eqs. (34) and (31) that time independence of the medium results in the simple relation between wave action and energy,

$$W = \omega J^0 . \tag{43}$$

To see this explicitly in the case $\omega_p^2 = \text{const}$ we note that the wave energy density that can be derived by Eq. (38) is

$$W = \frac{1}{16\pi}\left(\frac{1}{c^2}|A_t|^2 + \frac{\omega_p^2}{c^2}|A|^2 + |\nabla\times A|^2\right). \tag{44}$$

From Eq. (41) we have

$$J^0 = (\omega/8\pi c^2)|A|^2 . \tag{45}$$

To show Eq. (43) we use the dispersion relation $\Delta = (\omega^2 - \omega_p^2)\,(\omega^2 - \omega_p^2 - k^2c^2) = 0$, with branches corresponding to cold plasma oscillations and electromagnetic transverse waves. Using $\mathbf{A} \sim e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}$ in Eq. (44) and assuming either transverse waves ($\mathbf{k}\cdot\mathbf{A} = 0$) or longitudinal waves ($\mathbf{k}\times\mathbf{A} = 0$) results in the following:

$$W = (\omega^2/c^2)(|A|^2/8\pi) = \omega J^0 . \tag{46}$$

V. DISCUSSION

In the preceding sections we have shown that Lagrange identities for Hermitian operators determine conservation laws that are related to action, energy, and momentum conservation. We are mainly concerned with the Vlasov–Maxwell theory of wave propagation in an inhomogeneous time-independent plasma equilibrium. Here Hermitian operators arise for wave propagation perpendicular to the magnetic field and the Lagrange identities can be expected to be valid.[3,4] In general, the plasma response can be written as a differential operator of infinite order. If this operator is applied to plane waves the resulting series often have an infinite radius of convergence and accurate approximations can often be obtained for truncations of finite orders. It is assumed that a corresponding truncation of the differential operator can be justified, leading to differential equations of an arbitrary high order. An example can be found in Refs. 3, 4, and 6 if wave propagation in a thermal plasma perpendicular to a uniform magnetic field is considered. (We know that it is only for perpendicular propagation that Hermiticity is satisfied, otherwise Landau damping introduces an anti-Hermitian component to the wave operators.) For waves with a given frequency the action (31) and the energy (34) as well as the corresponding flows are related by a constant factor. The demonstration of global energy conservation for these Hermitian systems is the main result of this work.

For time-dependent Hermitian operators one can readily generalize the three-dimensional spatial conservation law to a continuity equation for the wave action. The energy then becomes an independent quantity, which in general is no longer conserved. One should, however, notice that the Hermitian form of the time-dependent Vlasov–Maxwell operator is less obvious and this structure may apply to special cases with an adiabatically slow time variation only.

ACKNOWLEDGMENTS

This work was supported by the Deutsche Forschungsgemeinschaft (Bonn, Federal Republic of Germany) and by the U.S. Department of Energy Contract No. DE-FG05-80ET-53088.

APPENDIX: OPERATOR IDENTITIES

In this appendix we consider vector systems of differential equations of arbitrary order n and derive explicit expressions for the adjoint operator and the concomitant. The operator has the general form

$$L_{ij} = \sum_{m=0}^{n} \sum_{\alpha_1\cdots\alpha_m=1}^{3} a_{ij}^{\alpha_1\cdots\alpha_m}\,\partial_{x_{\alpha_1}}\cdots\partial_{x_{\alpha_m}} \tag{A1}$$

and for simplicity of notation is written in the symbolic form

$$\mathbf{L} = \sum_{m=0}^{n} \mathbf{L}^m = \sum_{m=0}^{n} \mathbf{a}^m \nabla^m , \tag{A2}$$

where

$$\mathbf{a}^m \equiv a_{ij}^{\alpha_1\cdots\alpha_m}, \quad \nabla^m = \underbrace{\nabla\otimes\nabla\otimes\cdots\otimes\nabla}_{m\ \text{times}} .$$

Here summation over the indices of the gradients is always implicitly assumed. We start with the derivative of mth order, $\mathbf{L}^m$. Using repeatedly the product rule one obtains, for arbitrary fields $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$, the identity

$$\begin{aligned}\boldsymbol{\phi}^*\cdot\mathbf{L}^m\cdot\boldsymbol{\psi} &= \nabla(\boldsymbol{\phi}^*\cdot\mathbf{a}^m\nabla^{m-1}\cdot\boldsymbol{\psi}) + [(-\nabla)\boldsymbol{\phi}^*\cdot\mathbf{a}^m]\nabla^{m-1}\cdot\boldsymbol{\psi}\\ &= \nabla\left(\sum_{p=1}^{m}[(-\nabla)^{p-1}\boldsymbol{\phi}^*\cdot\mathbf{a}^m]\nabla^{m-p}\cdot\boldsymbol{\psi}\right)\\ &\quad + [(-\nabla)^m\boldsymbol{\phi}^*\cdot\mathbf{a}^m]\cdot\boldsymbol{\psi} .\end{aligned} \tag{A3}$$

Comparison with Eq. (2) and (A2) shows that

$$\begin{aligned}\mathbf{L}^{m\dagger} &= (-\nabla)^m\mathbf{a}^{m\dagger} ,\\ i\mathbf{J}(\boldsymbol{\phi},\boldsymbol{\psi}) &= \sum_{p=1}^{m}[(-\nabla)^{p-1}\boldsymbol{\phi}^*\cdot\mathbf{a}^m]\nabla^{m-p}\cdot\boldsymbol{\psi} ,\end{aligned} \tag{A4}$$

with $(a_{ij}^{\alpha_1\cdots\alpha_m})^\dagger = (a_{ji}^{\alpha_1\cdots\alpha_m})^*$. If the matrix $a_{ij}^{\alpha_1\cdots\alpha_m}$ is not completely symmetric with respect to the indices $\alpha_1\cdots\alpha_m$, the expression (A4) is not uniquely defined and depends on the ordering in which the product rule in (A3) is applied to different gradients. On the other hand, $\mathbf{L}^m$ and $\mathbf{L}^{m\dagger}$ are unique since $\mathbf{a}^m$ and $\mathbf{a}^{m\dagger}$ are here contracted with the completely symmetric tensor ∇^m. As a consequence two different representations for the concomitant can only differ by the curl of a bilinear vector field, $\nabla\times\mathbf{b}(\boldsymbol{\phi},\boldsymbol{\psi})$. An example of this ambiguity is discussed in Sec. V.

For the general operator (A2) the identity (A3) holds for each term from $m = 1$ up to $m = n$, while the term $m = 0$ produces no boundary term. Summation over m then yields

$$\begin{aligned}\mathbf{L}^\dagger &= \sum_{m=0}^{n}(-\nabla)^m\mathbf{a}^{m\dagger} ,\\ i\mathbf{J}(\boldsymbol{\phi},\boldsymbol{\psi}) &= \sum_{m=1}^{n}\sum_{p=1}^{m}[(-\nabla)^{p-1}\boldsymbol{\phi}^*\cdot\mathbf{a}^m]\nabla^{m-p}\cdot\boldsymbol{\psi} .\end{aligned} \tag{A5}$$

For Hermitian operators Eq. (A5) does not express explicitly the Hermitian form.

A general form for Hermitian operators is

$$\mathbf{L}_H = \sum_{m,n=0}^{N} (-\nabla)^m \mathbf{G}^{mn} \nabla^n , \tag{A6}$$

where the matrix $\mathbf{G}^{mn}$ satisfies the condition

$$\mathbf{G}_{ij}^{mn} = (G_{ji}^{nm})^* . \tag{A7}$$

Defining

$$\begin{aligned} \mathbf{T}(\psi) &= \sum_{n=0}^{N} (-1)^m \mathbf{G}^{mn} \nabla^n \psi , \\ Q(\phi,\psi) &= \sum_{m=0}^{N} [(-\nabla)^m \phi^*] \cdot \mathbf{T}(\psi) , \\ \mathbf{I}(\phi,\psi) &= \sum_{m=1}^{N} \sum_{p=1}^{m} [(-\nabla)^{p-1} \phi^*] \nabla^{m-p} \cdot \mathbf{T}^m(\psi) , \end{aligned} \tag{A8}$$

we obtain by the same procedure as in Eq. (A3) the identity

$$\phi^* \cdot \mathbf{L}_H \cdot \psi = Q(\phi,\psi) + \nabla \cdot \mathbf{I}(\phi,\psi) . \tag{A9}$$

The expression $Q(\phi,\psi)$ is a Hermitian form satisfying

$$\begin{aligned} Q^*(\psi,\phi) &= \sum_{m,n} [(\nabla^m \psi^*) \cdot \mathbf{G}^{mn} \cdot (\nabla^n \phi)]^* \\ &= \sum_{m,n} (\nabla^m \psi) \cdot \mathbf{G}^{mn*} \cdot (\nabla^n \phi^*) \\ &= \sum_{mn} (\nabla^n \phi^*) \cdot \mathbf{G}^{mn} \cdot (\nabla^m \psi) = Q(\phi,\psi) . \end{aligned} \tag{A10}$$

Interchanging ϕ and ψ in Eq. (A9) and taking the complex conjugate yield

$$\psi \cdot (\mathbf{L}_H \cdot \phi)^* = Q^*(\psi,\phi) + \nabla \cdot \mathbf{I}^*(\psi,\phi) . \tag{A11}$$

Subtracting Eq. (A11) from Eq. (A9) and observing Eq. (A10) one obtains the Lagrange identity for $\mathbf{L}_H$ with the concomitant

$$i\mathbf{J}(\phi,\psi) = \mathbf{I}(\phi,\psi) - \mathbf{I}^*(\psi,\phi) . \tag{A12}$$

We give the form of the algebraic operators that correspond to $\mathbf{L}$ and $\mathbf{L}^\dagger$ in the asymptotic limit. These are obtained by substituting $\nabla \to i\mathbf{k}$ in Eqs. (A2) and (A5) yielding

$$\begin{aligned} \mathbf{L} &\to \sum_{m=0}^{n} \mathbf{a}^m (i\mathbf{k})^m = \mathbf{D}(\mathbf{k}) , \\ \mathbf{L}^\dagger &\to \sum_{m=0}^{n} \mathbf{a}^{m\dagger} (-i\mathbf{k})^m = \mathbf{D}^\dagger(\mathbf{k}) . \end{aligned} \tag{A13}$$

When $\mathbf{L}$ is Hermitian, then $\mathbf{D}$ is Hermitian for real $\mathbf{k}$, but is not, in general, Hermitian for complex $\mathbf{k}$.

Finally note that it follows from Eq. (A13) that $D_{ij}^*(\mathbf{k}) = D_{ji}(\mathbf{k}^*)$. Therefore $D^{ij*}(\mathbf{k}) = D^{ji}(\mathbf{k}^*)$. This leads to the relation used in the text,

$$\hat{\psi}^{j*}(\mathbf{k}) = \frac{D^{jr*}(\mathbf{k})}{D^{rr*}(\mathbf{k}')} \hat{\psi}^{r*}(\mathbf{k}) = \hat{\psi}^{r*}(\mathbf{k}) \frac{D^{rj}(\mathbf{k}^*)}{D^{rr}(\mathbf{k}^*)} . \tag{A14}$$

[1] E. L. Ince, *Ordinary Differential Equations* (Dover, New York, 1956).
[2] R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Interscience, New York, 1953).
[3] H. L. Berk, R. R. Dominguez, and E. K. Maschke, Phys. Fluids **24**, 2245 (1981).
[4] R. R. Dominguez and H. L. Berk, Phys. Fluids **27**, 1142 (1984).
[5] J. Heading, *An Introduction to Phase-Integral Methods* (Wiley, New York, 1962).
[6] P. L. Colestock and R. J. Kashuba, Nucl. Fusion **23**, 763 (1983).
[7] I. N. Gelfand and S. V. Fomin, *Calculus of Variations* (Prentice–Hall, Englewood Cliffs, NJ, 1963).
[8] G. B. Whitham, *Linear and Nonlinear Waves* (Wiley, New York, 1974).
[9] H. L. Berk and D. Pfirsch, J. Math. Phys. **21**, 2054 (1980).