The free energy of Maxwell–Vlasov equilibria

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A previously derived expression [Phys. Rev. A **40**, 3898 (1989)] for the energy of arbitrary perturbations about arbitrary Vlasov–Maxwell equilibria is transformed into a very compact form. The new form is also obtained by a canonical transformation method for solving Vlasov's equation, which is based on Lie group theory. This method is simpler than the one used before and provides better physical insight. Finally, a procedure is presented for determining the existence of negative-energy modes. In this context the question of why there is an accessibility constraint for the particles, but not for the fields, is discussed.

I. INTRODUCTION

A Vlasov–Maxwell equilibrium is said to possess potentially usable free energy if, in a reference frame where the energy is lowest, there exists a perturbed state that is dynamically accessible from the unperturbed one, with an energy that is lower than that of the equilibrium. We denote this energy difference, the object of main interest in this paper, by $\delta^2 F$, i.e., $\delta^2 F$ is the perturbation or wave energy. The notation is appropriate for linearized theories where the wave energy is of second order and is obtained from a nonlinear constant of motion.

If an equilibrium possesses free energy we can expect the existence of several kinds of instabilities; these are either linear dissipationless instabilities with $\delta^2 F = 0$ or instabilities caused by drawing out energy from perturbations with $\delta^2 F < 0$, in which case the amplitudes of such perturbations must grow. This can occur either by dissipation or by coupling of "negative-energy waves" with $\delta^2 F < 0$ to positive-energy waves with $\delta^2 F > 0$ of the same system via nonlinear terms in the equations. The latter is exemplified in a very transparent way by Cherry's nonlinearly coupled oscillators,[1,2] which are described by the following Hamiltonian:

$$H = -\tfrac{1}{2}\omega_1(p_1^2 + q_1^2) + \tfrac{1}{2}\omega_2(p_2^2 + q_2^2) + (\alpha/2)[2q_1p_1p_2 - q_2(q_1^2 - p_1^2)]\,,$$

where the constants α, ω_1, and ω_2 are real, the latter two being positive, and (p_i, q_i) $(i = 1, 2)$ are the canonically conjugate variables. For $\alpha = 0$ we have two uncoupled oscillators, of negative and positive energy, respectively. This situation corresponds to a charged particle on a "mountain" with potential $V(x,y) = -(x^2 + y^2)/2$, whose equilibrium position $x = 0$, $y = 0$ is stabilized by a superimposed constant vertical magnetic field. If $\omega_2 = 2\omega_1$ we have a third-order resonance. Cherry found for this case the following exact two-parameter solution set:

$$q_1 = [-\sqrt{2}/(\epsilon - \alpha t)] \sin(\omega_1 t + \gamma),$$
$$p_1 = [\sqrt{2}/(\epsilon - \alpha t)] \cos(\omega_1 t + \gamma),$$
$$q_2 = [-1/(\epsilon - \alpha t)] \sin(2\omega_1 t + 2\gamma),$$
$$p_2 = [-1/(\epsilon - \alpha t)] \cos(2\omega_1 t + 2\gamma),$$

where ϵ and γ are determined by the initial conditions and α is an arbitrary parameter that measures the effect of the nonlinearity. These solutions show explosive instability, whereas the linearized theory gives only stable oscillations with the two real frequencies ω_1 and $\omega_2 = 2\omega_1$. The assumed resonance corresponds to the conservation law $\omega_1 + \omega_2 + \omega_3 = 0$ for the three-wave interaction in the Vlasov–Maxwell case.

In the past, discussions of such nonlinear electrostatic instabilities in homogeneous plasma[3–7] and their relation to the existence of free energy were based on the well-known expression[8,9]

$$\delta^2 F = \frac{1}{16\pi} |\mathbf{E}(\mathbf{k},\omega)|^2 \omega \frac{\partial \epsilon_{\rm H}}{\partial \omega}, \quad \omega = \omega(\mathbf{k}).$$

This expression is a special case of

$$\delta^2 F = \frac{1}{16\pi} \mathbf{E}^*(\mathbf{k},\omega)\cdot\frac{1}{\omega}\frac{\partial}{\partial\omega}[\omega^2 \boldsymbol{\epsilon}_{\rm H}(\mathbf{k},\omega)]\cdot\mathbf{E}(\mathbf{k},\omega),$$
$$\omega = \omega(k),$$

which is valid for general electromagnetic perturbations of homogeneous equilibria. Here $\boldsymbol{\epsilon}_{\rm H}$ is the Hermitian part of the dielectic tensor, whose anti-Hermitian part $\boldsymbol{\epsilon}_{\rm A}$ must be negligible. To evaluate this expression requires the knowledge of the Fourier transform of the perturbed electric field $\mathbf{E}(\mathbf{k},\omega)$ and the dispersion relation $\omega(\mathbf{k})$ and is therefore, in general, not easy to use. We note that there is also an extension to inhomogeneous equilibria, in which case $\boldsymbol{\epsilon}$ is an operator in x space.[10]

A different kind of energy expression, which allows a much simpler discussion, is known for one-dimensional Vlasov–Poisson systems with homogeneous monotonic equilibria,[11] namely,

$$\delta^2 F = \frac{1}{8\pi}\int \delta E^2\, dx - \sum_\nu \frac{m_\nu}{2}\int \frac{v\,\delta f_\nu^2}{\partial f_\nu^{(0)}/\partial v}\, dv\, dx,$$

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where ν is the species label, δE is the perturbed electric field, $f_\nu^{(0)}$ is the equilibrium distribution function, and δf_ν is the perturbed distribution function. More properly, this expression should be written as

$$\delta^2 F = \frac{1}{8\pi}\int \delta E^2\, dx - \sum_\nu \frac{m_\nu}{2}\int \hat{g}_\nu^2\, v\,\frac{\partial f_\nu^{(0)}}{\partial v}\, dv\, dx, \qquad (1)$$

with

$$\delta f_\nu = \hat{g}_\nu \frac{\partial f_\nu^{(0)}}{\partial v}, \quad \frac{\partial \hat{g}_\nu}{\partial t} + v\frac{\partial \hat{g}_\nu}{\partial x} = -\frac{e_\nu}{m_\nu}\delta E. \qquad (2)$$

(Below we will see that $\hat{g}_\nu$ is the spatial derivative of a generating function). The latter form, contrary to the first, is valid for arbitrary distribution functions $f^{(0)}(\mathbf{v})$.[12,13] Since the perturbed charge density

$$\delta\rho = \sum_\nu e_\nu \int \hat{g}_\nu \frac{\partial f_\nu^{(0)}}{\partial v}\, dv$$

can be made zero for any $\hat{g}_\nu^2$ (e.g., by picking $\hat{g}_\nu$ to be piecewise continuous), we obtain the minimization of $\delta^2 F$ for $\delta E = 0$. Hence $\delta^2 F < 0$ is possible if

$$v\frac{\partial f_\nu^{(0)}}{\partial v} > 0 \qquad (3)$$

holds for at least one particle species in some v interval, while in a frame of reference where the equilibrium obtains its minimum energy.

In a previous paper[13] we were able to derive a general expression for the energy of arbitrary perturbations of arbitrary three-dimensional Vlasov–Maxwell equilibria, from which we obtained a generalization of condition (3). Thus all interesting equilibria were shown to be either linearly unstable or possess negative-energy modes. In the present paper we complement the results obtained before in three respects. In Sec. II we transform the original expression for $\delta^2 F$, Eq. (69) of Ref. 13, into a more compact form. In Sec. III we rederive this expression by a new, much more elegant method, which provides better physical insight. The derivation begins with the well-known general nonlinear energy expression

$$H = \sum_\nu \int \frac{m_\nu}{2} v^2 f_\nu(\mathbf{x},\mathbf{v})\, d^3x\, d^3v + \frac{1}{8\pi}\int (E^2 + B^2)\, d^3x.$$

This expression is expanded up to the second order in the perturbations. The occurring first- and second-order distribution functions are represented by the generating function for a canonical transformation according to the Lie group formalism. The method allows, in addition, a simple direct proof that the quantity obtained in Ref. 13 is the second-order energy. In Sec. IV we describe a procedure for determining the existence of negative-energy perturbations. In discussing this procedure we address the questions, under which circumstances should we introduce a norm for the perturbations and which norm is appropriate? The section begins with an explanation of the question of why accessibility is a constraint for the particles but not for the fields, i.e., why it is only necessary to relate $\delta\mathbf{x}$ and $\delta\dot{\mathbf{x}}$, but not $\delta\mathbf{A}$ and $\delta\dot{\mathbf{A}}$.

II. SIMPLIFIED FREE-ENERGY EXPRESSION

In this section we begin with Eq. (68) of Ref. 13 and perform a sequence of manipulations that result in a simplified free-energy expression, the terms of which are physically identifiable. The expression of Ref. 13 is

$$\begin{aligned}
\delta^2 F = \sum_\nu \int \frac{d^3x\, d^3v}{2m_\nu} f_\nu^{(0)}(\mathbf{x},\mathbf{v}) \Bigg\{ & \left| \frac{\partial G_\nu}{\partial \mathbf{x}} + \frac{e_\nu}{m_\nu c}\frac{\partial G_\mu}{\partial v_i}\left(\frac{\partial \mathbf{A}^{(0)}}{\partial x_i} - \frac{\partial A_i^{(0)}}{\partial \mathbf{x}}\right) + \frac{e_\nu}{c}\delta\mathbf{A}\right|^2 \\
& + 2\left[\frac{\partial G_\nu}{\partial \mathbf{x}} + \frac{e_\nu}{m_\nu c}\frac{\partial G_\nu}{\partial v_i}\left(\frac{\partial \mathbf{A}^{(0)}}{\partial x_i} - \frac{\partial A_i^{(0)}}{\partial \mathbf{x}}\right) + \frac{e_\nu}{c}\delta\mathbf{A}\right]\cdot d\frac{\partial G_\nu}{\partial \mathbf{v}} - \frac{2e_\nu}{c}\frac{\partial G_\nu}{\partial v_i} v_k \left(\frac{\partial \delta A_k}{\partial x_i} - \frac{\partial \delta A_i}{\partial x_k}\right) \\
& - \frac{e_\nu}{m_\nu c}\frac{\partial G_\nu}{\partial v_i}\frac{\partial G_\nu}{\partial v_j}\left[-\frac{\partial^2 \Phi^{(0)}}{\partial x_i\, \partial x_j} + \frac{v_k}{c}\frac{\partial}{\partial x_i}\left(\frac{\partial A_k^{(0)}}{\partial x_j} - \frac{\partial A_j^{(0)}}{\partial x_k}\right)\right] \\
& + \frac{e_\nu}{m_\nu c}\frac{\partial G_\nu}{\partial v_i}\left(\frac{\partial A_i^{(0)}}{\partial x_j} - \frac{\partial A_j^{(0)}}{\partial x_i}\right) d\frac{\partial G_\nu}{\partial v_j}\Bigg\} + \frac{1}{8\pi}\int (\delta E^2 + \delta B^2)\, d^3x, \qquad (4)
\end{aligned}$$

where $G_\nu(\mathbf{x},\mathbf{v})$ is the generating function for particle displacement and velocity perturbations, the perturbed field quantities are denoted by δ, and equilibrium quantities are denoted by the superscript (0). [In Ref. 13 the expression of Eq. (4) was referred to as $\delta^2 H$.] The operator d is given by

$$d = \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}} + \frac{e_\nu}{m_\nu}\left(\mathbf{E}^{(0)} + \frac{1}{c}\mathbf{v}\times\mathbf{B}^{(0)}\right)\cdot\frac{\partial}{\partial \mathbf{v}}. \qquad (5)$$

Since Eq. (4) is written so as to make gauge invariance obvious, we comment on this now before altering its form. If we let

$$\begin{aligned} &\mathbf{A}^{(0)} \to \mathbf{A}^{(0)} + \nabla\psi^{(0)}, \\ &\delta\mathbf{A}^{(0)} \to \delta\mathbf{A} + \nabla\delta\psi, \quad G_\nu \to G_\nu - (e_\nu/c)\delta\psi, \end{aligned} \qquad (6)$$

Eq. (4) remains unchanged. The transformation of (6) that involves G_ν arises because the canonical momentum is not gauge invariant [cf. Ref. 1, Eq. (51)]. Note that the transformations

$$\Phi^{(0)} \to \Phi^{(0)} - \frac{1}{c}\frac{\partial \psi^{(0)}}{\partial t}, \quad \delta\Phi \to \delta\Phi - \frac{1}{c}\frac{\partial \delta\psi}{\partial t} \qquad (7)$$

are satisfied since $\partial\psi^{(0)}/\partial t$ should vanish in order that our equilibrium quantities be time independent; the second expression is obvious since $\delta^2 F$ only depends on $\delta\Phi$ through δE.

Now we rewrite Eq. (4) by transforming from the variable $\mathbf{v}$ to the canonical momentum $\mathbf{p}$, defined by

$$\mathbf{p} = m_\nu \mathbf{v} + (e_\nu/c)\mathbf{A}^{(0)}. \tag{8}$$

The generating function and equilibrium distribution function are transformed according to

$$g_\nu(\mathbf{x},\mathbf{p}) = G_\nu(\mathbf{x},\mathbf{v}), \quad \bar{f}_\nu^{(0)}(\mathbf{x},\mathbf{p}) = m_\nu^{-3} f_\nu^{(0)}(\mathbf{x},\mathbf{v}). \tag{9}$$

We will drop the overbar on $\bar{f}_\nu^{(0)}$ below. The chain rule implies

$$\begin{aligned}
\left.\frac{\partial G_\nu}{\partial \mathbf{v}}\right|_{\mathbf{x}} &= m_\nu \left.\frac{\partial g_\nu}{\partial \mathbf{p}}\right|_{\mathbf{x}}, \\
\left.\frac{\partial G_\nu}{\partial \mathbf{x}}\right|_{\mathbf{v}} &= \left.\frac{\partial g_\nu}{\partial \mathbf{x}}\right|_{\mathbf{p}} + \left.\frac{\partial g_\nu}{\partial p_i}\right|_{\mathbf{x}} \frac{\partial p_i}{\partial \mathbf{x}} \\
&= \left.\frac{\partial g_\nu}{\partial \mathbf{x}}\right|_{\mathbf{p}} + \frac{e_\nu}{c}\frac{\partial A_i^{(0)}}{\partial \mathbf{x}} \left.\frac{\partial g_\nu}{\partial p_i}\right|_{\mathbf{x}}.
\end{aligned} \tag{10}$$

Using Eqs. (8)–(10), Eq. (4) becomes

$$\begin{aligned}
\delta^2 F = \sum_\nu \int \frac{d^3x\, d^3v}{2m_\nu} f^{(0)}(\mathbf{x},\mathbf{p}) \Bigg\{ &\left| \frac{\partial g_\nu}{\partial \mathbf{x}} + \frac{e_\nu}{c}\frac{\partial g_\nu}{\partial p_i}\frac{\partial \mathbf{A}^{(0)}}{\partial x_i} + \frac{e_\nu}{c}\delta\mathbf{A} \right|^2 + 2m_\nu \left(\frac{\partial g_\nu}{\partial \mathbf{x}} + \frac{e_\nu}{c}\frac{\partial g_\nu}{\partial p_i}\frac{\partial \mathbf{A}^{(0)}}{\partial x_i} + \frac{e_\nu}{c}\delta\mathbf{A} \right) \\
&\cdot d\frac{\partial g_\nu}{\partial \mathbf{p}} - 2\frac{e_\nu m_\nu}{c}\frac{\partial g_\nu}{\partial p_i} v_k \left(\frac{\partial \delta A_k}{\partial x_i} - \frac{\partial \delta A_i}{\partial x_k} \right) - e_\nu m_\nu \frac{\partial g_\nu}{\partial p_i}\frac{\partial g_\nu}{\partial p_j} \left[-\frac{\partial^2 \Phi^{(0)}}{\partial x_i \partial x_j} + \frac{v_k}{c}\frac{\partial}{\partial x_i}\left(\frac{\partial A_k^{(0)}}{\partial x_j} - \frac{\partial A_j^{(0)}}{\partial x_k} \right) \right] \\
&+ \frac{e_\nu m_\nu}{c}\frac{\partial g_\nu}{\partial p_i}\left(\frac{\partial A_i^{(0)}}{\partial x_j} - \frac{\partial A_j^{(0)}}{\partial x_i} \right) d\frac{\partial g_\nu}{\partial p_i} \Bigg\} + \frac{1}{8\pi}\int (\delta E^2 + \delta B^2)\, d^3x,
\end{aligned} \tag{11}$$

where v_k is to be thought of as shorthand for $m_\nu^{-1}(p_k - e_\nu A_k^{(0)}/c)$ and the operator d becomes

$$d = \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}} - e_\nu \frac{\partial \Phi^{(0)}}{\partial \mathbf{x}}\cdot\frac{\partial}{\partial \mathbf{p}} + \frac{e_\nu}{c} v_k \frac{\partial A_k^{(0)}}{\partial x_i}\frac{\partial}{\partial p_i}. \tag{12}$$

In Appendix A we simplify Eq. (11) by performing a sequence of integrations by parts. The final result is

$$\begin{aligned}
\delta^2 F = \sum_\nu \int d^3x\, d^3p \bigg(&\frac{1}{2}[g_\nu, f_\nu^{(0)}]\,[H_\nu^{(0)}, g_\nu] \\
&- \frac{e_\nu}{m_\nu c}[g_\nu, f_\nu^{(0)}]\delta\mathbf{A}\cdot[\mathbf{p} - (e_\nu/c)\mathbf{A}^{(0)}] \\
&+ \frac{e_\nu^2}{2m_\nu c^2} f_\nu^{(0)} |\delta\mathbf{A}|^2 \bigg) \\
&+ \frac{1}{8\pi}\int (\delta E^2 + \delta B^2)\, d^3x,
\end{aligned} \tag{13}$$

where [,], the Poisson bracket, is defined by

$$[f,g] = \frac{\partial f}{\partial \mathbf{x}}\cdot\frac{\partial g}{\partial \mathbf{p}} - \frac{\partial f}{\partial \mathbf{p}}\cdot\frac{\partial g}{\partial \mathbf{x}}, \tag{14}$$

and $H_\nu^{(0)}$ is the unperturbed Hamiltonian,

$$H_\nu^{(0)} = (1/2m_\nu)|\mathbf{p} - (e_\nu/c)\mathbf{A}^{(0)}|^2 + e_\nu \Phi^{(0)}. \tag{15}$$

In the next section the physical meaning of the various terms in this expression will become clear.

III. CANONICAL DERIVATION OF THE FREE ENERGY

We now give a very simple derivation of the free-energy expression of Eq. (13). The amount of calculation required is far less than that of Appendix A. In Appendix B we calculate the first- and second-order perturbed distribution functions, which arise because the equilibrium particle orbits are perturbed. Since the orbit equations are Hamiltonian the perturbation of the orbits is completely determined by a generating function g_ν that we expand to second order. Thus the first- and second-order distribution functions are given, according to Eqs. (B6) and (B7), by

$$\delta^{(1)} f_\nu \equiv [g_\nu^{(1)}, f_\nu^{(0)}], \tag{16}$$

$$\delta^{(2)} f_\nu = [g_\nu^{(2)}, f_\nu^{(0)}] + \tfrac{1}{2}[g_\nu^{(1)}, [g_\nu^{(1)}, f_\nu^{(0)}]]. \tag{17}$$

With these relations we easily obtain the second-order energy from the exact one.

The total energy of the Maxwell–Vlasov system is given by

$$\begin{aligned}
H = \sum_\nu \int \frac{1}{2m_\nu}\left| \mathbf{p} - \frac{e_\nu}{c}\mathbf{A}(\mathbf{x},t) \right|^2 f_\nu(\mathbf{x},\mathbf{p},t)\, d^3x\, d^3p \\
+ \frac{1}{8\pi}\int (E^2 + B^2)\, d^3x.
\end{aligned} \tag{18}$$

The second-order energy is evidently

$$\begin{aligned}
\delta^2 H = \sum_\nu \int \bigg[&\frac{1}{2m_\nu}\left| \mathbf{p} - \frac{e_\nu}{c}\mathbf{A}^{(0)} \right|^2 \delta^{(2)} f_\nu - \frac{e_\nu}{m_\nu c}\delta^{(1)}\mathbf{A} \\
&\cdot\left(\mathbf{p} - \frac{e_\nu}{c}\mathbf{A}^{(0)} \right)\delta^{(1)} f_\nu - \frac{e_\nu}{m_\nu c}\delta^{(2)}\mathbf{A}\cdot\left(\mathbf{p} - \frac{e_\nu}{c}\mathbf{A}^{(0)} \right) f_\nu^{(0)} \\
&+ \frac{e_\nu^2}{2m_\nu c^2}|\delta^{(1)}\mathbf{A}|^2 f_\nu^{(0)} \bigg] d^3x\, d^3p + \frac{1}{8\pi}\int (|\delta^{(1)}\mathbf{E}|^2 \\
&+ |\delta^{(1)}\mathbf{B}|^2 + 2\delta^{(2)}\mathbf{E}\cdot\mathbf{E}^{(0)} + 2\delta^{(2)}\mathbf{B}\cdot\mathbf{B}^{(0)})\, d^3x.
\end{aligned} \tag{19}$$

For Hamiltonian systems it is always possible to write the second-order energy in terms of first-order quantities. The third terms and the last term indeed cancel; this is seen by integrating the last term by parts:

$$\frac{1}{4\pi}\int \delta^{(2)}\mathbf{A}\cdot\mathbf{J}^{(0)}\,d^3x = \sum_\nu \frac{e_\nu}{m_\nu c}\int\left(\mathbf{p}-\frac{e_\nu}{c}\mathbf{A}^{(0)}\right)\cdot\delta^{(2)}\mathbf{A}\,f_\nu^{(0)}\,d^3x\,d^3p.$$

Similarly, by making use of Poisson's equation, we obtain

$$\frac{1}{4\pi}\int \delta^{(2)}\mathbf{E}\cdot\mathbf{E}^{(0)}\,d^3x = \sum_\nu e_\nu \int \Phi^{(0)}\delta^{(2)}f_\nu\,d^3x\,d^3p, \tag{20}$$

which combines with the first term of Eq. (19) to give $\sum_\nu \int H_\nu^{(0)}\,\delta^{(2)}f_\nu\,d^3x\,d^3p$. The contribution of $[g_\nu^{(2)},f_\nu^{(0)}]$ to $\delta^{(2)}f_\nu$ in Eq. (17) yields zero after partial integration since $[H_\nu^{(0)},f_\nu^{(0)}]=0$. Thus all second-order quantities cancel and Eq. (19) becomes

$$\delta^2F=\sum_\nu\int\Bigg[\frac{1}{2}H_\nu^{(0)}\,[g_\nu,[g_\nu,f_\nu^{(0)}]] - \frac{e_\nu}{m_\nu c}\delta\mathbf{A}\cdot\left(\mathbf{p}-\frac{e_\nu}{c}\mathbf{A}^{(0)}\right)[g_\nu,f_\nu^{(0)}] + \frac{e_\nu}{2m_\nu c^2}|\delta\mathbf{A}|^2 f_\nu^{(0)}\Bigg]\,d^3x\,d^3p + \frac{1}{8\pi}\int(\delta E^2+\delta B^2)\,d^3x. \tag{21}$$

Here we have dropped the superscript (1) on the first-order quantities and changed the name to δ^2F, since now the Hamiltonian constraints are built in. Equation (21) is seen to be identical to Eq. (13) upon integration of the first term by parts.

To conclude this section we show directly that Eq. (21) is conserved by the linearized equations of motion. The fields satisfy

$$\frac{\partial\delta\mathbf{E}}{\partial t}=c\nabla\times\delta\mathbf{B}-4\pi\delta\mathbf{J}, \tag{22}$$

$$\frac{\partial\delta\mathbf{A}}{\partial t}=-c\delta\mathbf{E}-c\nabla\delta\Phi, \tag{23}$$

while the generating function g_ν satisfies

$$\frac{\partial g_\nu}{\partial t}+[g_\nu,H_\nu^{(0)}]=\delta H_\nu, \tag{24}$$

the linearized Hamiltonian–Jacobi equation. In Eq. (24),

$$\delta H_\nu = e_\nu\delta\Phi-(e_\nu/m_\nu c)\delta\mathbf{A}\cdot[\mathbf{p}-(e_\nu/c)\mathbf{A}^{(0)}]. \tag{25}$$

It is straightforward to show that if g_ν satisfies (24), then $\delta f_\nu = [g_\nu,f_\nu^{(0)}]$ satisfies the linearized Vlasov equation. Taking the time derivative of δ^2F and inserting Eqs. (22)–(24) yields

$$\begin{aligned}\frac{d}{dt}\delta^2F &= \sum_\nu\int\Bigg(-\frac{1}{2}[[g_\nu,H_\nu^{(0)}],f_\nu^{(0)}][H_\nu^{(0)},g_\nu]+\frac{1}{2}[\delta H_\nu,f_\nu^{(0)}][H_\nu^{(0)},g_\nu]-\frac{1}{2}[g_\nu,f_\nu^{(0)}]\,[H_\nu^{(0)},[g_\nu,H_\nu^{(0)}]]\\ &\quad+\frac{1}{2}[g_\nu,f_\nu^{(0)}]\,[H_\nu^{(0)},\delta H_\nu]+\frac{e_\nu}{c}\delta\mathbf{A}\cdot\mathbf{v}[[g_\nu,H_\nu^{(0)}],f_\nu^{(0)}]\\ &\quad-\frac{e_\nu}{c}\delta\mathbf{A}\cdot\mathbf{v}[\delta H_\nu,f_\nu^{(0)}]-\frac{e_\nu}{c}[g_\nu,f_\nu^{(0)}]\delta\dot{\mathbf{A}}\cdot\mathbf{v}+\frac{e_\nu^2}{m_\nu c^2}\delta\mathbf{A}\cdot\delta\dot{\mathbf{A}}\Bigg)\,d^3x\,d^3p\\ &\quad+\frac{1}{4\pi}\int[\delta\mathbf{E}\cdot(-4\pi\delta\mathbf{J}+c\nabla\delta\mathbf{B})+\delta\mathbf{B}\cdot(-c\nabla\times\mathbf{E})]\,d^3x.\end{aligned} \tag{26}$$

The first and third terms of Eq. (26) can be shown to cancel, while the second and fourth combine to yield

$$\sum_\nu\int[H_\nu^{(0)},\delta H_\nu]\,[g_\nu,f_\nu^{(0)}]\,d^3x\,d^3p.$$

Also, the last two terms cancel. Using

$$\begin{aligned}-\int\delta\mathbf{E}\cdot\delta\mathbf{J} &= \sum_\nu\int\Bigg[\frac{1}{c}\dot{\mathbf{A}}\cdot\left(e_\nu\mathbf{v}[g_\nu,f_\nu^{(0)}]-\frac{e_\nu^2}{m_\nu c^2}\delta\mathbf{A}f_\nu^{(0)}\right)\\ &\quad+e_\nu\mathbf{v}\cdot\nabla\delta\Phi[g_\nu,f_\nu^{(0)}]\\ &\quad-\frac{e_\nu^2}{m_\nu c^2}\delta\mathbf{A}\cdot\nabla\delta\Phi f_\nu^{(0)}\Bigg]\,d^3x\,d^3p,\end{aligned}$$

together with the above cancellations and combination, results in the following:

$$\begin{aligned}\frac{d}{dt}\delta^2F &= \sum_\nu\int\Bigg([H_\nu^{(0)},\delta H_\nu][g_\nu,f_\nu^{(0)}]+\frac{e_\nu}{c}\delta\mathbf{A}\\ &\quad\cdot\mathbf{v}[[g_\nu,H_\nu^{(0)}],f_\nu^{(0)}]-\frac{e_\nu}{c}[\delta\mathbf{A}\cdot\mathbf{v},\delta H_\nu]f_\nu^{(0)}\\ &\quad+e_\nu\mathbf{v}\cdot\nabla\delta\Phi[g_\nu,f_\nu^{(0)}]\\ &\quad-\frac{e_\nu^2}{m_\nu c^2}f_\nu^{(0)}\,\delta\mathbf{A}\cdot\nabla\delta\Phi\Bigg)\,d^3x\,d^3p.\end{aligned} \tag{27}$$

Using the Jacobi identity, $[a,[b,c]]+[b,[c,a]]+[c,[a,b]]=0$, on the second term, and Eq. (25) we see that the terms linear in g_ν cancel. We are left with

$$\frac{d}{dt}\delta^2F=\sum_\nu\int\Bigg(-\frac{e_\nu}{c}[\delta\mathbf{A}\cdot\mathbf{v},\delta H_\nu]f_\nu^{(0)} - \frac{e_\nu^2}{m_\nu c^2}\delta\mathbf{A}\cdot\nabla\delta\Phi f_\nu^{(0)}\Bigg)\,d^3x\,d^3p. \tag{28}$$

Since

$$\begin{aligned}[\delta\mathbf{A}\cdot\mathbf{v},\delta H_\nu] &= e_\nu[\delta\mathbf{A}\cdot\mathbf{v},\delta\Phi]\\ &= -e_\nu\nabla\delta\Phi\cdot\frac{\partial}{\partial\mathbf{p}}(\delta\mathbf{A}\cdot\mathbf{v})\end{aligned}$$

$$= -\frac{e_\nu}{m_\nu}\nabla\delta\Phi\cdot\delta\mathbf{A},$$

Eq. (28) is seen to vanish.

IV. EXTREMIZATION OF THE FREE ENERGY

Examination of Eq. (13) does not immediately reveal conditions that are necessary or sufficient for the definiteness of δ^2F. The first term can have either sign, while the sign of the second term depends on both the magnetic field and particle perturbations. The remaining terms are all positive definite, but it is not clear when these terms dominate. Further, things are complicated because $\delta\mathbf{E}$ is not completely independent; it must be consistent with the constraint imposed by Poisson's equation. For these reasons we describe in this section a procedure for determining the existence of negative-energy perturbations.

In δ^2F the quantities g_ν and $\delta\mathbf{A}$, $\delta\dot{\mathbf{A}}$, and $\delta\Phi$ can be chosen independently, provided only that $\delta\Phi$ and $\delta\dot{\mathbf{A}}$ satisfy the constraint $\nabla\cdot\delta\mathbf{E}=4\pi\delta\rho$. It is evident by now that the particle perturbations $\delta\mathbf{x}$ and $\delta\mathbf{p}$ are not completely independent; since we have insisted on the Hamiltonian constraints, they are generated through the single function g_ν. Recall that this guarantees that $\delta\mathbf{x}$ and $\delta\mathbf{p}$ will conserve phase space volume. For equilibria of interest this phase space volume is finite, contrary to the case where all particles have zero velocity. We can view the equilibria of interest as being states of minimum energy subject to the Hamiltonian constraints. However, we may question why we have taken $\delta\mathbf{A}$ and its canonical conjugate, which in essence is $\delta\dot{\mathbf{A}}$, to be independent. These quantities are independent of the particle quantity g_ν, because Maxwell's equations allow for the production of a displacement current that makes a given particle-field configuration consistent. They are independent of each other because Maxwell's equations are second order in time, but since the field equations are Hamiltonian, we could require that $\delta\mathbf{A}$ and $\delta\dot{\mathbf{A}}$ be derived from a generating functional in a manner analogous to the particle perturbations. We do not do this explicitly, but it will be seen below that the constraints are satisfied. The reason is that the phase space volume and other constraints for the equilibria of interest are identically zero. This follows because, generally, $\dot{\mathbf{A}}^{(0)}=0$. If the equilibrium state of interest was one where $\dot{\mathbf{A}}^{(0)}\neq 0$, then we might want to change this.

The first part of our procedure is to treat the Poisson constraint. We find it convenient to take variations with respect to $\delta\mathbf{E}$ and use the method of Lagrange multipliers. Since the positive semidefinite electric field energy contribution can be considered independently, we vary as follows:

$$\delta\int\left(\frac{\delta E^2}{2}-\delta U(\mathbf{x})(\nabla\cdot\delta\mathbf{E}-4\pi\delta\rho)\right)d^3x=0, \tag{29}$$

where $\delta U(\mathbf{x})$ is the Lagrange multiplier. Equation (29) yields $\delta\mathbf{E}=-\nabla\delta U$, which is satisfied by

$$\delta\dot{\mathbf{A}}=0,\quad \delta\Phi=\delta U, \tag{30}$$

with

$$\nabla^2\delta U=-4\pi\delta\rho[g_\nu]. \tag{31}$$

In Eq. (31) we have explicitly displayed the g_ν dependent. The reason for this is that we should view Eqs. (30) and (31) as a means for eliminating $\delta\mathbf{E}$ in δ^2F by an expression in terms of g_ν. We write $\delta\mathbf{E}[g_\nu]$ to indicate this.

In order to illustrate the remaining portion of our procedure we first consider the minimization of a simple algebraic example where things can be worked out explicitly. The following quadratic form will serve our purpose:

$$f(x,a)=\alpha x^2/2+\beta xa+\gamma a^2/2. \tag{32}$$

Here x is analogous to the generating function g_ν and a plays the role of $\delta\mathbf{A}$. The parameter γ in our example is assumed to be positive, while α and β can have either sign. Unlike the real problem, here it is trivial to see that f has a minimum only when

$$\alpha>\beta^2/\gamma>0. \tag{33}$$

In the case of δ^2F we have a quadratic form with both differential and integral operators and therefore it is difficult to use the straightforward approach. Instead we extremize with respect to a norm. The first step is to look at the sign of α. If it is negative a minimum does not exist and we are through. If it is positive then we do two things: first we extremize with respect to a and solve for $a(x)$. This yields

$$a(x)=-\beta x/\gamma. \tag{34}$$

Second, we insert Eq. (34) into Eq. (32) and extremize with respect to x subject to a norm. A norm is introduced for the purpose of probing the vicinity of the equilibrium point. This artifice allows us to find the extremal value of the function f at a fixed but arbitrarily small distance from the equilibrium. Only if this extremal value is positive does the equilibrium correspond to a minimum.

A convenient norm is provided by $\alpha x^2/2$, since we have already ascertained that it is positive semidefinite. We thereby obtain an eigenvalue problem upon variation of the following quantity:

$$g(\lambda,x)=-\beta^2x^2/2\gamma+\lambda(\alpha x^2/2); \tag{35}$$

in particular, variation with respect to x yields

$$(\alpha\lambda-\beta^2/\gamma)x=0, \tag{36}$$

which has a nontrivial solution if the solvability condition

$$\lambda=\beta^2/\alpha\gamma \tag{37}$$

is satisfied.

Using Eqs. (32) and (36) we evaluate the extremal value of f subject to the constraint. This yields

$$f^*=(\alpha x^2/2)(1-\lambda). \tag{38}$$

Inserting Eq. (37) yields

$$f^*=(1-\beta^2/\alpha\gamma)(\alpha x^2/2). \tag{39}$$

This quantity is positive and therefore possesses a minimum when

$$\alpha>\beta^2/\gamma>0,$$

the same condition as that obtained previously.

Now return to the real problem. The second step of our procedure is to examine the first term of Eq. (13), the one quadratic in g_ν. If there exists a g_ν that makes the quantity negative, then we have a negative-energy perturbation and our procedure ends. (We assume here that the reference frame is one of minimum energy.) If there is a g_ν that makes the first term negative, then there exists a perturbation where this term dominates the stabilizing δE^2 term. Thus picking $\delta\mathbf{A}=0$ we see that $\delta^2 F<0$. This is analogous to the case in our simple example where $\alpha<0$.

Assuming the first term of Eq. (13) is positive semidefinite, we turn to the third step of the procedure: extremization with respect to $\delta\mathbf{A}$. Recall that in Eq. (13) $\delta\mathbf{E}=-\nabla\delta\Phi$ and $\delta\mathbf{B}=\nabla\times\delta\mathbf{A}$. Thus variation with respect to $\delta\mathbf{A}$ yields

$$\nabla\times\nabla\delta\mathbf{A}=(4\pi/c)\delta\mathbf{J}[g_\nu,\delta\mathbf{A}], \qquad (40)$$

where

$$\delta\mathbf{J}=\sum_\nu\int d^3p\left[\frac{e_\nu}{m_\nu}[g_\nu,f_\nu^{(0)}]\left(\mathbf{p}-\frac{e_\nu}{c}\mathbf{A}^{(0)}\right)-\frac{e_\nu^2}{2m_\nu c}f_\nu^{(0)}\,\delta\mathbf{A}\right]. \qquad (41)$$

Observe that the extremal $\delta\mathbf{A}$ is a neighboring equilibrium state. In Eq. (40) we have explicitly displayed the g_ν and $\delta\mathbf{A}$ dependence. The reason for this is that we should view Eq. (40) as a means for eliminating $\delta\mathbf{A}$ in δ^2F by an expression in terms of g_ν. To indicate this we write $\delta\mathbf{A}[g_\nu]$ or $\delta\mathbf{B}[g_\nu]$.

The fourth and last step of our procedure is to seek a minimum of the quadratic form in g_ν that results upon inserting $\delta\mathbf{A}[g_\nu]$ and $\delta\mathbf{E}[g_\nu]$ into Eq. (13). Upon making use of Eqs. (25), (30), (31), (40), and (41), Eq. (13) becomes

$$\delta^2F'=\sum_\nu\frac{1}{2}\int d^3x\,d^3p\big([g_\nu,f_\nu^{(0)}]\;[H_\nu^{(0)},g_\nu]+\delta H_\nu[g_\nu,f_\nu^{(0)}]\big), \qquad (42)$$

where the prime denotes that we have already extremized with respect to the fields $\delta\mathbf{A}$ and $\delta\Phi$. Since evaluation of purely quadratic expressions like Eq. (42) at their external points always yields zero, we resort to the norm technique used in our simple example. Recall that the first term of Eq. (42) is at this point assumed to be positive semidefinite, otherwise our procedure would have ended at step two. Thus this quantity, reminiscent of the kinetic energy norm of the usual MHD energy principle, is a natural norm. The quantity analogous to Eq. (35) is

$$G[\lambda,g_\nu]=\lambda\sum_\nu\frac{1}{2}\int d^3x\,d^3p[g_\nu,f_\nu^{(0)}]\;[H_\nu^{(0)},g_\nu]+\sum_\nu\frac{1}{2}\int d^3x\,d^3p\,\delta H_\nu[g_\nu,f_\nu^{(0)}]. \qquad (43)$$

Recall that

$$\delta H_\nu=-\frac{e_\nu}{m_\nu c}\left(\mathbf{p}-\frac{e_\nu}{c}\mathbf{A}^{(0)}\right)\cdot\delta\mathbf{A}[g_\nu]+e_\nu\delta\Phi[g_\nu].$$

Therefore $G[\lambda,g_\nu]$ is a real bilinear functional of the g_ν's. Thus after variation of Eq. (43) with respect to $g_\mu(\mathbf{x},\mathbf{p})$, the following Hermitian eigenvalue problem is obtained:

$$\sum_\nu\frac{1}{2}\int d^3x'\,d^3p'\,\frac{\delta\delta H_\nu(\mathbf{x}',\mathbf{p}')}{\delta g_\mu(\mathbf{x},\mathbf{p})}[g_\nu(\mathbf{x}',\mathbf{p}'),f_\nu^{(0)}(\mathbf{x}',\mathbf{p}')]-\tfrac{1}{2}[\delta H_\mu(\mathbf{x},\mathbf{p}),f_\nu^{(0)}(\mathbf{x},\mathbf{p})]-\lambda\,[f_\mu^{(0)}(\mathbf{x},\mathbf{p}),[g_\mu(\mathbf{x},\mathbf{p}),H_\mu^{(0)}(\mathbf{x},\mathbf{p})]]=0. \qquad (44)$$

Multiplication of this equation by $g_\mu(\mathbf{x},\mathbf{p})$, integration over $\mathbf{x}$, $\mathbf{p}$, and summation over μ yields

$$\sum_\mu\int d^3x\,d^3p\,\delta H_\mu[g_\mu,f_\mu^{(0)}]=-\lambda\sum_\mu\int d^3x\,d^3p[g_\mu,f_\mu^{(0)}]\;[H_\mu^{(0)},g_\mu]. \qquad (45)$$

When this is inserted into Eq. (42), the following expression for the stationary values of δ^2F results:

$$\delta^2F=(1-\lambda)\sum_\nu\frac{1}{2}\int d^3x\,d^3p[g_\nu,f_\nu^{(0)}]\times[H_\nu^{(0)},g_\nu]. \qquad (46)$$

The minimum of δ^2F is therefore obtained for the largest eigenvalue $\lambda=\lambda_{\max}$ and δ^2F is negative if $\lambda_{\max}>1$.

To proceed further we can either attempt to find $\lambda_{\max}$ directly by solving the eigenvalue problem (44), or we can use trial functions in the following way. Because of Eq. (45) we have $G[\lambda,g_\nu]=0$ when λ,g_ν are solutions of the eigenvalue problem (44). Upon writing Eq. (43) as

$$G=\lambda A+B, \qquad (47)$$

the eigenvalues λ can be expressed as

$$\lambda=-B/A. \qquad (48)$$

Varying this relation around the solutions of $\delta G=0$ yields, with (48),

$$\delta\lambda=-\frac{1}{A}\left(\delta B-\frac{B}{A}\delta A\right)=-\frac{1}{A}(\delta B+\lambda\delta A). \qquad (49)$$

Thus $\delta(B/A)=0$ is equivalent to $\delta G=0$. Hence if $-B/A$ can be made larger than unity by the insertion of some trial function, then there must exist a true eigenvalue with $\lambda>1$. In this case we have negative-energy modes.

Before closing we comment on expression (42). Making use of Eqs. (16) and (24) results in

$$\delta^2F'=\frac{1}{2}\sum_\nu\int d^3x\,d^3p\,\delta f_\nu\frac{\partial g_\nu}{\partial t}=\frac{1}{2}\sum_\nu\int d^3x\,d^3p\,\delta f_\nu\Delta H_\nu, \qquad (50)$$

where ΔH_ν is the first-order difference between the exact Hamiltonian evaluated on the exact trajectories and the unperturbed Hamiltonian evaluated on the unperturbed trajectories. Here $\partial g_\nu/\partial t=\Delta H_\nu$ results because g_ν is to first order the time-dependent part of the mixed variable generating function for the canonical transformation from the perturbed to the unperturbed system. Therefore the minimum energy state depends only on the particle configuration; it is independent of the field quantities. In particular, in light of Eq. (30) there is no radiation field. This makes sense since given any configuration with a radiation field, we can obtain a lower energy state by keeping the particle configuration the same and eliminating the radiation.

V. CONCLUDING REMARKS

The generally valid expressions for the wave energy in the framework of Vlasov–Maxwell theory obtained in this and our previous paper,[13] can be used in a manner similar to the potential energy expression of MHD, i.e., δW. However, $\delta^2 F < 0$ does not immediately tell you that the system is linearly unstable, but indicates the possibility of nonlinear instability. The presented energy expressions are preferable to previous expressions not only because they are not restricted to particular types of equilibria, but, in general, they are far more practicable. We end with the remark that, since the class of equilibria admitting negative-energy waves is much larger than the class of equilibria admitting linear instability, and since there are many more negative-energy modes present in linearly unstable equilibria than linearly unstable modes, it might well be that the explanation of anomalous transport requires that the potential nonlinear instabilities associated with the negative-energy modes be taken into account.

Note added in proof: We would like to mention that Wong[17] has independently found the expression of Eq. (13), but in the context of strictly linearized dynamics.

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APPENDIX A: EQUIVALENCY OF THE TWO EXPRESSIONS FOR $\delta^2 F$

The purpose of this appendix is to fill in the steps between Eqs. (11) and (13). This is done by a sequence of integrations by parts and neglect of surface terms. It is important to remember that $\mathbf{v}$ is shorthand for $[\mathbf{p}] - (e_\nu/c)\mathbf{A}^{(0)}]/m$. Thus $\partial v_k/\partial x_j \neq 0$ and $\partial v_k/\partial p_j \neq 0$. Also, the fact that $f^{(0)}$ is an equilibrium distribution function implies $df^{(0)} = 0$. Working out the square and combining like terms yields the following for Eq. (11):

$$\begin{aligned}
\delta^2 F = & -\sum_\nu \int \frac{d^3x\, d^3p}{2m_\nu} f_\nu^{(0)} \Bigg[\left|\frac{\partial g_\nu}{\partial \mathbf{x}}\right|^2 + 2m_\nu \frac{\partial g_\nu}{\partial \mathbf{x}} \cdot d\frac{\partial g_\nu}{\partial \mathbf{p}} + e_\nu m_\nu \frac{\partial g_\nu}{\partial p_i}\frac{\partial g_\nu}{\partial p_j}\frac{\partial^2 \Phi^{(0)}}{\partial x_i\, \partial x_j} + \frac{e_\nu^2}{c^2}|\delta \mathbf{A}|^2 + \frac{2e_\nu}{c}\frac{\partial g_\nu}{\partial \mathbf{x}} \cdot \delta \mathbf{A} \\
& + \frac{2e_\nu^2}{c^2}\frac{\partial g_\nu}{\partial p_i}\frac{\partial \mathbf{A}^{(0)}}{\partial x_i} \cdot \delta \mathbf{A} + \frac{2e_\nu m_\nu}{c}\delta \mathbf{A} \cdot d\frac{\partial g_\nu}{\partial \mathbf{p}} - 2\frac{e_\nu m_\nu}{c}\frac{\partial g_\nu}{\partial p_i} v_k \left(\frac{\partial \delta A_k}{\partial x_i} - \frac{\partial \delta A_i}{\partial x_k}\right) + \frac{e_\nu^2}{c^2}\frac{\partial g_\nu}{\partial p_i}\frac{\partial \mathbf{A}_k^{(0)}}{\partial x_i}\frac{\partial g_\nu}{\partial p_j}\frac{\partial A_k^{(0)}}{\partial x_j} \\
& + 2\frac{e_\nu}{c}\frac{\partial g_\nu}{\partial p_i}\frac{\partial g_\nu}{\partial x_k}\frac{\partial A_k^{(0)}}{\partial x_i} + \frac{m_\nu e_\nu}{c}\frac{\partial g_\nu}{\partial p_i}\left(\frac{\partial A_j^{(0)}}{\partial x_i} + \frac{\partial A_i^{(0)}}{\partial x_j}\right) d\frac{\partial g_\nu}{\partial p_j} - \frac{m_\nu e_\nu}{c}\frac{\partial g_\nu}{\partial p_i}\frac{\partial g_\nu}{\partial p_j} \\
& \times v_k \frac{\partial}{\partial x_i}\left(\frac{\partial A_k^{(0)}}{\partial x_j} - \frac{\partial A_j^{(0)}}{\partial x_k}\right)\Bigg] + \frac{1}{8\pi}\int (\delta E^2 + \delta B^2)\, d^3x \equiv I_1 + I_2 + I_3. \qquad \text{(A1)}
\end{aligned}$$

In Eq. (A1) we define I_1 to be the sum of the first three terms; I_2 is the next six terms, the ones that depend on $\delta \mathbf{A}$; and I_3 is the last six terms. In the rest of this appendix we neglect the field contribution.

Consider first I_1, which can be written as follows:

$$\begin{aligned}
I_1 = \int \frac{d^3x\, d^3p}{2m} f^{(0)}(\mathbf{x}, \mathbf{p}) \Bigg(& \left|\frac{\partial g}{\partial \mathbf{x}}\right|^2 + 2m \frac{\partial g}{\partial x_k} v_i \frac{\partial^2 g}{\partial x_i\, \partial p_k} \\
& - 2me \frac{\partial g}{\partial x_k}\frac{\partial \Phi^{(0)}}{\partial x_i}\frac{\partial^2 g}{\partial p_k\, \partial p_i} \\
& + me \frac{\partial g}{\partial p_i}\frac{\partial g}{\partial p_j}\frac{\partial^2 \Phi^{(0)}}{\partial x_i\, \partial x_j} \\
& + 2\frac{me}{c}\frac{\partial g}{\partial x_k} v_j \frac{\partial A_j^{(0)}}{\partial x_i}\frac{\partial^2 g}{\partial p_i\, \partial p_k}\Bigg). \qquad \text{(A2)}
\end{aligned}$$

Here, for convenience, the species label has been dropped. Integrating half of the second term of (A2) by parts in x_i and the other half by parts in p_k and performing a cancellation and a combination yields

$$\begin{aligned}
I_1 = \int \frac{d^3x\, d^3p}{2m} \Bigg[& -mv_j \frac{\partial g}{\partial x_i}\frac{\partial g}{\partial x_k}\frac{\partial f^{(0)}}{\partial p_k} - mf^{(0)} v_i \frac{\partial}{\partial x_k}\left(\frac{\partial g}{\partial x_i}\frac{\partial g}{\partial p_k}\right) - mv_i \frac{\partial f^{(0)}}{\partial x_i}\frac{\partial g}{\partial p_k}\frac{\partial g}{\partial x_k} + \frac{e}{c} f^{(0)} \frac{\partial g}{\partial p_k}\frac{\partial g}{\partial x_k}\frac{\partial A_i^{(0)}}{\partial x_i} \\
& - 2em \frac{\partial g}{\partial x_k}\frac{\partial \Phi^{(0)}}{\partial x_i}\frac{\partial^2 g}{\partial p_i \partial p_k} f^{(0)} + me \frac{\partial g}{\partial p_i}\frac{\partial g}{\partial p_k}\frac{\partial^2 \Phi^{(0)}}{\partial x_i\, \partial x_k} f^{(0)} + 2\frac{me}{c}\frac{\partial g}{\partial x_k} v_j \frac{\partial A_j^{(0)}}{\partial x_i}\frac{\partial^2 g}{\partial p_i \partial p_k} f^{(0)}\Bigg]. \qquad \text{(A3)}
\end{aligned}$$

Now integrating the second term of (A3) by parts in x_k, and combining the terms involving $\Phi^{(0)}$, results in

$$I_1 = \int \frac{d^3x\, d^3p}{2m}\Big[-mv_i \frac{\partial g}{\partial x_i} [f^{(0)},g] + mef^{(0)} \frac{\partial g}{\partial p_k} \frac{\partial}{\partial x_k}\Big(\frac{\partial g}{\partial p_i}\frac{\partial \Phi^{(0)}}{\partial x_i}\Big)$$
$$- mef^{(0)} \frac{\partial g}{\partial x_k}\frac{\partial \Phi^{(0)}}{\partial x_i}\frac{\partial^2 g}{\partial p_i\, \partial p_k} - mef^{(0)}\frac{\partial \Phi^{(0)}}{\partial x_i}\frac{\partial}{\partial p_i}\Big(\frac{\partial g}{\partial x_k}\frac{\partial g}{\partial p_k}\Big) - mv_i \frac{\partial f^{(0)}}{\partial x_i}\frac{\partial g}{\partial p_k}\frac{\partial g}{\partial x_k}$$
$$+ \frac{e}{c} f^{(0)} \frac{\partial A_i^{(0)}}{\partial x_i}\frac{\partial g}{\partial p_k}\frac{\partial g}{\partial x_k} - \frac{e}{c} f^{(0)}\frac{\partial A_i^{(0)}}{\partial x_k}\frac{\partial g}{\partial x_i}\frac{\partial g}{\partial p_k} + 2\frac{me}{c} f^{(0)} v_j \frac{\partial A_j^{(0)}}{\partial x_i}\frac{\partial g}{\partial x_k}\frac{\partial^2 g}{\partial p_i\, \partial p_k}\Big]. \tag{A4}$$

Integrating the second term by parts in x_k and then combining this result with the third term, integrating the fourth term by parts in p_i and combining this result with the fifth term, and making use of the identity $df^{(0)} = 0$, yields

$$I_1 = \int \frac{d^3x\, d^3p}{2m}\Big[m[f^{(0)},g]\,[g,H^{(0)}] - \frac{me}{c}[f^{(0)},g]\, v_k \frac{\partial A_k^{(0)}}{\partial x_i}\frac{\partial g}{\partial p_i} - \frac{e}{c} f^{(0)} \frac{\partial A_i^{(0)}}{\partial x_k}\frac{\partial g}{\partial x_i}\frac{\partial g}{\partial p_k}$$
$$+ \frac{me}{c}\frac{\partial g}{\partial p_l}\frac{\partial g}{\partial x_l}\frac{\partial}{\partial p_i}\Big(v_k \frac{\partial A_k^{(0)}}{\partial x_i} f^{(0)}\Big) + 2\frac{me}{c} f^{(0)} v_k \frac{\partial A_k^{(0)}}{\partial x_i}\frac{\partial g}{\partial x_j}\frac{\partial^2 g}{\partial p_i\, \partial p_j}\Big]. \tag{A5}$$

Now we combine I_1 and I_3. Exploiting a couple of combinations yields

$$I_1 + I_3 = \int \frac{d^3x\, d^3p}{2}\Big[[f^{(0)},g]\,[g,H^{(0)}] - \frac{e}{c}[f^{(0)},g]\, v_k \frac{\partial A_k^{(0)}}{\partial x_i}\frac{\partial g}{\partial p_i} + \frac{e}{mc} f^{(0)} \frac{\partial g}{\partial p_k}\frac{\partial g}{\partial x_i}\frac{\partial A_i^{(0)}}{\partial x_k}$$
$$+ \frac{e}{c}\frac{\partial g}{\partial p_l}\frac{\partial g}{\partial x_l}\frac{\partial}{\partial p_i}\Big(v_k \frac{\partial A_k^{(0)}}{\partial x_i} f^{(0)}\Big) + 2\frac{e}{c} f^{(0)} v_k \frac{\partial A_k^{(0)}}{\partial x_i}\frac{\partial g}{\partial x_j}\frac{\partial^2 g}{\partial p_i\, \partial p_j} + \frac{e^2}{mc^2}\frac{\partial g}{\partial p_i}\frac{\partial g}{\partial p_j}\frac{\partial A_k^{(0)}}{\partial x_i}\frac{\partial A_k^{(0)}}{\partial x_j} f^{(0)}$$
$$+ \frac{e}{c} f^{(0)} \frac{\partial g}{\partial p_i}\frac{\partial A_k^{(0)}}{\partial x_i}\, d\frac{\partial g}{\partial p_j} + \frac{e}{c} f^{(0)}\frac{\partial g}{\partial p_i}\frac{\partial A_i^{(0)}}{\partial x_j}\, d\frac{\partial g}{\partial p_j}$$
$$- \frac{e}{c} f^{(0)} \frac{\partial g}{\partial p_i}\frac{\partial g}{\partial p_j} v_k \frac{\partial^2 A_k^{(0)}}{\partial x_i \partial x_j} + \frac{e}{c} f^{(0)}\frac{\partial g}{\partial p_i}\frac{\partial g}{\partial p_i} v_k \frac{\partial^2 A_j^{(0)}}{\partial x_i\, \partial x_k}\Big]. \tag{A6}$$

The seventh, eighth, and tenth terms cancel, the sixth and ninth combine, and we integrate the fourth by parts to obtain

$$I_1 + I_3 = \int \frac{d^3x\, d^3p}{2}\Big[[f^{(0)},g]\,[g,H^{(0)}] - \frac{e}{c}[f^{(0)},g]\, v_k \frac{\partial A_k^{(0)}}{\partial x_i}\frac{\partial g}{\partial p_i} - \frac{e}{mc} f^{(0)}\frac{\partial g}{\partial p_k}\frac{\partial g}{\partial x_l}\frac{\partial A_l^{(0)}}{\partial x_k}$$
$$+ \frac{e}{c} v_k \frac{\partial A_k^{(0)}}{\partial x_i} f^{(0)} \frac{\partial^2 g}{\partial p_i\, \partial p_l}\frac{\partial g}{\partial x_l} - \frac{e}{c} v_k \frac{\partial A_k^{(0)}}{\partial x_i} f^{(0)}\frac{\partial^2 g}{\partial x_l\, \partial p_i}\frac{\partial g}{\partial p_l} - \frac{e}{c} f^{(0)}\frac{\partial g}{\partial p_i}\frac{\partial g}{\partial p_l}\frac{\partial}{\partial x_l}\Big(v_k \frac{\partial A_k^{(0)}}{\partial x_i}\Big)\Big]. \tag{A7}$$

The last four terms of (A8) combine as follows:

$$I_1 + I_3 = \int \frac{d^3x\, d^3p}{2}\Big([f^{(0)},g]\,[g,H^{(0)}]$$
$$- \frac{e}{c}[f^{(0)},g]\, v_k \frac{\partial A_k^{(0)}}{\partial x_i}\frac{\partial g}{\partial p_i}$$
$$+ \frac{e}{c} f^{(0)}\Big[g, v_k \frac{\partial A_k^{(0)}}{\partial x_i}\frac{\partial g}{\partial p_i}\Big]\Big). \tag{A8}$$

After canceling the second and third terms, Eq. (A8) reduces to

$$I_1 + I_3 = \int d^3x\, d^3p\Big(\frac{1}{2}[f^{(0)},g]\,[g,H^{(0)}]\Big). \tag{A9}$$

Now consider the integral I_2, after canceling its fourth and sixth terms,

$$I_3 = \int d^3x\, d^3p\, f^{(0)}\Big(\frac{e^2}{2mc^2}|\delta\mathbf{A}|^2 + \frac{e}{mc}\frac{\partial g}{\partial x_i}\delta A_i$$
$$+ \frac{e^2}{mc^2}\frac{\partial g}{\partial p_i}\frac{\partial A_j^{(0)}}{\partial x_i}\delta A_j - \frac{e}{c}\frac{\partial g}{\partial p_i} v_k \frac{\partial \delta A_k}{\partial x_i}\Big). \tag{A10}$$

Equation (A10) is evidently equivalent to

$$I_3 = \int d^3x\, d^3p\Big(\frac{e^2}{2mc^2} f^{(0)}|\delta A|^2 - \frac{e}{c} f^{(0)}[\mathbf{v}\cdot\delta\mathbf{A}, g]\Big)$$
$$= \int d^3x\, d^3p\Big(\frac{e^2}{2mc^2} f^{(0)}|\delta A|^2 - \frac{e}{c}\mathbf{v}\cdot\delta\mathbf{A}[g, f^{(0)}]\Big). \tag{A11}$$

Combining (A9) and (A11) we obtain Eq. (13), our desired result.

APPENDIX B: LIE GROUP EXPRESSIONS

In this appendix we derive the expressions of Eqs. (16) and (17) for the first- and second-order distribution functions. The derivation uses some basic elements of Lie group theory. We start with the observation that there always exists a canonical transformation from the canonical variables **x**, **p**, which obey the equations of motion generated by the perturbed Hamiltonian $H(\mathbf{x}, \mathbf{p}, t)$, to the variables $\mathbf{x}^{(0)}$, $\mathbf{p}^{(0)}$, which obey the equations generated by the unperturbed Hamiltonian $H^{(0)}(\mathbf{x}^{(0)}, \mathbf{p}^{(0)})$. This follows from the group property of canonical transformation and from the fact that the equations of motion for any Hamiltonian generate canonical transformations. Thus going backward for a time t with the perturbed Hamiltonian H and forward again by the same amount of time t with the unperturbed Hamiltonian $H^{(0)}(\mathbf{x}^{(0)}, \mathbf{p}^{(0)})$ results in a canonical transformation leading from $\mathbf{x}, \mathbf{p}$ to $\mathbf{x}^{(0)}, \mathbf{p}^{(0)}$. Since these transformations are elements of a group it is evident that a single time-dependent transformation relates $\mathbf{x}, \mathbf{p}$ directly to $\mathbf{x}^{(0)}, \mathbf{p}^{(0)}$.

Instead of constructing such a transformation by a mixed variable generating function we use a basic result of Lie group theory (see, for example, Refs. 14–16); i.e., that an element of a group can be represented by exponentiating an operator:

$$\mathbf{x}^{(0)} = e^{[K(\mathbf{x},\mathbf{p},t;\epsilon),\ ]}\mathbf{x}, \quad \mathbf{p}^{(0)} = e^{[K(\mathbf{x},\mathbf{p},t;\epsilon),\ ]}\mathbf{p}. \tag{B1}$$

Here $K(\mathbf{x}, \mathbf{p}, t; \epsilon)$ is the generating function for the transformation, ϵ is a small parameter, and $[K,\ ]$ is a differential operator, which is defined by

$$[K(\mathbf{x}, \mathbf{p}, t; \epsilon),\ ] = \frac{\partial K}{\partial \mathbf{x}} \cdot \frac{\partial}{\partial \mathbf{p}} - \frac{\partial K}{\partial \mathbf{p}} \cdot \frac{\partial}{\partial \mathbf{x}}. \tag{B2}$$

This operator has the following property for any function $F(\mathbf{x}, \mathbf{p})$:

$$F(e^{[K,\ ]}\mathbf{x}, e^{[K,\ ]}\mathbf{p}) = e^{[K,\ ]}F(\mathbf{x}, \mathbf{p}). \tag{B3}$$

The unperturbed distribution function, $f^{(0)}(\mathbf{x}^{(0)}, \mathbf{p}^{(0)})$, is constant along the unperturbed orbits, which is equivalent to being a solution of the unperturbed Vlasov equation. Upon inserting into this function $\mathbf{x}^{(0)}, \mathbf{p}^{(0)}$ from (B1) we obtain, with the property (B3),

$$f^{(0)}(e^{[K,\ ]}\mathbf{x}, e^{[K,\ ]}\mathbf{p}) = e^{[K,\ ]}f^{(0)}(\mathbf{x}, \mathbf{p}) \equiv f(\mathbf{x}, \mathbf{p}, t), \tag{B4}$$

where correspondingly now $f(\mathbf{x}, \mathbf{p}, t)$ is constant along the perturbed orbits. It is thus an exact solution to the perturbed Vlasov equation.

We are interested in small perturbations characterized by the small parameter ϵ. We begin by expanding K as follows:

$$K(\mathbf{x}, \mathbf{p}, t;\epsilon) = \epsilon K^{(1)}(\mathbf{x}, \mathbf{p}, t) + \epsilon^2 K^{(2)}(\mathbf{x}, \mathbf{p}, t) + \cdots. \tag{B5}$$

This allows us to expand Eq. (B4) and obtain

$$\begin{aligned} f(\mathbf{x}, \mathbf{p}, t) &= f^{(0)}(\mathbf{x}, \mathbf{p}) + [K, f^{(0)}] + \tfrac{1}{2}[K, [K, f^{(0)}]] + \cdots \\ &= f^{(0)}(\mathbf{x}, \mathbf{p}) + \epsilon[K^{(1)}, f^{(0)}] + \epsilon^2([K^{(2)}, f^{(0)}] \\ &\quad + \tfrac{1}{2}[K^{(1)}, [K^{(1)}, f^{(0)}]]) + \cdots. \end{aligned} \tag{B6}$$

Equations (16) and (17) follow upon defining

$$g^{(1)} = \epsilon K^{(1)}, \quad g^{(2)} = \epsilon^2 K^{(2)}. \tag{B7}$$

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