The effect of viscosity on the resistive tearing mode with the presence of shear flow

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The effect of small viscosity on the "constant ψ" tearing mode in the presence of shear flow is analyzed using the boundary-layer approach. It is found that the influence of viscosity depends upon the parameter $G'(0)/F'(0)$, where $G'(0)$ and $F'(0)$ denote the flow shear and magnetic field shear at the magnetic null plane, respectively. When $|G'(0)/F'(0)| \ll 1$, the tearing mode growth rate is suppressed by the viscosity, but not completely stabilized. When $|G'(0)/F'(0)| \sim \mathscr{O}(1)$ and the viscosity is comparable to the resistivity, the growth rate vanishes as $[1 - G'(0)^2/F'(0)^2]^{1/3}$, when $G'(0)^2 \to F'(0)^2$ from below. In the case where $1 - [G'(0)^2/F'(0)^2] < 0$ matching cannot be achieved and the tearing mode vanishes.

I. INTRODUCTION

Resistive tearing instability is important in many contexts, ranging from transferring magnetic energy to kinetic energy in solar physics to sawtooth phenomena in tokamaks. The presence of shear flow is of interest in rotating plasmas and astrophysics, for instance, coronal loops, the magnetopause boundary, plasma streams in the solar wind, and extragalactic jets. In this paper we generalize previous work[1] by including, in addition to equilibrium shear flow, viscosity. Generally, the viscosity is described by a complicated tensor. However, since plasma motion tends to exhibit transverse gradients near the magnetic null plane, the appropriate viscosity is the transverse component.[2-5] Since this viscosity is often comparable to the resistivity in laboratory plasmas[2] and much larger than the resistivity in astrophysical plasmas, such as those that occur in the solar wind and active coronal regions,[3] one expects this to be an important effect. Moreover, since the tearing instability produces vorticity, and equilibrium shear flow can enhance this production, the diffusive nature of viscosity should have a significant influence, one that depends upon the equilibrium shear flow.

Previously, the effect of viscosity on the resistive tearing mode without flow was treated.[2,4] These authors found that the growth is suppressed while the width of the singular layer is increased. Also, Dobrowolny *et al.*[3] have given scalings in the case where it is assumed that the viscosity is comparable to the resistivity. Assuming that the viscous term dominates the inertial term in the singular layer, and that the shear flow is small, Bondeson and Persson[6] solved the "constant ψ" tearing mode problem by making use of Fourier transforms. Recently, Einaudi and Rubini[5] have investigated this problem numerically.

Here we generalize the work of Ref. 6 by allowing the equilibrium shear flow to be large. As in this reference, we Fourier transform the internal singular layer equations in order to derive growth rate expressions. We find that for small viscosity there is a general tendency to diminish the growth rate. Also, when the viscosity becomes comparable to the resistivity and the flow shear is larger than the magnetic field shear at the magnetic null plane, there is no "constant ψ" tearing mode. Matching in this case cannot be achieved. Another result of this paper is to justify the constant ψ approximation.

In Sec. II the basic equations are set up and we briefly discuss the role of viscosity in both the external region and the internal singular layer. In Sec. III the internal singular layer equations are discussed in two shear flow limits. Finally, in Sec. IV we summarize.

II. BASIC EQUATIONS

We begin with the following incompressible dissipative magnetohydrodynamic (MHD) equations:

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) = -\nabla p + \mathbf{j}\times\mathbf{B} + \rho\nu\nabla_\perp^2\mathbf{v} + \mathbf{S}_p,$$

$$\nabla\times\mathbf{B} = 4\pi\mathbf{J},$$

$$\mathbf{E} + \mathbf{v}\times\mathbf{B} = \eta\mathbf{J},$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla\times\mathbf{E},$$

$$\nabla\cdot\mathbf{B} = 0,$$

$$\nabla\cdot\mathbf{v} = 0,$$

where $\mathbf{S}_p$ is a momentum source. Another implicit source term is the external electrical field, which is the inductive part of $\mathbf{E}$. The source terms are introduced to compensate for the dissipation in the equilibrium state; they do not appear in the linearized equations. We assume that the equilibrium magnetic and velocity fields have the form

$$\mathbf{B}_0 = B_{0x}(y)\hat{x} + B_{0z}(y)\hat{z},$$

$$\mathbf{v}_0 = v_{0x}(y)\hat{x} + v_{0z}(y)\hat{z},$$

and we denote the perturbed quantities by subscript 1 and write them as follows:

$$f_1 = f_1(y)e^{i(k_x x + \kappa_z z) + i\omega t}.$$

Introducing the scalings

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$$\mu = y/a, \quad \alpha = ka, \quad k = \sqrt{k_x^2 + k_z^2},$$
$$\tau_R = \frac{4\pi a^2}{\eta}, \quad \tau_v = \frac{a^2}{\nu}, \quad \tau_H = \frac{\sqrt{4\pi\rho}a}{B},$$
$$F = \frac{\mathbf{k}\cdot\mathbf{B}_0}{kB}, \quad G = \frac{\mathbf{k}\cdot\mathbf{v}_0}{kv_A}, \quad v_A = \frac{a}{\tau_H},$$
$$W = \frac{v_{1y}}{v_A}, \quad \psi = \frac{B_{1y}}{B}, \quad S_R = \frac{\tau_R}{\tau_H},$$
$$S_V = \tau_v/\tau_H, \quad \gamma = i\omega\tau_H,$$

where the prime denotes the differentiation with respect to μ, B is the measure of the magnetic field, and a is the magnetic shear length, results in the following linearized perturbation equations:

$$(\gamma + i\alpha G)(W'' - \alpha^2 W) - i\alpha G'' W = i\alpha F(\psi'' - \alpha^2\psi) - i\alpha F''\psi + \frac{1}{S_V}\frac{\partial^4 W}{\partial\mu^4}, \tag{1}$$

$$(\gamma + i\alpha G)\psi - i\alpha F W = (1/S_R)(\psi'' - \alpha^2\psi). \tag{2}$$

We assume that both resistivity and viscosity are very small, i.e., $S_V \gg 1$, $S_R \gg 1$. For convenience, we choose a reference frame such that $G(0) = 0$, where $\mu = 0$ is the location of the magnetic null plane. Only the tearing mode is considered. Unlike the case without viscosity, the small resistivity and viscosity are not only important in the internal singular layer, but also important in a thin layer at the external physical boundary. This boundary layer will affect the matching quantity Δ', which is defined by $\Delta' \equiv (1/\psi)(d\psi/d\mu)|_{0^-}^{0^+}$. However, if the boundary is far away from the singular layer, this effect is negligible. In the internal singular layer, the viscosity influences magnetic diffusion by diffusing the vorticity produced in the tearing instability. We discuss this problem in the following section. Without loss of generality, we assume $F'(0) > 0$. Also, we assume $|G''(0)/F'(0)| \lesssim \mathscr{O}(1)$, $|F''(0)/F'(0)| \lesssim \mathscr{O}(1)$, and $\alpha \lesssim \mathscr{O}(1)$.

III. INTERNAL SINGULAR LAYER

Assuming that the scale length of the internal singular layer is ϵ, we consider Eqs. (1) and (2) near $\mu = 0$. Using the stretched variable $\zeta = \mu/\epsilon$, Eqs. (1) and (2) become

$$\left(\frac{\gamma}{\alpha F'(0)\epsilon} + i\frac{G'(0)}{F'(0)}\zeta + \frac{1}{2}i\frac{G''(0)}{F'(0)}\epsilon\zeta^2\right)\frac{\partial^2 W}{\partial\zeta^2} - i\epsilon\frac{G''(0)}{F'(0)}W = \left(i\zeta + \frac{1}{2}i\frac{F''(0)}{F'(0)}\epsilon\zeta^2\right)\frac{\partial^2\psi}{\partial\zeta^2} - i\epsilon\frac{F''(0)}{F'(0)}\psi + B\frac{\partial^4 W}{\partial\zeta^4} + \mathscr{O}(\epsilon^2), \tag{3}$$

$$\left(\frac{\gamma}{\alpha F'(0)\epsilon} + i\frac{G'(0)}{F'(0)}\zeta + \frac{1}{2}i\frac{G''(0)}{F'(0)}\epsilon\zeta^2\right)\psi - \left(i\zeta + \frac{1}{2}i\frac{F''(0)}{F'(0)}\epsilon\zeta^2\right)W = C\frac{\partial^2\psi}{\partial\zeta^2} + \mathscr{O}(\epsilon^2), \tag{4}$$

where B and C are defined as

$$B \equiv 1/\alpha F'(0)\epsilon^3 S_V, \qquad C \equiv 1/\alpha F'(0)\epsilon^3 S_R. \tag{5}$$

The quantities B and C measure, respectively, the diffusion of the vorticity and magnetic fields in the singular layer. When the viscosity is very small compared with resistivity, i.e., $B/C \ll 1$, magnetic diffusion dominates vorticity diffusion. In this case the viscosity only alters numerical coefficients of the tearing mode growth rate; the scaling is unchanged. Here we omit this case, but consider the more interesting case where viscosity is comparable to or larger than the resistivity and the "constant ψ" approximation is assumed. As discussed in Ref. 1, the "constant ψ" approximation requires that $|\gamma/\alpha F'(0)\epsilon| \ll 1$. This implies that the anticipated growth time is much longer than the local Alfvén time at the magnetic null plane. Also we consider the problem in two shear flow limits. One is the very small shear flow case, where flow shear is very small compared to the magnetic field shear at the null plane. In the other limit the flow shear is comparable to the magnetic field shear.

A. Very small shear flow

In this limit we assume $|G'(0)/F'(0)| \lesssim |\gamma/\alpha F'(0)\epsilon|$, which implies the flow shear is so small that the convection terms are at most comparable to the inertia terms. From the constraint imposed by the external solution, we have the ordering $W \sim [\gamma/\alpha F'(0)\epsilon]\psi$ in the internal singular layer.[1] Thus even though the viscosity is much larger than the resistivity, the magnetic diffusion still dominates the vorticity diffusion. In order to facilitate comparison of the orders of various terms, we replace W by a new variable φ defined by

$$\varphi = \frac{W}{-i[\gamma/\alpha F'(0)\epsilon]}, \tag{6}$$

and rewrite Eqs. (3) and (4) as

$$-\left(\frac{\gamma}{\alpha F'(0)\epsilon}\right)^2\left(1 + i\frac{\alpha G'(0)\epsilon}{\gamma}\right)\frac{\partial^2\varphi}{\partial\zeta^2} + B\left(\frac{\gamma}{\alpha F'(0)\epsilon}\right)\frac{\partial^4\varphi}{\partial\zeta^4} = \left(\zeta + \frac{1}{2}\frac{F''(0)}{F'(0)}\epsilon\zeta^2\right)\frac{\partial^2\psi}{\partial\zeta^2} - \epsilon\frac{F''(0)}{F'(0)}\psi + \mathscr{O}\left(\epsilon\frac{\gamma}{\alpha F'(0)\epsilon}\right), \tag{7}$$

$$\frac{\gamma}{\alpha F'(0)\epsilon}\left[\left(1 + i\frac{\alpha G'(0)\epsilon}{\gamma}\zeta\right)\psi - \zeta\varphi\right] + \frac{1}{2}i\frac{G''(0)}{F'(0)}\epsilon\zeta^2\psi = C\frac{\partial^2\psi}{\partial\zeta^2} + \mathscr{O}\left(\epsilon\frac{\gamma}{\alpha F'(0)\epsilon}\right). \tag{8}$$

In order to find the solutions matching the external solutions, we must have the scaling

$$B\sqrt{S_V/S_R} \sim C\sqrt{S_R/S_V} \sim 1,$$

instead of $[|\gamma|/\alpha F'(0)\epsilon]C \sim 1$ as required in the case of zero viscosity.[1,2] This implies that the viscous term dominates the inertial term in Eq. (7), and the width of the internal singular layer scales as

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$$\epsilon \sim \left[\alpha F'(0)\sqrt{S_R S_V}\right]^{-1/3}. \tag{9}$$

The natural expansion parameter in Eqs. (7) and (8) is $\{[\gamma/\alpha F'(0)\epsilon]\sqrt{S_R/S_V}\}$, instead of $[\gamma/\alpha F'(0)\epsilon]^2$ in the case of no viscosity.[1] This implies the following expansions:

$$\psi = \sum_n \left(\frac{\gamma}{\alpha F'(0)\epsilon}\sqrt{\frac{S_R}{S_V}}\right)^n \psi_n,$$
$$\varphi = \sum_n \left(\frac{\gamma}{\alpha F'(0)\epsilon}\sqrt{\frac{S_R}{S_V}}\right)^n \varphi_n. \tag{10}$$

Inserting Eqs. (10) into Eqs. (7) and (8), the leading order of Eqs. (7) and (8) yields $\psi_0 = \text{const}$, as expected. To the first order, Eqs. (7) and (8) yield

$$B\sqrt{\frac{S_V}{S_R}}\frac{\partial^4 \overline{\varphi}_0}{\partial \zeta^4} = \zeta\frac{\partial^2\psi_1}{\partial\zeta^2} - \frac{\epsilon}{\{[\gamma/\alpha F'(0)\epsilon]\sqrt{S_R/S_V}\}}\frac{F''(0)}{F'(0)}\psi_0, \tag{11}$$

$$\psi_0 - \zeta\overline{\varphi}_0 = C\sqrt{\frac{S_R}{S_V}}\frac{\partial^2\psi_1}{\partial\zeta^2}, \tag{12}$$

where

$$\overline{\varphi}_0 = \varphi_0 - i\frac{\alpha G'(0)\epsilon}{\gamma}\psi_0 - \frac{1}{2}i\frac{\epsilon}{\{[\gamma/\alpha F'(0)\epsilon]\sqrt{S_R/S_V}\}}\zeta\psi_0.$$

We reduce the order of Eqs. (11) and (12) by Fourier transformation,

$$\psi_1 = \int_{-\infty}^{\infty} e^{ik\zeta}\psi_1(k)dk,$$
$$\overline{\varphi}_0 = \int_{-\infty}^{\infty} e^{ik\zeta}\overline{\varphi}_0(k)dk.$$

The transformed equations are

$$B\sqrt{\frac{S_V}{S_R}}k^4\overline{\varphi}_0(k) = -i\frac{d}{dk}[k^2\psi_1(k)] - \frac{\epsilon}{\{[\gamma/\alpha F'(0)\epsilon]\sqrt{S_R/S_V}\}}\frac{F''(0)}{F'(0)}\psi_0\delta(k), \tag{13}$$

$$\psi_0\delta(k) - i\frac{d}{dk}\overline{\varphi}_0(k) = -C\sqrt{\frac{S_R}{S_V}}k^2\psi_1(k). \tag{14}$$

Integration of Eqs. (13) and (14) gives

$$\overline{\varphi}_0(0^+) - \overline{\varphi}_0(0^-) = -i\psi_0, \tag{15}$$

$$k^2\psi_1(k)|_{0^-}^{0^+} = i\frac{\epsilon}{\{[\gamma/\alpha F'(0)\epsilon]\sqrt{S_R/S_V}\}}\frac{F''(0)}{F'(0)}\psi_0. \tag{16}$$

Using Eq. (14), Eq. (16) yields

$$\frac{d}{dk}\overline{\varphi}_0(0^+) - \frac{d}{dk}\overline{\varphi}(0^-) = \frac{1}{\gamma S_R\epsilon}\frac{F''(0)}{F'(0)}\psi_0. \tag{17}$$

After Fourier transformation, the matching quantity Δ' becomes

$$\Delta' = \frac{1}{\epsilon\psi}\frac{d\psi}{d\zeta}\bigg|_{-\infty}^{\infty} = \frac{\gamma\sqrt{S_R/S_V}}{\alpha F'(0)\epsilon^2\psi_0}\int_{-\infty}^{\infty}\frac{d^2\psi_1(\zeta)}{d\zeta^2}d\zeta$$
$$= -i\pi\frac{\gamma\epsilon S_R}{\psi_0}\left(\frac{d\overline{\varphi}_0(0^+)}{dk} + \frac{d\overline{\varphi}_0(0^-)}{dk}\right). \tag{18}$$

Combining Eqs. (15), (17), and (18), we have

$$\left(\frac{\Delta'}{\pi} - i\frac{F''(0)}{F'(0)}\right)\frac{\varphi_0(0^+)}{d\varphi_0(0^+)/dk} - \left(\frac{\Delta'}{\pi} + i\frac{F''(0)}{F'(0)}\right) \times\frac{\varphi_0(0^-)}{d\varphi_0(0^-)/dk} = -2\gamma S_R\epsilon. \tag{19}$$

When $k \neq 0$, we obtain from Eqs. (13) and (14)

$$\frac{d^2\overline{\varphi}_0(k)}{dk^2} - BCk^4\overline{\varphi}_0(k) = 0, \tag{20}$$

with the boundary condition that $\overline{\varphi}_0(k)$ vanishes at infinity. Equation (20) is a special case of Eq. (A1), which is solved in the Appendix in terms of Kummer functions. Applying Eq. (A6) yields the quantities $\varphi_0(0^\pm)/[d\varphi_0(0^\pm)/dk]$, which are then substituted into Eq. (19), yielding

$$\gamma = \frac{1}{\pi}\left(\frac{1}{18}\alpha F'(0)\right)^{1/3}\frac{\Gamma(\frac{1}{3})^2}{\Gamma(\frac{2}{3})^2}S_R^{-2/3}\left(\frac{S_V}{S_R}\right)^{1/6}\Delta'. \tag{21}$$

Neglecting the small shear flow in the corresponding result of Ref. 6 produces a growth rate equal to that given in Eq. (21).

Now we check the validity of our assumption $|[\gamma/\alpha F'(0)\epsilon]\sqrt{S_R/S_V}| \ll 1$, the assumption that implies the "constant ψ" approximation. Equations (9) and (21) give

$$|[\gamma/\alpha F'(0)\epsilon]\sqrt{S_R/S_V}| \sim |\epsilon\Delta'| \ll 1,$$

verifying the assumption. The form of the above is the same as the case without viscosity; however, viscosity increases the scale length of the internal singular layer. Obviously, our approximation is not valid when $\Delta' \to \infty$. This case is the regime with nonconstant ψ tearing mode.[1] We do not consider this case here.

In the case without viscosity, the small flow shear $G'(0)$ contributes a destabilizing correction to the growth rate of

$$\mathscr{O}\left[\left(\frac{G'(0)/F'(0)}{\gamma/\alpha F'(0)\epsilon}\right)^2\right]$$

(see Refs. 1 and 7). When viscosity is included and it is assumed that the viscous term dominates the inertial and convection terms, not only are the scalings changed, but also the correction to the growth rate due to $G'(0)$ is changed to $\mathscr{O}[G'(0)/F'(0)]$, and thus neglected. In the numerical work of Ref. 5, the scaling of Eq. (21) was obtained in the limit $G'(0) = 0$. We also want to emphasize that even though the flow shear at the magnetic null plane is small, the flow in the external ideal region could be large, which can significantly change the matching quantity Δ'.[1] For the pro-

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file (5b) of Ref. 5 where $G'(0)=0$, when the flow scale length is small the $|\Delta'|$ value is too large for the validity of our "constant ψ" assumption.[1] In this case a mixture of tearing and Kelvin–Helmholtz instabilities occurs and it turns out[5] that small viscosity has no significant influence. However, a very large viscosity will stabilize the instability.

B. Comparable shear flow

In this limit, we have the ordering $G'(0)/F'(0)\sim\mathscr{O}(1)$, and $\psi\sim W$ in the internal singular layer.[1] Thus the convection term now dominates the inertial term, and the vorticity diffusion is greatly enhanced. The relative magnitude of resistivity and viscosity becomes important and decides which diffusive effect dominates in the singular layer. We consider the case where viscosity is comparable to the resistivity. Thus in Eqs. (3) and (4), assume $B\sim C\sim 1$. This gives the scale length of the singular layer

$$\epsilon\sim[\alpha F'(0)S_R]^{-1/3}\sim[\alpha F'(0)S_V]^{-1/3}, \tag{22}$$

and the appropriate expansions become

$$\psi=\sum_n\left(\frac{\gamma}{\alpha F'(0)\epsilon}\right)^n\psi_n,$$

$$W=\sum_n\left(\frac{\gamma}{\alpha F'(0)\epsilon}\right)^n W_n.$$

Inserting the above into Eqs. (3) and (4), the leading order leads to

$$[G'(0)/F'(0)]\psi_0=W_0=\text{const},$$

which means in addition to the "constant ψ" approximation, we have a "constant W" approximation in this limit.

To first order, Eqs. (3) and (4) yield

$$\begin{aligned}iB\frac{\partial^4\overline{W}_1}{\partial\zeta^4}+\frac{G'(0)}{F'(0)}\zeta\frac{\partial^2\overline{W}_1}{\partial\zeta^2}-\frac{\epsilon}{\gamma/\alpha F'(0)\epsilon}\frac{G''(0)}{F'(0)}W_0\\=\left(1-\frac{G'(0)^2}{F'(0)^2}\right)\zeta\frac{\partial^2\psi_1}{\partial\zeta^2}-iB\frac{G'(0)}{F'(0)}\frac{\partial^4\psi_1}{\partial\zeta^4}\\-\frac{\epsilon}{\gamma/\alpha F'(0)\epsilon}\frac{F''(0)}{F'(0)}\psi_0,\end{aligned} \tag{23}$$

$$\psi_0-i\zeta\overline{W}_1=C\frac{\partial^2\psi_1}{\partial\zeta^2}, \tag{24}$$

where

$$\begin{aligned}\overline{W}_1=W_1-\frac{G'(0)}{F'(0)}\psi_1+\frac{1}{2}\frac{\epsilon}{\gamma/\alpha F'(0)\epsilon}\\\times\zeta\frac{G'(0)F''(0)-F'(0)G''(0)}{F'(0)^2}\psi_0.\end{aligned}$$

Equations (23) and (24) are similar to Eqs. (11) and (12). Again, Fourier transforming and following the same procedure as in the previous limit yields equations analogous to Eqs. (19) and (20),

$$\begin{aligned}&\left(\frac{\Delta'}{\pi}+i\frac{F'(0)F''(0)-G'(0)G''(0)}{G'(0)^2-F'(0)^2}\right)\frac{\overline{W}(0^+)}{d\overline{W}(0^+)/dk}\\&-\left(\frac{\Delta'}{\pi}-i\frac{F'(0)F''(0)-G'(0)G''(0)}{G'(0)^2-F'(0)^2}\right)\\&\times\frac{\overline{W}_1(0^-)}{d\overline{W}_1(0^-)/dk}=-2\gamma\epsilon S_R;\end{aligned} \tag{25}$$

when $k\neq 0$,

$$\begin{aligned}\frac{[G'(0)^2/F'(0)^2]-1}{C}\frac{d^2\overline{W}_1(k)}{dk^2}-\frac{G'(0)}{F'(0)}\frac{B+C}{C}\\\times k^2\frac{d\overline{W}_1(k)}{dk}+\left(Bk^4-2k\frac{G'(0)}{F'(0)}\right)\overline{W}_1(k)=0,\end{aligned} \tag{26}$$

with the boundary condition that $\overline{W}_1(k)$ vanishes at infinity.

Equation (26) is exactly the same as Eq. (A1) in the Appendix with $A=G'(0)/F'(0)$. When $G'(0)^2/F'(0)^2>1$, there are no appropriate solutions to Eq. (26) that satisfy the boundary condition. Thus no "constant ψ" tearing mode exists. When $G'(0)^2/F'(0)^2<1$, applying Eq. (A6) to Eq. (25) yields

$$\begin{aligned}\gamma=&\frac{3^{-2/3}}{2\pi}\frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{2}{3})}\left[\alpha F'(0)\left(1-\frac{G'(0)^2}{F'(0)^2}\right)\right]^{1/3}\left[4\frac{S_R}{S_V}+\frac{G'(0)^2}{F'(0)^2}\left(1-\frac{S_R}{S_V}\right)^2\right]^{-1/6}S_R^{-2/3}\left[\Delta'\left(\frac{\Gamma(\frac{1}{3}-\frac{1}{3}\delta)}{\Gamma(\frac{2}{3}-\frac{1}{3}\delta)}+\frac{\Gamma(\frac{1}{3}+\frac{1}{3}\delta)}{\Gamma(\frac{2}{3}+\frac{1}{3}\delta)}\right)\right.\\&\left.+i\pi\frac{F'(0)F''(0)-G'(0)G''(0)}{G'(0)^2-F'(0)^2}\left(\frac{\Gamma(\frac{1}{3}-\frac{1}{3}\delta)}{\Gamma(\frac{2}{3}-\frac{1}{3}\delta)}-\frac{\Gamma(\frac{1}{3}+\frac{1}{3}\delta)}{\Gamma(\frac{2}{3}+\frac{1}{3}\delta)}\right)\right],\end{aligned} \tag{27}$$

where

$$\delta=\frac{G'(0)/F'(0)(S_R-S_V)}{\sqrt{G'(0)^2/F'(0)^2(S_R-S_V)^2+4S_RS_V}},$$

$$|\delta|<1.$$

The above result is very different from the case without viscosity,[1] but if $G'(0)/F'(0)=0$ the growth rate expression for small shear flow, Eq. (21), is obtained. The assumption $|\gamma/\alpha F'(0)\epsilon|\ll 1$ requires $|\epsilon\Delta'|\ll 1$ as before. However, the special case $1-[G'(0)^2/F'(0)^2]\to 0$ needs to be discussed.

If $F'(0)F''(0)-G'(0)G''(0)\neq 0$ and $S_R\neq S_V$, there is a singularity at $1-[G'(0)^2/F'(0)^2]=0$ for the growth rate, i.e., as $1-[G'(0)^2/F'(0)^2]\to 0$ (from above), the imaginary part diverges, while the real part approaches zero as $[1-G'(0)^2/F'(0)^2]^{1/3}$. The "constant ψ" approximation is not valid for any values of Δ', due to the large imaginary part of the growth rate.

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If $F'(0)F''(0) - G'(0)G''(0) = 0$, or $S_R = S_V$, there is no imaginary part of the growth rate expression, and the "constant ψ" approximation is valid for all values of Δ' when $1 - [G'(0)^2/F'(0)^2] \to 0$.

Einaudi and Rubini[5] have not explored this limit in detail. Qualitatively, their results agree with ours in that the growth rate is suppressed when viscosity is comparable to resistivity and $G'(0)/F'(0) \sim \mathscr{O}(1)$.

When the viscosity is much larger than the resistivity, vorticity diffusion dominates magnetic field diffusion. In this case there is streamline as well as magnetic field reconnection, and the viscosity enhances the growth rate. This has been shown in the numerical work of Ref. 5.

IV. SUMMARY AND DISCUSSION

We have investigated the effect of viscosity on the "constant ψ" tearing mode, with the presence of equilibrium shear flow. This problem has been treated in two shear flow limits. When the flow shear is much smaller than the magnetic field shear at the magnetic null plane magnetic diffusion dominates vorticity diffusion and the scale length of the internal singular layer is changed from $S_R^{-2/5}$ to $(S_R S_V)^{-1/6}$, while the scaling of the growth rate is changed from $S_R^{-3/5}$ to $S_R^{-2/3}(S_V/S_R)^{1/6}$. The influence of $G'(0)$ is negligible, however, the flow in the external ideal region can be large and significantly change the matching quantity Δ'. When the flow shear is comparable to the magnetic shear, and the viscosity is comparable to the resistivity, vorticity diffusion is as important as magnetic diffusion in the singular layer. The scaling of the growth rate is changed from $S_R^{-1/2}$ to $S_R^{-2/3}$ and the scaling of the singular layer remains as $S_R^{-1/3}$. Moreover, the $\Delta' > 0$ instability criterion, which is removed in the case of no viscosity and $G'(0)G''(0) - F'(0)F''(0) \neq 0$, is restored. When $G'(0)^2/F'(0)^2 > 1$, there is no "constant ψ" tearing mode.

We have also discussed the special case where $1 - [G'(0)^2/F'(0)^2] \to 0$. In this case, if $G'(0) \times G''(0) - F'(0)F''(0) = 0$, or $S_V = S_R$, the "constant ψ" approximation is valid for all values of Δ', and the growth rate goes to zero with a factor $[1 - G'(0)^2/F'(0)^2]^{1/3}$, while if $G'(0)G''(0) - F'(0)F''(0) \neq 0$ and $S_V \neq S_R$, there is a singularity at $1 - [G'(0)^2/F'(0)^2] = 0$ for the imaginary part of the growth rate, and the "constant ψ" approximation is not valid for any value of Δ'. Thus our calculation is not valid in this case. However, we can still conclude that the tearing instability is totally suppressed when $1 - [G'(0)^2/F'(0)^2] \to 0$, since there is no tearing mode when $G'(0)^2/F'(0)^2 > 1$, and the growth rate has a factor of $[1 - G'(0)^2/F'(0)^2]^{1/3}$ when $G'(0)^2/F'(0)^2 < 1$.

Finally, in the case $G'(0)/F'(0) \sim \mathscr{O}(1)$ with viscosity much larger than resistivity, vorticity diffusion dominates magnetic field diffusion and viscosity enhances the growth rate.[5]

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APPENDIX: SOLUTION OF THE GENERAL SECOND-ORDER SINGULAR LAYER EQUATION

We consider the equation

$$\frac{A^2 - 1}{C}\frac{d^2 f}{dk^2} - \frac{A(B+C)}{C}k^2\frac{df}{dk} + (Bk^4 - 2Ak)f = 0, \tag{A1}$$

where A, B, and C are real parameters and $B > 0$, $C > 0$. We seek solutions of Eq. (A1) subject to the boundary condition $\lim_{|k| \to \infty} f(k) = 0$ and allowing discontinuity at $k = 0$.

Equation (A1) is similar to the equation discussed in Ref. 6. In terms of the variables

$$\begin{aligned} z &= \left[\sqrt{A^2(B-C)^2 + 4BC}/(A^2 - 1)\right]k^3, \\ d_\pm &= \tfrac{1}{2}\{[A(B+C)/\sqrt{A^2(B-C)^2 + 4BC}] \pm 1\}, \\ g &= e^{-d_+ z} f, \end{aligned} \tag{A2}$$

Eq. (A1) becomes

$$z\frac{d^2 g}{dz^2} + \left(\frac{2}{3} \pm z\right)\frac{dg}{dz} + \left(\pm\frac{1}{3} + \frac{1}{3}\delta\right)g = 0, \tag{A3}$$

where

$$\delta = A(B-C)/\sqrt{A^2(B-C)^2 + 4BC},$$
$$|\delta| < 1.$$

It is easily seen that the signs of d_+ and d_- are opposite if $A^2 < 1$, and the same if $A^2 > 1$.

Equation (A3) is a Kummer equation.[8] When $A^2 > 1$, there is no solution that satisfies the boundary condition. When $A^2 < 1$, the appropriate solution is

$$f = e^{d_- z}U(\tfrac{1}{3} - \tfrac{1}{3}\delta, \tfrac{2}{3}, z), \tag{A4}$$

if $k < 0$, and

$$f = e^{d_+ z}U(\tfrac{1}{3} + \tfrac{1}{3}\delta, \tfrac{2}{3}, -z), \tag{A5}$$

if $k > 0$. In the above, U is the Kummer function.

Using the expansion of Kummer's function for small arguments[8]

$$U(a,b,z) \approx \frac{\pi}{\sin(\pi b)}\left(\frac{1}{\Gamma(1+a-b)\Gamma(b)} - z^{1-b}\frac{1}{\Gamma(a)\Gamma(2-b)}\right), \quad 0 \leqslant b < 1.$$

We obtain from Eqs. (A4) and (A5)

$$\frac{f(0^-)}{df(0^-)/dk} = \frac{\Gamma(\frac{1}{3})\Gamma(\frac{1}{3} - \delta)}{\Gamma(\frac{2}{3})\Gamma(\frac{2}{3} - \frac{1}{3}\delta)} \times \left(\frac{1 - A^2}{3^2\sqrt{A^2(B-C)^2 + 4BC}}\right)^{1/3},$$

$$\frac{f(0^+)}{df(0^+)/dk} = -\frac{\Gamma(\frac{1}{3})\Gamma(\frac{1}{3} + \delta)}{\Gamma(\frac{2}{3})\Gamma(\frac{2}{3} + \frac{1}{3}\delta)} \times \left(\frac{1 - A^2}{3^2\sqrt{A^2(B-C)^2 + 4BC}}\right)^{1/3}. \tag{A6}$$

The above results will be used in Sec. III.

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[1] X. L. Chen and P. J. Morrison, Phys. Fluids B **2**, 495 (1990).
[2] H. P. Furth, J. Killeen, and M. N. Rosenbluth, Phys. Fluids **6**, 459 (1963).
[3] M. Dobrowolny, P. Veltri, and A. Mangeney, J. Plasma Phys. **29**, 303 (1983).
[4] F. Porcelli, Phys. Fluids **30**, 1734 (1987).
[5] G. Einaudi and F. Rubini, Phys. Fluids B **1**, 2224 (1989).
[6] A. Bondeson and M. Persson, Phys. Fluids **29**, 2997 (1986).
[7] R. B. Paris and W. N.-C. Sy, Phys. Fluids **26**, 2966 (1983).
[8] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1970).

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