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A Model for the Effects of Temperature Gradients and Magnetic Shear on the Drift Wave Monopole Solutions

P.J. Morrison, X.N. Su, and W. Horton
Department of Physics and Institute for Fusion Studies
The University of Texas at Austin
Austin, Texas 78712

Abstract

A model that incorporates both the effects of temperature gradients and magnetic shear on the drift wave monopole solutions is analyzed. In the case where the former effect is treated improperly and the latter is neglected, it was shown in Ref. 1 that there exist exact monopole solutions, which can further be shown [Ref. 4] to be equivalent to the existence of a point spectrum for a nonlinear eigenvalue problem. When both the effects are included, this spectrum becomes a banded continuous spectrum. An eigenvalue of this spectrum is associated with a localized vortex structure that undulates in space about a fixed level, eventually matching to a radiative ion acoustic tail. A novel separatrix crossing technique is used to investigate this problem.

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over the scale length L_s. Including both the sheared field and temperature gradient effects, the perpendicular and parallel momentum equations are

$$\left(\frac{1}{\tau(x)} - \nabla^2\right)\frac{\partial\varphi}{\partial t} \;+\; v_d\,\frac{\partial\varphi}{\partial y} + \frac{\tau'(x)}{\tau^2(x)}\,\varphi\,\frac{\partial\varphi}{\partial y} - [\varphi, \nabla^2\varphi] + Sx\,\frac{\partial v^i_{\|}}{\partial y} = 0 \tag{1}$$

$$\frac{\partial v^i_{\|}}{\partial t} \;+\; [\varphi, v^i_{\|}] + Sx\,\frac{\partial\varphi}{\partial y} = 0 \tag{2}$$

where the normalized electron temperature $\tau(x) \equiv T_e(x)/T_0$, the spatial coordinates are scaled by $\rho_{i0} \equiv (T_0/m_i\,\omega_{ci}^2)^{1/2}$ with $\omega_{ci} \equiv eB_0/cm_i$, time is scaled by ω_{ci} and as usual φ is made dimensionless by a factor e/T_0. The parallel velocity is scaled by $\rho_{i0}\,\omega_{ci} = c_{s0} = \sqrt{T_0/m_i}$, and the drift velocity is also scaled by c_{s0}. The quantity $S \equiv \rho_{i0}/L_s$ is a measure of the magnetic shear.

For small parallel speed, we take the ion acoustic dynamics as linear. Equation (2) then becomes

$$\frac{\partial v^i_{\|}}{\partial t} = -Sx\,\frac{\partial\varphi}{\partial y}\,. \tag{3}$$

Combining Eqs. (1) and (3), we get the following model equation:

$$\left(\frac{1}{\tau} - \nabla^2\right)\frac{\partial\varphi}{\partial t} + v_d\,\frac{\partial\varphi}{\partial y} + \frac{\tau'}{\tau^2}\,\varphi\,\frac{\partial\varphi}{\partial y} - [\varphi, \nabla^2\varphi] - S^2x^2\int_0^t \frac{\partial^2\varphi}{\partial y^2}\,dt = 0\,. \tag{4}$$

In the limit of $S \to 0$ and in the inconsistent limit $\frac{1}{\tau} \to 1$ while $\tau'/\tau^2 \to$ constant we obtain the incomplete Petviashvili model. In this "limit" the monopole solution is obtained upon substituting $\varphi = \varphi_0(r)$ where $r = [x^2 + (y-ut)^2]^{1/2}$ into (4). One obtains the following equation:

$$\frac{1}{r}\frac{d}{dr}\left(r\,\frac{d\varphi_0}{dr}\right) - 4k^2\varphi_0 + \frac{\alpha}{2u}\,\varphi_0^2 = 0 \tag{5}$$

where $k^2 = \frac{1}{4}\left(1 - \frac{v_d}{u}\right)$ and $\alpha = \tau'/\tau^2$. Note that the presence of shear and temperature gradient removes the radial symmetry of this equation. Assuming $\varphi(x, y-ut)$ Eq. (4) reduces exactly to

$$\nabla^2\varphi - \frac{1}{\tau(x)}\,\varphi - \ell n\, n_0(x) + \frac{S^2\,x^3}{3u} = F(\varphi - ux) \tag{6}$$

together with Eq. (8), define a nonlinear eigenvalue problem for γ. In the limit $s \to 0$ the solution of the model Eq. (7) is $\gamma(s=0) \equiv \gamma_0 = 1$ with the homoclinic orbit $\varphi_0 = \operatorname{sech}^2 t$ where v_d, u and α/S are restricted to $\widehat{k}^2 > 0$.

Equation (8) can be written in the form of Hamilton's equation for an effective particle with coordinate $q = \psi$, time t, momentum $p = \frac{d\psi}{dt}$ and effective potential $V(\psi,t) = -2\psi^2 + 2\gamma\psi^3 + \frac{s^2t^2}{2}\psi^2$. The Hamiltonian is

$$H(\psi,p,t) = \frac{1}{2}p^2 + 2\gamma\psi^3 + 2\psi^2\left(\frac{s^2t^2}{4} - 1\right) , \tag{10}$$

and the dynamical equations are

$$\begin{aligned} \dot{p} &= -\frac{\partial}{\partial\psi}\left(-2\psi^2 + 2\gamma\psi^3 + \frac{s^2t^2}{2}\psi^2\right) = 4\psi - 6\gamma\psi^2 - s^2\,t^2\phi \\ \dot{\psi} &= p\,. \end{aligned} \tag{11}$$

As noted above, the effect of shear is to couple the vortex solution to the ion wave by changing the potential energy $V(\psi,t)$ with time. The critical time for particles passing from the solitary wave potential into ion acoustic wave potential is $t_0 = \frac{2}{s}$. At the critical time, the potential can no longer contain trapped particles (see dashed line in Fig. 1(b)). For $t > t_0$ trapped particle orbits exist in the nieghborhood of the origin; i.e. in the ion acoustic potential.

From numerical integration of the model Eqs. (8) and (9), we obtain the spectrum of eigenvalues $\gamma_n(s)$. This is done by choosing a zero momentum initial condition and integrating beyond t_c to determine if there is trapping for all time in the ion acoustic potential. We are interested in the critical value of the amplitude for such trapping. Physically the trapping implies the radiative tailing at large t-values. See. Fig. 2. A detailed study of the numerical spectrum yielded the $\gamma_n(s)$ curve shown in Fig. 3. The curve shows the amplitude at $t = 0\,(\gamma)$ for eigenfunctions $\psi(t)$ that are bounded as $t \to \infty$, versus the shear parameters, s. Observe that the upper and lower boundaries of the "bounded region" oscillate as $s \to 0$

Boltzmann distribution $n_e = N(x)\exp(e\Phi/T_e) = N(1+\varphi)$ used in deriving Eq. (4) breaks down. The potential has depleted the electron density to the unphysical point where n_e becomes negative.

With $s \neq 0$, however, the potential V, depth V_m of the potential well, and position ψ_m of its bottom change with time. The potential well becomes shallower and shallower with increasing t until $V_m = 0$ and $\psi_m = 0$ when $t = t_0(s) = \dfrac{2}{s}$ as shown by dashed line in Fig. 1(b). When $t > \dfrac{2}{s}$, the potential well changes its shape into that shown by the solid line in Fig. 1(b). Therefore, the question of whether the ball becomes eventually trapped in the well shown by the solid line in Fig. 1(b), or it goes into the "Hell" as $t \to \infty$, is determined not only by the initial potential energy (the initial position ψ of the ball) but also by the magnitude of s. The shear parameter s serves as an inverse characteristic time for the change of the well. It is also obvious that the number of oscillations that the ball performs around the bottom of the potential well within $t < \dfrac{2}{s}$ is determined by value $\dfrac{2}{s}$. The first formula (for $\gamma > \dfrac{2}{3}$) of Eq. (12) or the upper branch of Fig. 3 corresponds to dropping the ball from the right side of the bottom of the potential well at $t = 0$, while the second one (for $\gamma < \dfrac{2}{3}$) of Eq. (12), or the lower branch of Fig. 3, from the left side of the bottom.

Although Eqs. (12) have not been proven, we understand that coupling of the waves at large t to the vortex at $t = 0$ results in a spectrum of vortices with an increasing number of oscillations of ψ in the nonlinear trapping region. Thus, the inhomogeneity acts to split up the vortex point spectrum into continuous bands given by $\gamma_n(s)$. The shear inhomogeneity is a defocusing effect.

By introducing the action-angle variables

$$J = \frac{1}{2\pi}\oint p\,dq = \frac{1}{2\pi}\oint \sqrt{2(H-V)}\,d\varphi$$

and separatrix crossing theory we can relate the action at $t = 0$ for the homoclinic orbit

$$J(t=0,\gamma) = \frac{1}{2\pi}\oint \sqrt{2H(t=0) - 4(\gamma\psi^3 - \psi^2)}\,d\psi \tag{14}$$

$t < \dfrac{2}{s}$ satisfies $n = \dfrac{1}{s}$ and when s is small ($s < .25$), the eigenvalue $\gamma_n(s)$ satisfy the recursion relations of Eq. (12).

Even though the present formulation is an oversimplified model, it exhibits the main physical features of coupling the integrable solitary wave to the ion acoustic wave due to the presence of the magnetic shear. The study of a more comprehensive model which involves the two fields, φ and v_z, will be presented elsewhere.

Acknowledgments

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Figure Captions

1. Effective potential for nonlinear drift waves.

   (a) Behavior of effective potential near the center of the drift wave structure: $t = 0$.

   (b) Behavior of effective potential far from the center of the drift wave structure: $t > 2/s$.

2. (a) Nonlinear eigenfunction for $s = \frac{1}{8}$ showing monopole vortex and wave solutions for $\gamma > \frac{2}{3}$ (upper bound).

   (b) Nonlinear eigenfunction for $s = \frac{1}{7}$ showing monopole vortex and wave solutions for $\gamma < \frac{2}{3}$ (lower bound).

3. Spectrum of critical amplitude γ versus shear s showing vortex branches $\gamma_n(s)$ and nonlinear wave solutions.

4. Comparison of $\gamma(s)$ from separatrix crossing theory with that from numerical integration rsults of Eq. (7). The solid lines represent the analytical results and the dashed lines, numerical results.

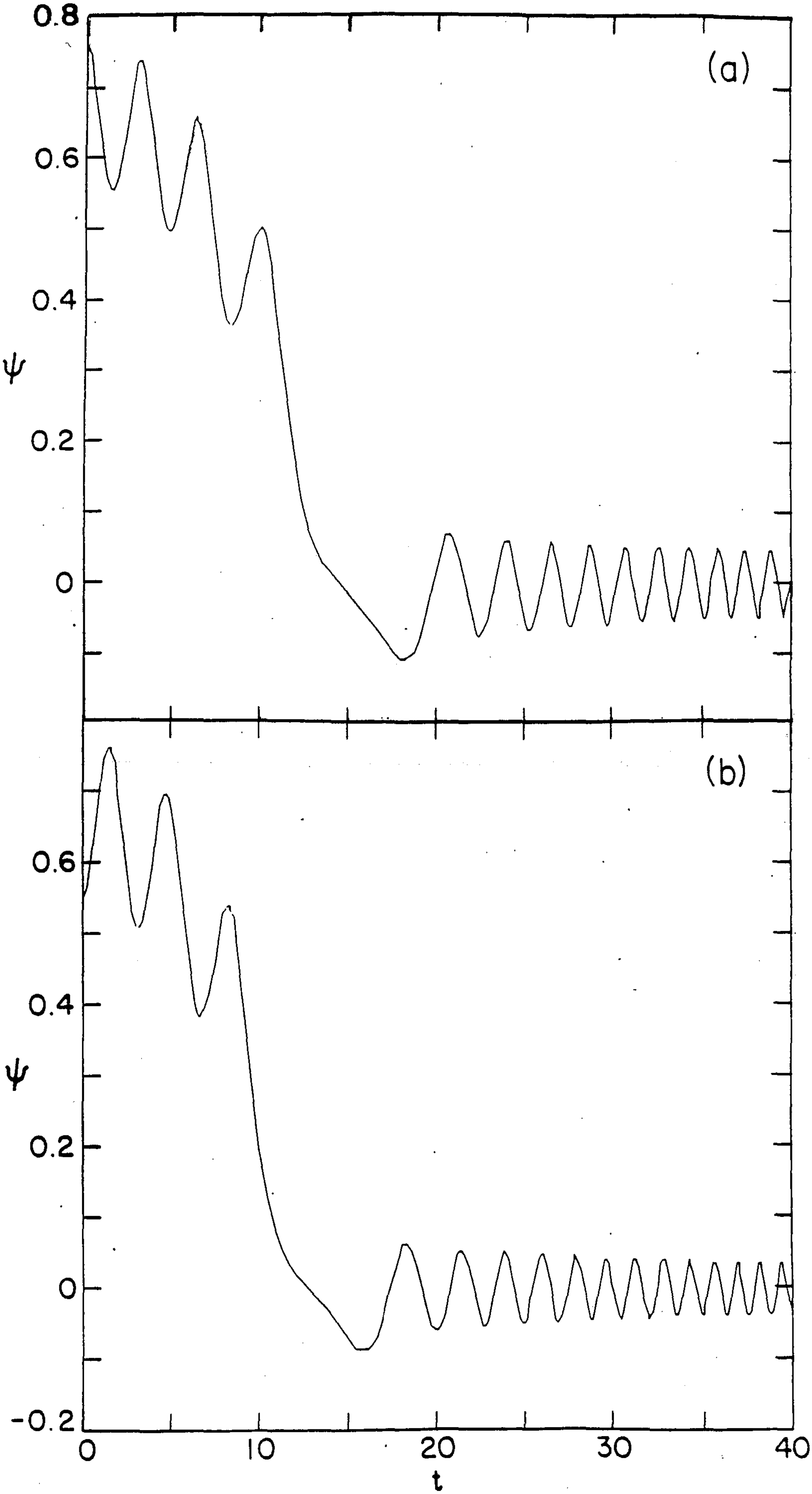
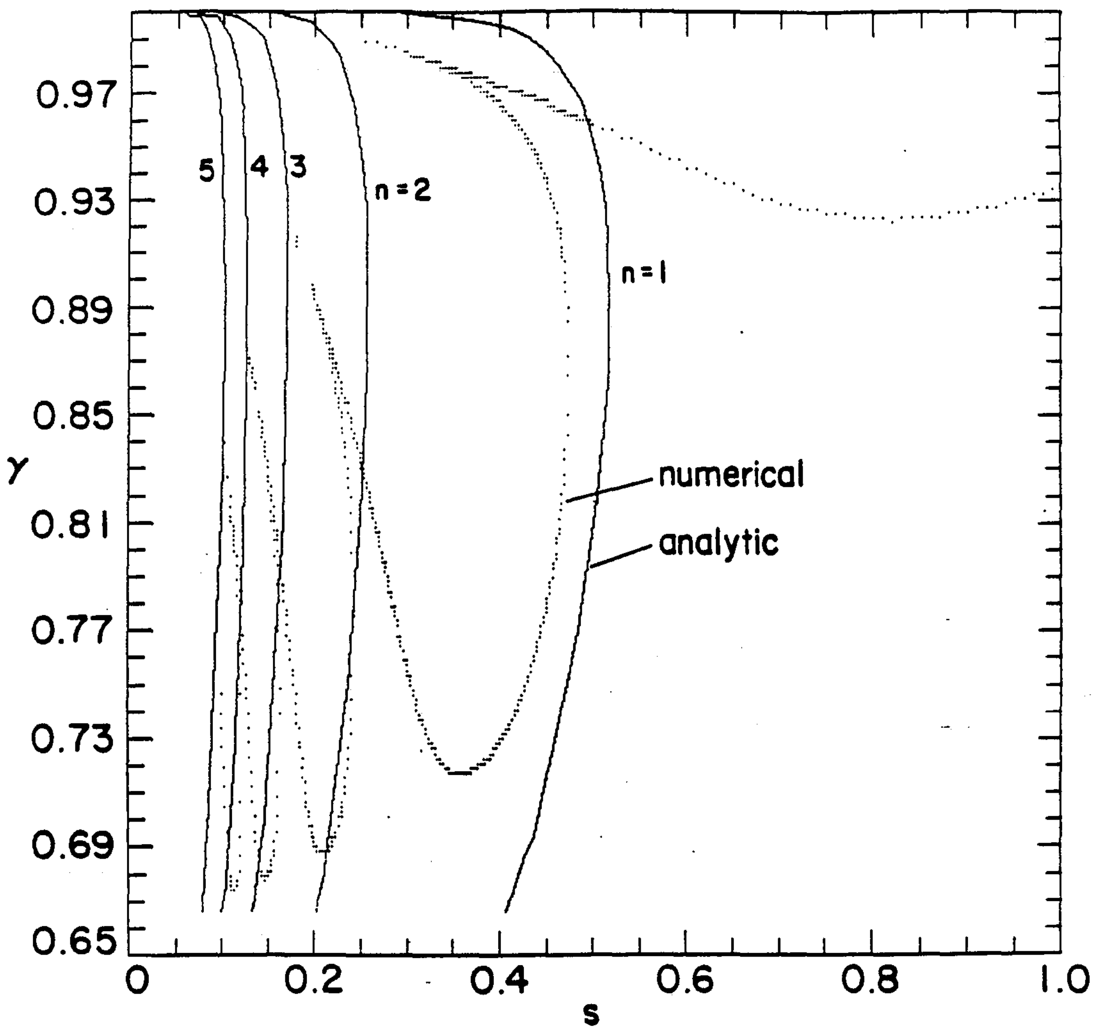


Fig. 2



0.97
0.93
0.89
0.85
0.81
0.77
0.73
0.69
0.65
γ
5
4
3
n=2
n=1
numerical
analytic
0
0.2
0.4
0.6
0.8
1.0
s

Fig. 4