ARTICLES

The energy-momentum tensor for the linearized Maxwell–Vlasov and kinetic guiding center theories

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A modified Hamilton–Jacobi formalism is introduced as a tool to obtain the energy-momentum and angular-momentum tensors for any kind of nonlinear or linearized Maxwell-collisionless kinetic theories. The emphasis is on linearized theories, for which these tensors are derived for the first time. The kinetic theories treated—which need not be the same for all particle species in a plasma—are the Vlasov and kinetic guiding center theories. The Hamiltonian for the guiding center motion is taken in the form resulting from Dirac's constraint theory for nonstandard Lagrangian systems. As an example of the Maxwell-kinetic guiding center theory, the second-order energy for a perturbed homogeneous magnetized plasma is calculated with initially vanishing field perturbations. The expression obtained is compared with the corresponding one of Maxwell–Vlasov theory.

I. INTRODUCTION

In two previous papers[1,2] different forms of generally valid expressions for the energy of perturbations of general Maxwell–Vlasov equilibria are derived by various methods. A consequence drawn from these expressions was that all inhomogeneous equilibria of interest allow negative-energy modes and are therefore potentially nonlinearly unstable. The proof of this result is based on infinitely strongly localized perturbations. A question therefore arises, to what degree is localization necessary for negative-energy waves. Perturbations with extents smaller than typical gyroradii of the different particle species could lead to anomalous collision terms in Fokker–Planck-like equations and might thus contribute to anomalous transport. It would, however, also be of interest to find out which equilibria allow negative-energy modes with wavelengths larger than the gyroradii. One can, of course, do this kind of investigation with the energy expressions mentioned above. A more appropriate procedure would be to use from the outset theories that have automatically eliminated all perturbations with wavelengths smaller than the gyroradii. The collisionless guiding center theories are of this type.

For the case of the nonlinear Maxwell-kinetic guiding center theory, which included all kinds of drift motions, especially polarization drift, we were able to obtain completely general expressions for the conserved energy, and also the full energy-momentum and angular-momentum tensors.[3,4] For relativistic theories these quantities were obtained independently by Similon[5] using a method different from ours. Our derivations made use of the Hamilton–Jacobi formalism for the particles. As mentioned in Ref. 1, there are, however, some difficulties in applying this formalism to general linearized theory. In Sec. II of this paper we present a modified Hamilton–Jacobi formalism which is simpler than the original one and circumvents these difficulties. It is applicable to linearized theories without restriction. For general Hamiltonians that depend upon the electromagnetic potentials $\phi(\mathbf{x},t)$, $\mathbf{A}(\mathbf{x},t)$, the electric and magnetic fields $\mathbf{E}(\mathbf{x},t)$, $\mathbf{B}(\mathbf{x},t)$, and are arbitrary functions of extended phase space variables, necessary for describing guiding center motion, the new method is used to derive the energy-momentum and angular-momentum tensors. In Sec. III the general linearized theory is presented, and in Sec. IV the corresponding energy-momentum and angular-momentum tensors are derived on the basis of the formalism described in Ref. 4. In Sec. V we specialize to the Maxwell–Vlasov case and obtain for the first time the full energy-momentum tensor for the linearized theory. In Sec. VI we introduce explicitly the Hamiltonian for the guiding center motion within the framework of Dirac's constraint theory for nonstandard Lagrangians.[6] We use the regularized Hamiltonian of Correa-Restrepo and Wimmel[7] and indicate in which way the derivations for the more familiar unregularized theory are related to the ones for the regularized theory. Thereby we make use of the results of Ref. 8, where Dirac's constraint theory was previously applied to the nonlinear theory within the original Hamilton–Jacobi formalism.

In Sec. VII the results of Sec. VI are used to derive for the Maxwell-kinetic guiding center theory rules for obtaining the energy-momentum tensor for each special case from its general form. We prefer to present the results in this way instead of writing out in full detail the very complicated expressions for the general form of this tensor. At the end of this section we give an example: the second-order energy for a perturbed homogeneous system with nonvanishing unperturbed magnetic field but vanishing unperturbed electric field; no initial field perturbations are assumed, i.e., all initial perturbations are perturbations of the distribution functions with vanishing corresponding charge density. The expression obtained is used to derive a sufficient condition for the existence of negative-energy modes. The result is compared with a corresponding one of the Maxwell–Vlasov theory. Finally in Sec. VIII we summarize.

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II. THE MODIFIED HAMILTON–JACOBI FORMALISM FOR THE MAXWELL–VLASOV AND KINETIC GUIDING CENTER THEORIES

Let $H_\nu(p_i,q_i,t)$ be the Hamiltonian for particles of species ν in a phase space $p_1,\dots,p_n$, $q_1,\dots,q_n$ with $(q_1,q_2,q_3)=(x_1,x_2,x_3)=\mathbf{x}$ and correspondingly $(p_1,p_2,p_3)=\mathbf{p}$, where $\mathbf{x}$ is the position in normal space; $n=4$ is needed for describing guiding center motion. The $\mathbf{x},t$ dependence of H_ν is given by the dependence of H_ν on the electromagnetic potentials $\phi(\mathbf{x},t)$ and $\mathbf{A}(\mathbf{x},t)$ and, for the kinetic guiding center theory, also on the electric and magnetic fields $\mathbf{E}(\mathbf{x},t)$ and $\mathbf{B}(\mathbf{x},t)$ and their various derivatives. The derivatives only occur, when Dirac's constraint theory formalism is used. They are absent in a formalism that avoids the necessity of constraint theory by introducing inertial terms with infinitesimally small masses (see Ref. 3). But even with Dirac's formalism the variation of these quantities makes vanishing contributions to the Euler–Lagrange equations and to the energy-momentum tensor [see remark after Eq. (122) in Sec. VII]. The general formalism is therefore equivalent to that for Hamiltonians not depending on the derivatives of $\mathbf{E}$ and $\mathbf{B}$.

In addition to H_ν, we introduce a reference Hamiltonian $H_\nu^{(0)}(P_i,Q_i,t)$ in the phase space $P_1,\dots,P_n$, $Q_1,\dots,Q_n$ that will later—at the beginning of Sec. III—be specified to be the equilibrium Hamiltonian and then be time independent. Let, furthermore, $S_\nu(P_i,q_i,t)$ be a mixed-variable generating function for a canonical transformation between p_i,q_i with corresponding Hamiltonian $H_\nu(p_i,q_i,t)$ and P_i,Q_i with corresponding $H_\nu^{(0)}(P_i,Q_i,t)$. The quantities p_i and Q_i are obtained from S_ν as

$$p_i=\frac{\partial S_\nu}{\partial q_i},\quad Q_i=\frac{\partial S_\nu}{\partial P_i},\tag{1}$$

and S_ν must be a solution of the equation

$$\frac{\partial S_\nu}{\partial t}+H_\nu\left(\frac{\partial S_\nu}{\partial q_i},q_i,t\right)=H_\nu^{(0)}\left(P_i,\frac{\partial S_\nu}{\partial P_i},t\right).\tag{2}$$

The original Hamilton–Jacobi theory is obtained when $H_\nu^{(0)}\equiv 0$. If this is the case, then for perturbation theory there is a problem of finding a solution $S_\nu^{(0)}$ of the unperturbed Hamilton–Jacobi equation with $\partial S_\nu^{(0)}/\partial q_i$ time-independent. This is needed for obtaining an energy expression. In the modified Hamilton–Jacobi formalism we can choose $H_\nu^{(0)}$ as the time-independent equilibrium Hamiltonian. The time-independent, zeroth-order solution $S_\nu^{(0)}$ of Eq. (2) is then simply $S_\nu^{(0)}=\Sigma P_i q_i$, which makes the new formalism applicable in a straightforward way with full generality.

We claim that, analogously to Refs. 3 and 4,

$$\begin{aligned}L=&-\sum_\nu\int dq\,dP\,\varphi_\nu(P_i,q_i,t)\\&\times\left[\frac{\partial S_\nu}{\partial t}+H_\nu\left(\frac{\partial S_\nu}{\partial q_i},q_i,t\right)-H_\nu^{(0)}\left(P_i,\frac{\partial S_\nu}{\partial P_i},t\right)\right]\\&+\frac{1}{8\pi}\int d^3x(\mathbf{E}^2-\mathbf{B}^2)\end{aligned}\tag{3}$$

is the Lagrangian for the Maxwell–Vlasov or kinetic guiding center theory, the criterion being that it leads to the correct "particle" contributions to the charge and current densities. The quantities to be varied are φ_ν, S_ν, $\mathbf{A}$, and ϕ. In expression (3)

$$dq\,dP\equiv dq_1\cdots dq_n\,dP_1\cdots dP_n.\tag{4}$$

In addition, we define $d\hat{q}$ as

$$d^3x\,d\hat{q}=dq.\tag{5}$$

The variational principle is

$$\delta\int_{t_1}^{t_2}L\,dt=0,\tag{6}$$

with $\delta\varphi=\delta S_\nu=\delta\phi=\delta\mathbf{A}=0$ at t_1,t_2 and some boundaries in q,P space. Gauge invariance requires that H_ν and, similarly, $H_\nu^{(0)}$ be of the following form:

$$\begin{aligned}&H_\nu(p_i,q_i,t)\\&\quad=\hat{H}_\nu(\mathbf{p}-(e_\nu/c)\mathbf{A},p_4\cdots p_n,q_4\cdots q_n,\mathbf{E},\mathbf{B})+e_\nu\phi.\end{aligned}\tag{7}$$

Variation with respect to φ_ν, S_ν, ϕ, and $\mathbf{A}$ in Eq. (6) then yields, respectively,

$$\frac{\partial S_\nu}{\partial t}+H_\nu\left(\frac{\partial S_\nu}{\partial q_i},q_i,t\right)-H_\nu^{(0)}\left(P_i,\frac{\partial S_\nu}{\partial P_i},t\right)=0,\tag{8}$$

$$\frac{\partial\varphi_\nu}{\partial t}+\frac{\partial}{\partial q_i}\left(\frac{\partial H_\nu}{\partial p_i}\varphi_\nu\right)-\frac{\partial}{\partial P_i}\left(\frac{\partial H_\nu^{(0)}}{\partial Q_i}\varphi_\nu\right)=0,\tag{9}$$

$$\begin{aligned}&-\sum_\nu e_\nu\int\varphi_\nu\,d\hat{q}\,dP-\frac{\partial}{\partial\mathbf{x}}\cdot\sum_\nu\int\frac{\partial H_\nu}{\partial\mathbf{E}}\varphi_\nu\,d\hat{q}\,dP\\&\quad+\frac{1}{4\pi}\frac{\partial}{\partial\mathbf{x}}\cdot\mathbf{E}=0,\end{aligned}\tag{10}$$

$$\begin{aligned}&-\sum_\nu\frac{e_\nu}{c}\int\frac{\partial H_\nu}{\partial\mathbf{p}}\varphi_\nu\,d\hat{q}\,dP+\frac{1}{c}\frac{\partial}{\partial t}\sum_\nu\int\frac{\partial H_\nu}{\partial\mathbf{E}}\varphi_\nu\,d\hat{q}\,dP\\&\quad+\operatorname{curl}\sum_\nu\int\frac{\partial H_\nu}{\partial\mathbf{B}}\varphi_\nu\,d\hat{q}\,dP\\&\quad+\frac{1}{4\pi c}\frac{\partial}{\partial t}\mathbf{E}+\frac{1}{4\pi}\operatorname{curl}\mathbf{B}=0,\end{aligned}\tag{11}$$

with

$$\frac{\partial H_\nu}{\partial p_i}\equiv\left.\frac{\partial H_\nu(p_i,q_i,t)}{\partial p_i}\right|_{p_i=\partial S_\nu/\partial q_i},\tag{12a}$$

$$\frac{\partial H_\nu^{(0)}}{\partial Q_i}\equiv\left.\frac{\partial H_\nu^{(0)}(P_i,Q_i,t)}{\partial Q_i}\right|_{Q_i=\partial S_\nu/\partial P_i}.\tag{12b}$$

In Eq. (9), and often in the following, we use the summation convention in the form

$$\sum_i a_i b^i\equiv a_i b^i \text{ and } \sum_i a_i b_i\equiv a_i b_i.$$

Equation (8) is Eq. (2) again. Equations (10) and (11) are the inhomogeneous Maxwell equations with "particle," polarization, and magnetization contributions to the charge and current densities. These equations do not have contributions arising from $H_\nu^{(0)}$ since this quantity depends only on equilibrium field variables that are not dynamical variables.

That the "particle" contributions, which are the first terms in Eqs. (10) and (11), are correct follows from the properties of the density functions φ_ν: In Appendix A we prove that the modified Van Vleck determinant

$$\hat{\varphi}_\nu = \det\left|\left|\frac{\partial^2 S_\nu}{\partial q_i\,\partial P_k}\right|\right| \tag{13}$$

solves the mixed-variable continuity equation (9). Its general solution can then be written as

$$\varphi_\nu(P_i,q_i,t) = \hat{\varphi}_\nu \hat{f}_\nu(P_i,q_i,t), \tag{14}$$

where, as shown in Appendix B, $\hat{f}_\nu$ can be represented as

$$\hat{f}_\nu(P_i,q_i,t) = f_\nu\left(\frac{\partial S_\nu}{\partial q_i},q_i,t\right), \tag{15}$$

or

$$\hat{f}_\nu(P_i,q_i,t) = f_\nu^{(0)}\left(P_i,\frac{\partial S_\nu}{\partial P_i},t\right), \tag{16}$$

and where $f_\nu(p_i,q_i,t)$ solves the "Vlasov" equation

$$\frac{\partial f_\nu}{\partial t} + \frac{\partial H_\nu(p_i,q_i,t)}{\partial p_i}\frac{\partial f_\nu}{\partial q_i} - \frac{\partial H_\nu}{\partial q_i}\frac{\partial f_\nu}{\partial p_i} = \frac{\partial f_\nu}{\partial t} - [H_\nu,f_\nu] = 0, \tag{17}$$

and $f_\nu^{(0)}(P_i,Q_i,t)$ solves the "Vlasov" equation for the reference system

$$\frac{\partial f_\nu^{(0)}}{\partial t} + \frac{\partial H_\nu^{(0)}(P_i,Q_i,t)}{\partial P_i}\frac{\partial f_\nu^{(0)}}{\partial Q_i} - \frac{\partial H_\nu^{(0)}}{\partial Q_i}\frac{\partial f_\nu^{(0)}}{\partial P_i} = \frac{\partial f_\nu^{(0)}}{\partial t} - [H_\nu^{(0)},f_\nu^{(0)}] = 0. \tag{18}$$

The brackets [ ] are the corresponding Poisson brackets. The representation (15) yields for any function $G(p_i,q_i,t)$

$$\begin{aligned}\int G\left(\frac{\partial S_\nu}{\partial q_i},q_i,t\right)\varphi_\nu\,d\hat{q}\,dP &= \int G\left(\frac{\partial S_\nu}{\partial q_i},q_i,t\right) f_\nu\left(\frac{\partial S_\nu}{\partial q_i},q_i,t\right) \times \left|\left|\frac{\partial^2 S_\nu}{\partial q_i\,\partial P_k}\right|\right| d\hat{q}\,dP \\ &= \int G(p_i,q_i,t) f_\nu(p_i,q_i,t)\,d\hat{q}\,dp,\end{aligned} \tag{19}$$

which shows that Eqs. (10) and (11) contain the correct "particle" contributions to the charge and current densities. Altogether we can now replace Eqs. (8)–(11), in agreement with Refs. 3 and 4, by the following set of equations:

$$\frac{\partial f_\nu}{\partial t} - [H_\nu,f_\nu] = 0, \tag{20}$$

$$\rho = \sum_\nu e_\nu \int f_\nu\,d\hat{q}\,dp + \operatorname{div}\sum_\nu \int \frac{\partial H_\nu}{\partial \mathbf{E}} f_\nu\,d\hat{q}\,dp, \tag{21}$$

$$\mathbf{j} = \sum_\nu e_\nu \int \frac{\partial H_\nu}{\partial \mathbf{p}} f_\nu\,d\hat{q}\,dp - \frac{\partial}{\partial t}\sum_\nu \int \frac{\partial H_\nu}{\partial \mathbf{E}} f_\nu\,d\hat{q}\,dp - c\,\operatorname{curl}\sum_\nu \int \frac{\partial H_\nu}{\partial \mathbf{B}} f_\nu\,d\hat{q}\,dp. \tag{22}$$

This section is concluded by rewriting the theory in a way that facilitates derivations to come. We introduce the following notation:

$$(x^\mu) = (x^0,\ldots,x^3) = (ct,\mathbf{x}), \quad (A_\mu) = (-\phi,\mathbf{A}), \tag{23a}$$

$$F_{\mu\lambda} = \frac{\partial A_\mu}{\partial x^\lambda} - \frac{\partial A_\lambda}{\partial x^\mu} \equiv A_{\mu,\lambda} - A_{\lambda,\mu}, \tag{23b}$$

$$E_i = \tfrac{1}{2}(F_{0i} - F_{i0}), \quad B_i = -\tfrac{1}{2}\mathbf{e}_i\cdot[\mathbf{e}_k\times\mathbf{e}_l]F^{kl}, \tag{23c}$$

where $\mathbf{e}_i$ is the unit vector in the i direction,

$$F_{kl} = -[\mathbf{e}_k\times\mathbf{e}_l]\cdot\mathbf{B}, \tag{23d}$$

$$\mathbf{E}^2 - \mathbf{B}^2 = -\tfrac{1}{2}F_{\mu\lambda}F^{\mu\lambda}, \tag{23e}$$

$$\frac{\partial}{\partial A_{\lambda,\mu}} = \frac{\partial F_{\sigma\rho}}{\partial A_{\lambda,\mu}}\frac{\partial}{\partial F_{\sigma\rho}} = 2\frac{\partial}{\partial F_{\lambda\mu}}, \tag{23f}$$

$$\frac{\partial}{\partial F_{0i}} = -\frac{\partial}{\partial F_{i0}} = \frac{1}{2}\frac{\partial}{\partial E_i}, \tag{23g}$$

$$\frac{\partial}{\partial F_{kl}} = -\frac{1}{2}[\mathbf{e}_k\times\mathbf{e}_l]\cdot\frac{\partial}{\partial \mathbf{B}}, \quad k,l{:}1,2,3, \tag{23h}$$

$$(\tilde{q}_i) = (\tilde{q}_0,\ldots,\tilde{q}_n) = (ct,\mathbf{x},q_4,\ldots,q_n),$$

where "$\sim$" is used when time is included,

$$\begin{aligned}(\tilde{p}_i) &= (\tilde{p}_0,\ldots,\tilde{p}_n) = (p_0,\mathbf{p},p_4,\ldots,p_n),\\ (\tilde{Q}_i) &= (\tilde{Q}_0,\ldots,\tilde{Q}_n) = (ct,\mathbf{x},Q_4,\ldots,Q_n),\\ (\tilde{P}_i) &= (\tilde{P}_0,\ldots,\tilde{P}_n) = (P_0,\mathbf{p},P_4,\ldots,P_n),\end{aligned} \tag{23i}$$

$$\mathscr{H}_\nu(\tilde{p}_i,\tilde{q}_i) = \mathscr{H}_\nu(\tilde{p}_0,\ldots,\tilde{p}_n,\tilde{q}_0,\ldots,\tilde{q}_n) = cp_0 + H_\nu(p_1,\ldots,p_n,q_1,\ldots,q_n,t), \tag{23j}$$

$$\mathscr{H}_\nu^{(0)}(\tilde{P}_i,\tilde{Q}_i) = \mathscr{H}_\nu^{(0)}(\tilde{P}_0,\ldots,\tilde{P}_n,\tilde{Q}_0,\ldots,\tilde{Q}_n) = cP_0 + H_\nu^{(0)}(P_1,\ldots,P_n,Q_1,\ldots,Q_n,t), \tag{23k}$$

$$A_i \equiv 0 \quad \text{for } i>3, \tag{23l}$$

$$d\tilde{q}\,d\tilde{P} = d\tilde{q}_1\cdots d\tilde{q}_n\,d\tilde{P}_1\cdots d\tilde{P}_n = dq\,dP, \tag{23m}$$

$$\frac{\partial \varphi_\nu}{\partial \tilde{P}_0} = \frac{\partial \varphi_\nu}{\partial P_0} \equiv 0. \tag{23n}$$

Note $\mathscr{H}$ is a function of $\tilde{p}_i - A_i e_\nu/c$, $i = 0,\ldots n$, and $F_{\mu\lambda}$. The Lagrangian for our theory is then

$$L = \sum_\nu \int d\tilde{q}\,d\tilde{P}\,\varphi_\nu\left[\mathscr{H}_\nu\left(\frac{\partial S_\nu}{\partial \tilde{q}_i},\tilde{q}_i\right) - \mathscr{H}_\nu^{(0)}\left(\tilde{P}_i,\frac{\partial S_\nu}{\partial \tilde{P}_i}\right)\right] - \frac{1}{16\pi}\int d^3x\,F_{\mu\lambda}F^{\mu\lambda} \tag{24}$$

and the corresponding Euler–Lagrange equations (8)–(11) become

$$\mathscr{H}_\nu\left(\frac{\partial S_\nu}{\partial \tilde{q}_i},\tilde{q}_i\right) - \mathscr{H}_\nu^{(0)}\left(\tilde{P}_i,\frac{\partial S_\nu}{\partial \tilde{P}_i}\right) = 0, \tag{25}$$

$$\frac{\partial}{\partial \tilde{q}_i}\left(\varphi_\nu\frac{\partial \mathscr{H}_\nu}{\partial \tilde{p}_i}\right) - \frac{\partial}{\partial \tilde{P}_i}\left(\varphi_\nu\frac{\partial \mathscr{H}_\nu^{(0)}}{\partial \tilde{Q}_i}\right) = 0, \tag{26}$$

$$\sum_\nu \frac{e_\nu}{c}\int d\hat{\tilde{q}}\,d\tilde{P}\varphi_\nu\frac{\partial \mathscr{H}_\nu}{\partial p_\lambda} + 2\sum_\nu \int d\hat{\tilde{q}}\,d\tilde{P}\frac{\partial}{\partial x^\mu}\times\left(\varphi_\nu\frac{\partial \mathscr{H}_\nu}{\partial F_{\lambda\mu}}\right) - \frac{1}{4\pi}\frac{\partial F^{\lambda\mu}}{\partial x^\mu} = 0. \tag{27}$$

III. THE LINEARIZED THEORY

The equilibria considered in this section are represented by

$H_\nu^{(0)}(P_i,Q_i)$, $\varphi_\nu^{(0)}(P_i,q_i)$, $S_\nu^{(0)}(P_i,q_i)$, $A_\mu^{(0)}(\mathbf{x})$,

while "primary" perturbations away from these equilibria are represented by

$$\varphi_\nu^{(1)}(P_i,q_i,t),\ S_\nu^{(1)}(P_i,q_i,t),\ A_\mu^{(1)}(\mathbf{x},t),$$

where the superscript (1) is used since later these perturbations will only be first-order quantities; however, this is not assumed from the outset. The primary perturbations lead to first-, second-, and higher-order expressions for the perturbed Hamiltonian $H_\nu(\partial S_\nu/\partial q_i,q_i,t)$ or $\mathscr{H}_\nu$, the unperturbed Hamiltonian $H_\nu^{(0)}(P_i,\partial S_\nu/\partial P_i)$ or $\mathscr{H}_\nu^{(0)}$, and the Lagrangian [see Eqs. (23k), (32) and (33) below]. The variations of the variational principle (3), (6) can then be done in terms of the quantities $\varphi_\nu^{(1)}$, $S_\nu^{(1)}$, $A_\mu^{(1)}$.

Variation of the first-order Lagrangian yields zero, because the unperturbed quantities are solutions to the variational principle and thus variations around them vanish. The lowest-order perturbation of the Lagrangian that is relevant is therefore of second order, and one can now consider the perturbations $\varphi_\nu^{(1)}$, $S_\nu^{(1)}$, and $A_\mu^{(1)}$ as being of first order only. The second-order Lagrangian in these perturbations is then the Lagrangian for the linearized theory.

As mentioned in Sec. II, the advantage of the modified Hamilton–Jacobi formalism over the original one is the simple and generally valid form of the time-independent, zeroth-order function $S_\nu^{(0)}(P_i,q_i)$, namely,

$$S_\nu^{(0)}(P_i,q_i) = \sum_{i=1}^{n} P_i q_i. \tag{28}$$

Up to first order we therefore have

$$\frac{\partial S_\nu}{\partial q_i} = P_i + \frac{\partial S_\nu^{(1)}}{\partial q_i}, \quad \frac{\partial S_\nu}{\partial P_i} = q_i + \frac{\partial S_\nu^{(1)}}{\partial P_i}. \tag{29}$$

In the following we again use the notations of Eqs. (23). In order to obtain the second-order Lagrangian we need

$$\mathscr{H}_\nu^{(1)} = \left(\frac{\partial S_\nu^{(1)}}{\partial \tilde{q}_i} - \frac{e_\nu}{c}A_i^{(1)}\right)\frac{\partial \mathscr{H}_\nu^{(0)}}{\partial \tilde{P}_i} + F_{\mu\lambda}^{(1)}\frac{\partial \mathscr{H}_\nu^{(0)}}{\partial F_{\mu\lambda}^{(0)}} + F_{\mu\lambda,\gamma}^{(1)}\frac{\partial \mathscr{H}_\nu^{(0)}}{\partial F_{\mu\lambda,\gamma}^{(0)}}, \tag{30}$$

$$\begin{aligned}\mathscr{H}_\nu^{(2)} = {}& \frac{1}{2}\left(\frac{\partial S_\nu^{(1)}}{\partial \tilde{q}_i} - \frac{e_\nu}{c}A_i^{(1)}\right)\left(\frac{\partial S_\nu^{(1)}}{\partial \tilde{q}_k} - \frac{e_\nu}{c}A_k^{(1)}\right) \\ &\times \frac{\partial^2 \mathscr{H}_\nu^{(0)}}{\partial \tilde{P}_i \partial \tilde{P}_k} + \left(\frac{\partial S_\nu^{(1)}}{\partial \tilde{q}_i} - \frac{e_\nu}{c}A_i^{(1)}\right) F_{\mu\lambda}^{(1)}\frac{\partial^2 \mathscr{H}_\nu^{(0)}}{\partial \tilde{P}_i \partial F_{\mu\lambda}^{(0)}} \\ &+ \frac{1}{2}F_{\mu\lambda}^{(1)}F_{\sigma\rho}^{(1)}\frac{\partial^2 \mathscr{H}_\nu^{(0)}}{\partial F_{\mu\lambda}^{(0)}\partial F_{\sigma\rho}^{(0)}} + \left(\frac{\partial S_\nu^{(1)}}{\partial \tilde{q}_i} - \frac{e_\nu}{c}A_i^{(1)}\right) \\ &\times F_{\mu\lambda,\gamma}^{(1)}\frac{\partial^2 \mathscr{H}_\nu^{(0)}}{\partial \tilde{P}_i \partial F_{\mu\lambda,\gamma}^{(0)}} + \frac{1}{2}F_{\mu\lambda,\gamma}^{(1)}F_{\sigma\rho,\tau}^{(1)}\frac{\partial^2 \mathscr{H}_\nu^{(0)}}{\partial F_{\mu\lambda,\gamma}^{(0)}\partial F_{\sigma\rho,\tau}^{(0)}},\end{aligned} \tag{31}$$

$$\mathscr{H}_\nu^{(0)(1)} = \frac{\partial S_\nu^{(1)}}{\partial \tilde{P}_i}\frac{\partial \mathscr{H}_\nu^{(0)}}{\partial \tilde{q}_i}, \tag{32}$$

$$\mathscr{H}_\nu^{(0)(2)} = \frac{1}{2}\frac{\partial S_\nu^{(1)}}{\partial \tilde{P}_i}\frac{\partial S_\nu^{(1)}}{\partial \tilde{P}_k}\frac{\partial^2 \mathscr{H}_\nu^{(0)}}{\partial \tilde{q}_i \partial \tilde{q}_k}. \tag{33}$$

Here $\mathscr{H}_\nu^{(0)(1)}$ and $\mathscr{H}_\nu^{(0)(2)}$ are the first- and second-order expressions in the expansion of

$$\mathscr{H}_\nu^{(0)}\left(\tilde{P}_i,\frac{\partial S_\nu}{\partial \tilde{P}_i}\right) = \mathscr{H}_\nu^{(0)}\left(\tilde{P}_i,\tilde{q}_i + \frac{\partial S_\nu^{(1)}}{\partial \tilde{P}_i}\right).$$

The terms containing the quantities $F_{\mu\lambda,\gamma}^{(1)} = \partial F_{\mu\lambda}^{(1)}/\partial x^\gamma$ and $F_{\sigma\rho,\tau}^{(1)}$ occur in the kinetic guiding center theory when Dirac's constraint theory formalism is used. Their variations do not, however, contribute to the Euler–Lagrange equations and the energy-momentum tensor and therefore do not influence the general formalism [see the beginning of Sec. II and the remark after Eq. (92) in Sec. VI].

The density of the second-order Lagrangian following from Eq. (24) is then

$$\begin{aligned}\mathscr{L}^2 = {}& -(1/16\pi)F_{\mu\lambda}^{(1)}F^{(1)\mu\lambda} \\ &- \sum_\nu \int d\tilde{q}\, d\tilde{P}\,[\varphi_\nu^{(0)}(\mathscr{H}_\nu^{(2)} - \mathscr{H}_\nu^{(0)(2)}) \\ &+ \varphi_\nu^{(1)}(\mathscr{H}_\nu^{(1)} - \mathscr{H}_\nu^{(0)(1)})].\end{aligned} \tag{34}$$

Variation with respect to $\varphi_\nu^{(1)}$, $S_\nu^{(1)}$, and $A_\mu^{(1)}$ in

$$\delta \int_{t_1}^{t_2} dt \int d^3x\, \mathscr{L}^{(2)} = 0$$

yields the first-order equations,

$$\begin{aligned}\mathscr{H}_\nu^{(1)} - \mathscr{H}_\nu^{(0)(1)} = {}& -\frac{e_\nu}{c}A_i^{(1)}\frac{\partial \mathscr{H}_\nu^{(0)}}{\partial \tilde{P}_i} \\ &+ F_{\mu\lambda}^{(1)}\frac{\partial H_\nu^{(0)}}{\partial F_{\mu\lambda}^{(0)}} + [S_\nu^{(1)},\mathscr{H}_\nu^{(0)}] = 0,\end{aligned} \tag{35}$$

$$\begin{aligned}&\frac{\partial}{\partial \tilde{q}_k}\left[\left(\frac{\partial S_\nu^{(1)}}{\partial \tilde{q}_i} - \frac{e_\nu}{c}A_i^{(1)}\right)\frac{\partial^2 \mathscr{H}_\nu^{(0)}}{\partial \tilde{P}_i \partial \tilde{P}_k}\varphi_\nu^{(0)}\right] \\ &- \frac{\partial}{\partial \tilde{P}_k}\left(\frac{\partial S_\nu^{(1)}}{\partial \tilde{P}_i}\frac{\partial^2 \mathscr{H}_\nu^{(0)}}{\partial \tilde{q}_i \partial \tilde{q}_k}\right) + \frac{\partial}{\partial \tilde{q}_k} \\ &\times \left(F_{\mu\lambda}^{(1)}\frac{\partial^2 \mathscr{H}_\nu^{(0)}}{\partial \tilde{P}_k \partial F_{\mu\lambda}^{(0)}}\varphi_\nu^{(0)}\right) + [\varphi_\nu^{(1)},\mathscr{H}_\nu^{(0)}] = 0,\end{aligned} \tag{36}$$

$$\begin{aligned}\sum_\nu \int d\tilde{q}\, d\tilde{P}\, \bigg\{&\varphi_\nu^{(0)}\frac{e_\nu}{c}\left(\frac{\partial S_\nu^{(1)}}{\partial \tilde{q}_k} - \frac{e_\nu}{c}A_k^{(1)}\right)\frac{\partial^2 \mathscr{H}_\nu^{(0)}}{\partial \tilde{P}_i \partial \tilde{P}_k} \\ &+ \varphi_\nu^{(0)}\frac{e_\nu}{c}F_{\mu\lambda}^{(1)}\frac{\partial^2 \mathscr{H}_\nu^{(0)}}{\partial \tilde{P}_i \partial F_{\mu\lambda}^{(0)}} + 2\frac{\partial}{\partial x^\lambda} \\ &\times \left[\left(\frac{\partial S_\nu^{(1)}}{\partial \tilde{q}_k} - \frac{e_\nu}{c}A_k^{(1)}\right)\frac{\partial^2 \mathscr{H}_\nu^{(0)}}{\partial \tilde{P}_k \partial F_{i\lambda}^{(0)}}\varphi_\nu^{(0)}\right] \\ &+ 2\frac{\partial}{\partial x^\lambda}\left(F_{\sigma\rho}^{(1)}\frac{\partial^2 \mathscr{H}_\nu^{(0)}}{\partial F_{i\lambda}^{(0)} F_{\sigma\rho}^{(0)}}\right) + \frac{e_\nu}{c}\varphi_\nu^{(1)}\frac{\partial \mathscr{H}_\nu^{(0)}}{\partial \tilde{P}_i} \\ &+ 2\frac{\partial}{\partial x^\lambda}\left(\frac{\partial \mathscr{H}_\nu^{(0)}}{\partial F_{i\lambda}^{(0)}}\varphi_\nu^{(1)}\right)\bigg\} + \frac{1}{4\pi}\frac{\partial}{\partial x^\lambda}F^{(1)i\lambda} = 0.\end{aligned} \tag{37}$$

Here we have defined mixed variable Poisson brackets as

$$[a,b] = \frac{\partial a}{\partial \tilde{q}_i}\frac{\partial b}{\partial \tilde{P}_i} - \frac{\partial a}{\partial \tilde{P}_i}\frac{\partial b}{\partial \tilde{q}_i}. \tag{38}$$

Equation (28) yields for $\hat{\varphi}_\nu^{(0)}$, upon making use of Eq. (13),

$$\hat{\varphi}_\nu^{(0)} = 1 \tag{39}$$

and similarly the first-order contribution is

$$\hat{\varphi}_\nu^{(1)} = \frac{\partial^2 S_\nu^{(1)}}{\partial \tilde{P}_i\, \partial \tilde{q}_i}. \tag{40}$$

Furthermore, from Eq. (16), it follows that

$$\hat{f}_\nu^{(0)} = f_\nu^{(0)}(\tilde{P}_i,\tilde{q}_i),\quad \hat{f}_\nu^{(1)} = \frac{\partial f_\nu^{(0)}}{\partial \tilde{q}_i}\frac{\partial S_\nu^{(1)}}{\partial \tilde{P}_i}. \tag{41}$$

Note that Eqs. (41) embody the fact that perturbations of $\hat{f}_\nu$ are assumed to arise solely from changes in the particle orbits. With the foregoing equations we obtain from Eq. (14)

$$\varphi_\nu^{(0)} = f_\nu^{(0)}(\tilde{P}_i,\tilde{q}_i),\quad \varphi_\nu^{(1)} = \frac{\partial}{\partial \tilde{q}_i}\left(f_\nu^{(0)}\frac{\partial S_\nu^{(1)}}{\partial \tilde{P}_i}\right). \tag{42}$$

IV. THE ENERGY-MOMENTUM AND ANGULAR-MOMENTUM TENSORS FOR THE LINEARIZED THEORY

As in Ref. 4 one can show that the energy-momentum tensor T_ρ^λ is given by

$$T_\rho^\lambda = \sum_\nu \int d\hat{q}\, d\tilde{P}\left(\frac{\partial S_\nu}{\partial x^\rho} - \frac{e_\nu}{c}A_\rho\right)\frac{\delta \mathscr{L}}{\delta(\partial S_\nu/\partial x^\lambda)} + 2F_{\mu\rho}\frac{\partial \mathscr{L}}{\partial F_{\mu\lambda}} - \delta_\rho^\lambda \mathscr{L}, \tag{43}$$

with

$$T_\rho^k - T_k^\rho = 0,\quad k,\rho = 1,2,3 \tag{44}$$

and with

$$M_\rho^{\lambda k} = T_\rho^\lambda x^k - T_k^\lambda x^\rho,\quad \frac{\partial M_\rho^{\lambda k}}{\partial x^\lambda} = 0,\quad k,\rho = 1,2,3 \tag{45}$$

being the corresponding angular-momentum tensor. The energy-momentum tensor (43) has been derived without specifying $\mathscr{L}$. We can therefore use expression (43) for the linearized theory by simply replacing $\mathscr{L}$, S_ν, A_ρ, and $F_{\mu\rho}$ by $\mathscr{L}^{(2)}$, $S_\nu^{(1)}$, $A_\rho^{(1)}$, and $F_{\mu\rho}^{(1)}$. The result is

$$\begin{aligned} T_\rho^{(2)\lambda} = {} & -\sum_\nu \int d\hat{q}\, d\tilde{P}\left(\frac{\partial S_\nu^{(1)}}{\partial \tilde{q}_\rho} - \frac{e_\nu}{c}A_\rho^{(1)}\right)\Bigg[f_\nu^{(0)}\left(\frac{\partial S_\nu^{(1)}}{\partial \tilde{q}_k} - \frac{e_\nu}{c}A_k^{(1)}\right)\frac{\partial^2 \mathscr{H}_\nu^{(0)}}{\partial \tilde{P}_\lambda\, \partial \tilde{P}_k} \\ & + f_\nu^{(0)}F_{\tau\sigma}^{(1)}\frac{\partial^2 \mathscr{H}_\nu^{(0)}}{\partial \tilde{P}_\lambda\, \partial F_{\tau\sigma}^{(0)}} + \frac{\partial}{\partial \tilde{q}_i}\left(f_\nu^{(0)}\frac{\partial S_\nu^{(1)}}{\partial \tilde{P}_i}\right)\frac{\partial \mathscr{H}_\nu^{(0)}}{\partial \tilde{P}_\lambda}\Bigg] \\ & - 2F_{\mu\rho}^{(1)}\sum_\nu \int d\hat{q}\, d\tilde{P}\left[f_\nu^{(0)}\left(\frac{\partial S_\nu^{(1)}}{\partial \tilde{q}_k} - \frac{e_\nu}{c}A_k^{(1)}\right)\frac{\partial^2 \mathscr{H}_\nu^{(0)}}{\partial \tilde{P}_k\, \partial F_{\mu\lambda}^{(0)}} + f_\nu^{(0)}F_{\sigma\tau}^{(1)}\frac{\partial^2 \mathscr{H}_\nu^{(0)}}{\partial F_{\mu\lambda}^{(0)}\, \partial F_{\sigma\tau}^{(0)}}\right] \\ & - \frac{1}{4\pi}F_{\mu\rho}^{(1)}F^{(1)\mu\lambda} + \delta_\rho^\lambda\left(\sum_\nu \int d\hat{q}\, d\tilde{P} f_\nu^{(0)}(\mathscr{H}_\nu^{(2)} - \mathscr{H}_\nu^{(0)(2)}) + \frac{1}{16\pi}F_{\tau\sigma}^{(1)}F^{(1)\tau\sigma}\right). \end{aligned} \tag{46}$$

In this expression one has to use the Euler–Lagrange equation (35) together with Eq. (30) in order to eliminate $\partial S_\nu^{(1)}/\partial t$ wherever it occurs. Specifically these equations yield the following expression for this purpose:

$$\frac{\partial S_\nu^{(1)}}{\partial t} - \frac{e_\nu}{c}A_0^{(1)} = -[S_\nu^{(1)},H_\nu^{(0)}] + \frac{e_\nu}{c}\mathbf{A}^{(1)}\cdot\frac{\partial H_\nu^{(0)}}{\partial \mathbf{P}} - F_{\mu\lambda}^{(1)}\frac{\partial H_\nu^{(0)}}{\partial F_{\mu\lambda}^{(0)}}. \tag{47}$$

The angular-momentum tensor corresponding to $T_\rho^{(2)\lambda}$ is

$$M_\rho^{(2)\lambda k} = T_\rho^{(2)\lambda}x^k - T_k^{(2)\lambda}x^\rho. \tag{48}$$

Since $\mathscr{L}^{(2)}$ does not depend explicitly on time, we have

$$\frac{\partial T_0^{(2)\lambda}}{\partial x^\lambda} = 0, \tag{49}$$

which means that there is energy conservation. However, generally $\mathscr{L}^{(2)}$ depends explicitly on $\mathbf{x}$ and therefore one has

$$\frac{\partial T_\rho^{(2)\lambda}}{\partial x^\lambda} = -\left.\frac{\partial \mathscr{L}^{(2)}}{\partial x^\rho}\right|_{\text{explicit}},\quad \rho = 1,2,3. \tag{50}$$

Nevertheless, for certain symmetries of the equilibrium one can use the energy-momentum tensor to construct quantities, such as the angular-momentum tensor in the case of rotational symmetry, that obey a local conservation law of the form (49).

V. THE ENERGY-MOMENTUM TENSOR FOR THE LINEARIZED MAXWELL–VLASOV THEORY

In the Maxwell–Vlasov theory the extension of phase space introduced in the above formalism is not needed, i.e., $n = 3$,

$\int d\tilde{q} = 1$, and for greater clarity we now write $d^3\tilde{P}$ instead of $d\tilde{P}$. Furthermore, H_ν does not depend on $F_{\mu\nu}$. Equation (46) therefore reduces to

$$
\begin{aligned}
T_\rho^{(2)\lambda} = & -\sum_\nu \int d^3\tilde{P}\Bigg[\left(\frac{\partial S_\nu^{(1)}}{\partial \tilde{q}_\rho} - \frac{e_\nu}{c} A_\rho^{(1)}\right)\left(\frac{\partial S_\nu^{(1)}}{\partial \tilde{q}_k} - \frac{e_\nu}{c} A_k^{(1)}\right)\frac{\partial^2 H_\nu^{(0)}}{\partial \tilde{P}_\lambda\, \partial \tilde{P}_k} f_\nu^{(0)} \\
& + \left(\frac{\partial S_\nu^{(1)}}{\partial \tilde{q}_\rho} - \frac{e_\nu}{c} A_\rho^{(1)}\right)\frac{\partial}{\partial \mathbf{x}}\cdot\left(f_\nu^{(0)} \frac{\partial S_\nu^{(1)}}{\partial \mathbf{P}}\right)\frac{\partial \mathscr{H}_\nu^{(0)}}{\partial \tilde{P}_\lambda}\Bigg] - \frac{1}{4\pi} F_{\mu\rho}^{(1)} F^{(1)\mu\lambda} \\
& + \delta_\rho^\lambda \Bigg\{\sum_\nu \int d^3\tilde{P} f_\nu^{(0)} \frac{1}{2}\Bigg[\left(\frac{\partial S_\nu^{(1)}}{\partial \tilde{q}_i} - \frac{e_\nu}{c} A_i^{(1)}\right)\left(\frac{\partial S_\nu^{(1)}}{\partial \tilde{q}_k} - \frac{e_\nu}{c} A_k^{(1)}\right)\frac{\partial^2 H_\nu^{(0)}}{\partial \tilde{P}_i\, \partial \tilde{P}_k} \\
& - \frac{\partial S_\nu^{(1)}}{\partial \tilde{P}_i}\frac{\partial S_\nu^{(1)}}{\partial \tilde{P}_k}\frac{\partial^2 H_\nu^{(0)}}{\partial \tilde{q}_i\, \partial \tilde{q}_k}\Bigg] + \frac{1}{16\pi} F_{\tau\sigma}^{(1)} F^{(1)\tau\sigma}\Bigg\},
\end{aligned}
\tag{51}
$$

with

$$
\left(\frac{\partial \mathscr{H}_\nu^{(0)}}{\partial \tilde{P}_\lambda}\right) = \left(c, \frac{\partial H_\nu^{(0)}}{\partial \mathbf{P}}\right).
\tag{52}
$$

Equation (52) denotes a vector with four components: the timelike component $\lambda = 0$ has the value c; the spacelike components $\lambda = 1,2,3$ are the components of the particle velocity of species ν.

Of special interest is, of course, the energy, which we can compare with results obtained in Refs. 1 and 2. For $\rho = \lambda = 0$ we have, expressed in terms of the quantities without tilde,

$$
\begin{aligned}
T_0^{(2)0} = & -\sum_\nu \int d^3P\Bigg[f_\nu^{(0)}\frac{1}{2}\left(\frac{\partial S_\nu^{(1)}}{\partial q_i} - \frac{e_\nu}{c} A_i^{(1)}\right) \\
& \times\left(\frac{\partial S_\nu^{(1)}}{\partial q_k} - \frac{e_\nu}{c} A_k^{(1)}\right)\frac{\partial^2 H_\nu^{(0)}}{\partial P_i\, \partial P_k} \\
& - \frac{1}{2} f_\nu^{(0)} \frac{\partial S_\nu^{(1)}}{\partial P_i}\frac{\partial S_\nu^{(1)}}{\partial P_k}\frac{\partial^2 H_\nu^{(0)}}{\partial x^i\, \partial x^k}\Bigg] \\
& - \left(\frac{\partial S_\nu^{(1)}}{\partial t} - e_\nu A_0^{(1)}\right)\frac{\partial}{\partial \mathbf{x}}\cdot\left(f_\nu^{(0)}\frac{\partial S_\nu^{(1)}}{\partial \mathbf{P}}\right) \\
& + \frac{1}{8\pi}\left(\mathbf{E}^{(1)2} + \mathbf{B}^{(1)2}\right),
\end{aligned}
\tag{53}
$$

with

$$
\frac{\partial S_\nu^{(1)}}{\partial t} - \frac{e_\nu}{c} A_0^{(1)} = -\left[S_\nu^{(1)}, H_\nu^{(0)}\right] + \frac{e_\nu}{c}\mathbf{A}^{(1)}\cdot\frac{\partial H_\nu^{(0)}}{\partial \mathbf{P}}
\tag{54}
$$

from Eq. (47). The perturbation of the energy $F^{(2)}$ is then

$$
F^{(2)} = \int T_0^{(2)0}\, d^3x.
\tag{55}
$$

It will be given in a form that can immediately be compared with an expression in Ref. 2.

To this end we add to the right-hand side of Eq. (55) the vanishing expression

$$
\begin{aligned}
& \int d^3x\, d^3P \frac{\partial}{\partial \mathbf{P}}\cdot\Bigg\{\Bigg(-\left[S_\nu^{(1)}, H_\nu^{(0)}\right] \\
& \quad + \frac{e_\nu}{c}\mathbf{A}^{(1)}\cdot\frac{\partial H_\nu^{(0)}}{\partial \mathbf{P}}\Bigg)\frac{\partial S_\nu^{(1)}}{\partial \mathbf{x}} f_\nu^{(0)}\Bigg\} \\
& \quad = \int d^3x\, d^3P\Bigg\{\frac{\partial}{\partial \mathbf{P}}\Bigg(-\left[S_\nu^{(1)}, H_\nu^{(0)}\right] \\
& \qquad + \frac{e_\nu}{c}\mathbf{A}^{(1)}\cdot\frac{\partial H_\nu^{(0)}}{\partial \mathbf{P}}\Bigg)\cdot\frac{\partial S_\nu^{(1)}}{\partial \mathbf{x}} f_\nu^{(0)} + \Bigg(-\left[S_\nu^{(1)}, H_\nu^{(0)}\right] \\
& \qquad + \frac{e_\nu}{c}\mathbf{A}^{(1)}\cdot\frac{\partial H_\nu^{(0)}}{\partial \mathbf{P}}\Bigg)\frac{\partial}{\partial \mathbf{P}}\cdot\left(\frac{\partial S_\nu^{(1)}}{\partial \mathbf{x}} f_\nu^{(0)}\right)\Bigg\},
\end{aligned}
\tag{56}
$$

and we write

$$
H_{\nu 1} = e_\nu A_0^{(1)} - \frac{e_\nu}{c}\mathbf{A}^{(1)}\cdot\frac{\partial H_\nu^{(0)}}{\partial \mathbf{P}} \equiv e_\nu A_0^{(1)} + \hat{H}_{\nu 1},
\tag{57a}
$$

$$
H_{\nu 2}\ \frac{1}{2}\frac{e_\nu^2}{c^2} A_i^{(1)} A_k^{(1)} \frac{\partial^2 H_\nu^{(0)}}{\partial P_i\, \partial P_k}.
\tag{57b}
$$

We then obtain

$$
\begin{aligned}
F^{(2)} = & \sum_\nu \int d^3x\, d^3P\Bigg\{\Bigg(\frac{1}{2}\frac{\partial S_\nu^{(1)}}{\partial x^i}\frac{\partial S_\nu^{(1)}}{\partial x^k}\frac{\partial^2 H_\nu^{(0)}}{\partial P_i\, \partial P_k} \\
& - \frac{1}{2}\frac{\partial S_\nu^{(1)}}{\partial P_i}\frac{\partial S_\nu^{(1)}}{\partial P_k}\frac{\partial^2 H_\nu^{(0)}}{\partial x^i\, \partial x^k} \\
& + H_{\nu 2} - \frac{\partial S_\nu^{(1)}}{\partial \mathbf{x}}\cdot\frac{\partial}{\partial \mathbf{P}}\left[S_\nu^{(1)}, H_\nu^{(0)}\right]\Bigg) f_\nu^{(0)} \\
& + \left(-\left[S_\nu^{(1)}, H_\nu^{(0)}\right] - H_{\nu 1}\right)\left[S_\nu^{(1)}, f_\nu^{(0)}\right]\Bigg\} \\
& + \frac{1}{8\pi}\int\left(\mathbf{E}^{(1)2} + \mathbf{B}^{(1)2}\right) d^3x.
\end{aligned}
\tag{58}
$$

With

$$\frac{\partial S_\nu^{(1)}}{\partial \mathbf{x}}\cdot\frac{\partial}{\partial \mathbf{P}}[S_\nu^{(1)},H_\nu^{(0)}]=\frac{1}{2}[S_\nu^{(1)},[S_\nu^{(1)},H_\nu^{(0)}]]+\frac{1}{2}\frac{\partial S_\nu^{(1)}}{\partial \mathbf{x}}\cdot\frac{\partial}{\partial \mathbf{P}}[S_\nu^{(1)},H_\nu^{(0)}]+\frac{1}{2}\frac{\partial S_\nu^{(1)}}{\partial \mathbf{P}}\cdot\frac{\partial}{\partial \mathbf{x}}[S_\nu^{(1)},H_\nu^{(0)}]$$

$$=\frac{1}{2}[S_\nu^{(1)},[S_\nu^{(1)},H_\nu^{(0)}]]+\frac{1}{2}\frac{\partial S_\nu^{(1)}}{\partial x^i}\frac{\partial S_\nu^{(1)}}{\partial x^k}\frac{\partial^2 H_\nu^{(0)}}{\partial P_i\,\partial P_k}$$

$$-\frac{1}{2}\frac{\partial S_\nu^{(1)}}{\partial P_i}\frac{\partial S_\nu^{(1)}}{\partial P_k}\frac{\partial^2 H_\nu^{(0)}}{\partial x^i\,\partial x^k}+\frac{1}{2}\left[H_\nu^{(0)},\frac{\partial S_\nu^{(1)}}{\partial \mathbf{x}}\cdot\frac{\partial S_\nu^{(1)}}{\partial \mathbf{P}}\right], \tag{59}$$

$$\int d^3x\,d^3P\left[H_\nu^{(0)},\frac{\partial S_\nu^{(1)}}{\partial \mathbf{x}}\cdot\frac{\partial S_\nu^{(1)}}{\partial \mathbf{P}}\right]f_\nu^{(0)}$$

$$=-\int d^3x\,d^3P\frac{\partial S_\nu^{(1)}}{\partial \mathbf{x}}\cdot\frac{\partial S_\nu^{(1)}}{\partial \mathbf{P}}[H_\nu^{(0)},f_\nu^{(0)}]=0, \tag{60}$$

and

$$\int d^3x\,d^3P\,[S_\nu^{(1)},[S_\nu^{(1)},H_\nu^{(0)}]]f_\nu^{(0)}$$

$$=\int d^3x\,d^3P\,[S_\nu^{(1)},f_\nu^{(0)}][H_\nu^{(0)},S_\nu^{(1)}], \tag{61}$$

one obtains with Eq. (57a)

$$F^{(2)}=\sum_\nu\int d^3x\,d^3P\left\{\frac{1}{2}[S_\nu^{(1)},f_\nu^{(0)}][H_\nu^{(0)},S_\nu^{(1)}]\right.$$

$$\left.-\hat{H}_{\nu 1}[S_\nu^{(1)},f_\nu^{(0)}]+H_{\nu 2}f_\nu^{(0)}\right\}. \tag{62}$$

Relation (62) agrees with Ref. 2 if one identifies

$$\mathbf{P}_{\text{here}}=\mathbf{p}_{\text{Ref.2}}\quad\text{and}\quad g_{\nu\,\text{Ref.2}}=-S_\nu^{(1)}{}_{\text{here}}.$$

VI. HAMILTONIAN FOR THE GUIDING CENTER MOTION

We start with a Lagrangian for the guiding center motion. Such a Lagrangian was given by Littlejohn[9] and later in somewhat modified form by Wimmel.[10] Correa-Restrepo and Wimmel[7] observed a difficulty with these Lagrangians, namely that they are singular for large parallel velocities if $\mathbf{B}\cdot\text{curl}\,(\mathbf{B}/B)\neq 0$. This led them to propose a simple regularization method for removing the singular behavior while retaining the variational form of the theory. They applied this method to the nonrelativistic guiding center theory without polarization drift. Later, in Ref. 8 the same method was employed to derive regular kinetic guiding center theories by means of the original Hamiltonian–Jacobi theory.[8] Here we, too, apply the regularized Lagrangian that is only slightly more complicated than the unregularized one in order to avoid possible difficulties. In the following the unregularized theory is obtained, if preferred, simply by replacing the function $g(z)$ by z.

The Lagrangian is defined in terms of the variables

$$t,\quad \mathbf{x}=(q_1,q_2,q_3),\ \text{and}\ q_4, \tag{63}$$

where q_4 is an additional variable needed in guiding center theory. Here, L is of nonstandard form since it is not a convex function of $\dot{\mathbf{x}}$; it is given by the following linear function of $\dot{\mathbf{x}}$ (the index for the particle species being suppressed)

$$L=(e/c)\mathbf{A}^*\cdot\dot{\mathbf{x}}-e\phi^*, \tag{64a}$$

where

$$\mathbf{A}^*=\mathbf{A}+(mc/e)[v_0 g(q_4/v_0)\mathbf{b}+\mathbf{v}_E], \tag{64b}$$

$$e\phi^*=e\phi+\mu B+(m/2)(q_4^2+v_E^2), \tag{64c}$$

$$\mathbf{v}_E=c(\mathbf{E}\times\mathbf{B})/B^2, \tag{64d}$$

$$\mathbf{b}=\mathbf{B}/B, \tag{64e}$$

and μ is the magnetic moment of the gyrating particle.

The antisymmetric function $g(z)$ with $z=q_4/v_0$ does the regularization, where v_0 is some constant velocity. The nonregularized theory is obtained for $g(z)=z$, in which case the solution of Eq. (68) below for q_4 resulting from the Lagrangian (64) is $q_4=v_\parallel=\mathbf{b}\cdot\dot{\mathbf{x}}$. In the regularized theory $g(z)\approx z$ should still hold for small $|z|$. For large $|z|$, however, g must stay finite such that with $v_0\gg v_{\text{thermal}}$ one has

$$v_0 g(\infty)\ll v_c\equiv\frac{(eB)/(mc)}{\mathbf{b}\cdot\text{curl}\,\mathbf{b}}. \tag{65}$$

A possible choice for $g(z)$ is

$$g(z)=\tanh z. \tag{66}$$

Upon varying with respect to $\mathbf{x}$, the variational principle with L given by Eq. (64) yields

$$-\frac{d}{dt}\left(\frac{e}{c}\mathbf{A}^*\right)+\frac{e}{c}\frac{\partial}{\partial \mathbf{x}}(\mathbf{A}^*\cdot\dot{\mathbf{x}})-e\frac{\partial\phi^*}{\partial \mathbf{x}}=0 \tag{67}$$

and varying with respect to q_4 yields

$$m(\mathbf{b}\cdot\dot{\mathbf{x}}\,g'(q_4/v_0)-q_4)=0, \tag{68}$$

where $g'=dg/dz$. In Eq. (67) one has

$$\frac{d}{dt}\left(\frac{e}{c}\mathbf{A}^*\right)=\frac{e}{c}\frac{\partial\mathbf{A}^*}{\partial t}+\frac{e}{c}\dot{\mathbf{x}}\cdot\frac{\partial}{\partial \mathbf{x}}\mathbf{A}^*+\frac{e}{c}\frac{\partial q_4}{\partial t}\frac{\partial\mathbf{A}^*}{\partial q_4}. \tag{69}$$

Therefore, by defining

$$\mathbf{E}^*=-\frac{1}{c}\frac{\partial\mathbf{A}^*}{\partial t}-\frac{\partial\phi^*}{\partial \mathbf{x}},\quad \mathbf{B}^*=\text{curl}\ \mathbf{A}^*,\quad \mathbf{v}=\dot{\mathbf{x}}, \tag{70}$$

we can rewrite Eq. (67) as

$$\mathbf{E}^*+(1/c)\mathbf{v}\times\mathbf{B}^*-(m/e)g'\dot{q}_4\mathbf{b}=0. \tag{71}$$

Crossing Eq. (71) with $\mathbf{b}$ yields

$$\mathbf{b}\times\mathbf{E}^*+(1/c)\mathbf{v}B_\parallel^*-(1/c)v_\parallel\mathbf{B}^*=0,\quad B_\parallel^*=\mathbf{b}\cdot\mathbf{B}^*. \tag{72}$$

From Eq. (68) we find

$$\mathbf{b}\cdot\dot{\mathbf{x}}=v_\parallel=q_4/g'. \tag{73}$$

When this is inserted in Eq. (72), we obtain the guiding center velocity $\mathbf{v} = \mathbf{v}_g$ as a function of t, $\mathbf{x}$, q_4, which will enter the Hamiltonian in Dirac's constraint theory:

$$\mathbf{v} = \mathbf{v}_g = (q_4/g'B^*_\parallel)\mathbf{B}^* + (c/B^*_\parallel)\mathbf{E}^*\times\mathbf{b}. \tag{74}$$

Another "velocity" that is needed is $\dot{q}_4 = V_4$, which follows from Eq. (71) upon multiplication by $\mathbf{B}^*$:

$$\dot{q}_4 = V_4 = (e/mg')(1/B^*_\parallel)\mathbf{E}^*\cdot\mathbf{B}^*. \tag{75}$$

The momenta canonical to $\mathbf{x}$ and q_4 follow from Eq. (64):

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{x}}} = \frac{e}{c}\mathbf{A}^*, \quad p_4 = \frac{\partial L}{\partial \dot{q}_4} = 0. \tag{76}$$

With these momenta the "primary" Hamiltonian H_p in the sense of Dirac's constraint theory[6] is

$$H_p = \dot{\mathbf{x}}\cdot\frac{\partial L}{\partial \dot{\mathbf{x}}} + \dot{q}_4\frac{\partial L}{\partial \dot{q}_4} - L = e\phi^*, \tag{77}$$

and thus Dirac's Hamiltonian is given by

$$H = e\phi^* + \mathbf{v}_g\cdot(\mathbf{p} - (e/c)\mathbf{A}^*) + V_4 p_4. \tag{78}$$

In addition to

$$\dot{\mathbf{x}} = \frac{\partial H}{\partial p} = \mathbf{v}_g, \quad \dot{q}_4 = \frac{\partial H}{\partial p_4} = V_4, \tag{79}$$

which are equivalent to Eqs. (67) and (68), one has the equations

$$\dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{x}} = -e\frac{\partial \phi^*}{\partial \mathbf{x}} - \left(\frac{\partial}{\partial \mathbf{x}}\mathbf{v}_g\right)\cdot\left(\mathbf{p} - \frac{e}{c}\mathbf{A}^*\right) + \frac{e}{c}\left(\frac{\partial}{\partial \mathbf{x}}\mathbf{A}^*\right)\cdot\mathbf{v}_g - \frac{\partial V_4}{\partial \mathbf{x}}p_4 \tag{80}$$

and

$$\dot{p}_4 = -\frac{\partial H}{\partial q_4} = -mq_4 - \frac{\partial \mathbf{v}_g}{\partial q_4}\cdot\left(\mathbf{p} - \frac{e}{c}\mathbf{A}^*\right) + v_\parallel mg'. \tag{81}$$

By using Eqs. (67) and (68) these two equations can be rewritten as

$$\frac{d}{dt}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}^*\right) = -\left(\frac{\partial}{\partial \mathbf{x}}\mathbf{v}_g\right)\cdot\left(\mathbf{p} - \frac{e}{c}\mathbf{A}^*\right) - \frac{\partial V_4}{\partial \mathbf{x}}p_4, \tag{82}$$

$$\dot{\mathbf{p}}_4 = -\frac{\partial \mathbf{v}_g}{\partial q_4}\cdot\left(\mathbf{p} - \frac{e}{c}\mathbf{A}^*\right). \tag{83}$$

This shows that relations (76) are possible solutions, but not the only ones, and that $\mathbf{p} - (e/c)\mathbf{A}^*$ and p_4 are not constants of motion. In order to guarantee that relations (76) are satisfied, the distribution function $f(q_i,p_i,t)$ must be of the form

$$f = \delta(p_4)\delta(\mathbf{p} - (e/c)\mathbf{A}^*)h(\mathbf{x},q_4 t), \tag{84}$$

where h cannot be a constant of motion, because p_4 and $\mathbf{p} - (e/c)\mathbf{A}^*$ are not constants of motion. However, it holds that

$$\delta(p_4)\delta(\mathbf{p} - (e/c)\mathbf{A}^*)dp_4\, d^3p = \text{const along orbits} \tag{85}$$

and, of course, also that

$d^3x\, dq_4\, d^3p\, dp_4 = \text{const}$ and $f = \text{const}$ along the orbits.

Hence it follows that

$$h(\mathbf{x},q_4,t)d^3x\, dq_4 = \text{const along the orbits.} \tag{86}$$

We therefore write

$$h(\mathbf{x},q_4,t) = \hat{h}(\mathbf{x},q_4,t)f_g(\mathbf{x},q_4,t), \tag{87}$$

with $\hat{h}$ being a density in $(\mathbf{x}, q_4)$ space and the guiding center distribution function f_g being a constant of motion.

The equation for f is

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial q_i}\left(\frac{\partial H}{\partial p_i}f\right) - \frac{\partial}{\partial p_i}\left(\frac{\partial H}{\partial q_i}f\right) = 0. \tag{88}$$

Integration of this equation over the full $(\mathbf{p}, p_4)$ space yields, with f given by Eq. (84), an equation for h:

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial \mathbf{x}}\cdot(\mathbf{v}_g h) + \frac{\partial}{\partial q_4}(V_4 h) = 0. \tag{89}$$

It was found in Refs. 7 and 8, corresponding to a result obtained by a different method in Ref. 3, that

$$\hat{h} = B^*_\parallel g'(q_4/v_0) \tag{90}$$

solves this equation. This can also be proved directly by means of Eqs. (74), (75), (73), and the "Maxwell" equations for $\mathbf{E}^*$ and $\mathbf{B}^*$ which follow from Eqs. (70) (note that $\partial\mathbf{A}^*/\partial t$ is a partial time derivative at constant $\mathbf{x}$ and constant q_4). Finally we arrive at

$$f = \delta(p_4)\delta(\mathbf{p} - (e/c)\mathbf{A}^*)B^*_\parallel g'(q_4/v_0)f_g(\mathbf{x},q_4,\mu,t), \tag{91}$$

where f_g is a solution of the drift-kinetic equation

$$\frac{\partial f_g}{\partial t} + \mathbf{v}_g\cdot\frac{\partial f_g}{\partial \mathbf{x}} + V_4\frac{\partial f_g}{\partial q_4} = 0. \tag{92}$$

In f_g a dependence on the magnetic moment μ has been added; this appears in the various expressions only as a parameter distinguishing between different "kinds" of particles. Later, one must sum over all these kinds of particles in order to obtain the total energy-momentum tensor, i.e., one integrates over μ. In the nonregularized case, q_4 is identical to $v_\parallel$. Note that the form (91) of f has the consequence that in the Lagrangian (3), any variation of $\mathbf{v}_g$ [see Eq. (78)] is multiplied by zero. Thus, although $\mathbf{v}_g$ also depends on the derivatives of $\mathbf{E}$ and $\mathbf{B}$, which is not the case with the rest of H, this dependence is unimportant for both the variational principle and the energy-momentum tensor.

Whereas Eq. (91) for f is sufficient in the nonlinear theory to pick out the correct solutions, this is not the case with the linearized theory. The constraints (76), which must hold along the orbits, mean for $p_4 = 0$ that

$$\frac{\partial S}{\partial q_4} = \frac{\partial S^{(0)}}{\partial q_4} + \frac{\partial S^{(1)}}{\partial q_4} = P_4 + \frac{\partial S^{(1)}}{\partial q_4} = 0 \text{ along the orbits.} \tag{93}$$

Here, $P_4 = 0$ is guaranteed by relation (91) when used for the unperturbed distribution function. Hence $S^{(1)}$ must obey

$$\frac{\partial S^{(1)}}{\partial q_4} = 0 \text{ along the orbits.} \tag{94}$$

The constraint for $\mathbf{p}$ means that

$$\frac{\partial S}{\partial \mathbf{x}} = \frac{e}{c}\mathbf{A}^* \text{ along the orbits} \tag{95}$$

or that

$$\frac{\partial S^{(0)}}{\partial \mathbf{x}}+\frac{\partial S^{(1)}}{\partial \mathbf{x}}$$
$$=\frac{e}{c}\mathbf{A}^{*(0)}(\mathbf{x},q_4)+\frac{e}{c}\mathbf{A}^{*(1)}(\mathbf{x},q_4,t)\ \text{ along the orbits.} \tag{96}$$

The equilibrium distribution function guarantees

$$\frac{\partial S^{(0)}}{\partial \mathbf{x}}=\mathbf{P}(t)=\frac{e}{c}\mathbf{A}^{*(0)}(\mathbf{Q}(t),Q_4(t)), \tag{97}$$

where $\mathbf{P}(t)$, $\mathbf{Q}(t)$, and $Q_4(t)$ refer to the unperturbed orbits. In Eqs. (95) and (96) $\mathbf{x}$, q_4 mean $\mathbf{x}(t)$, $q_4(t)$, which refer to the perturbed orbits.

Up to first order we can write

$$\mathbf{x}(t)=\mathbf{Q}(t)+\mathbf{x}^{(1)}(t),\quad q_4(t)=Q_4(t)+q_4^{(1)}(t) \tag{98}$$

and then find from Eq. (96)

$$\frac{\partial S^{(1)}}{\partial \mathbf{x}}=\frac{e}{c}\left(\mathbf{x}^{(1)}(t)\cdot\frac{\partial}{\partial \mathbf{Q}}+q_4^{(1)}\frac{\partial}{\partial Q_4}\right)\times\mathbf{A}^{*(0)}(\mathbf{Q}(t),Q_4(t))+\frac{e}{c}\mathbf{A}^{*(1)}(\mathbf{Q}(t),Q_4(t),t). \tag{99}$$

Furthermore, it holds that

$$\mathbf{Q}(t)=\frac{\partial S}{\partial \mathbf{P}}=\frac{\partial S^{(0)}}{\partial \mathbf{P}}+\frac{\partial S^{(1)}}{\partial \mathbf{P}}=\mathbf{x}(t)+\frac{\partial S^{(1)}}{\partial \mathbf{P}}, \tag{100a}$$

$$Q_4(t)=\frac{\partial S}{\partial P_4}=\frac{\partial S^{(0)}}{\partial P_4}+\frac{\partial S^{(1)}}{\partial P_4}=q_4(t)+\frac{\partial S^{(1)}}{\partial P_4}, \tag{100b}$$

from which it follows that

$$\frac{\partial S^{(1)}}{\partial \mathbf{P}}=-\mathbf{x}^{(1)}(t),\quad \frac{\partial S^{(1)}}{\partial P_4}=-q_4^{(1)}(t). \tag{101}$$

We can now consider for a certain instant of time $\hat{t}$ a distribution of perturbations $\mathbf{x}^{(1)}(\hat{t})$, $q_4^{(1)}(\hat{t})$ in ($\mathbf{x}$, q_4) space, which we denote by $\boldsymbol{\xi}(\mathbf{x},q_4,\hat{t})$, $\xi_4(\mathbf{x},q_4,\hat{t})$. Thus Eqs. (101) and (99) become

$$\frac{\partial S^{(1)}}{\partial \mathbf{P}}=-\boldsymbol{\xi},\quad \frac{\partial S^{(1)}}{\partial P_4}=-\xi_4, \tag{102}$$

$$\frac{\partial S^{(1)}}{\partial \mathbf{x}}=\frac{e}{c}\left(\boldsymbol{\xi}\cdot\frac{\partial}{\partial \mathbf{x}}+\xi_4\frac{\partial}{\partial q_4}\right)\mathbf{A}^{*(0)}(\mathbf{x},q_4)+\frac{e}{c}\mathbf{A}^{*(1)}. \tag{103}$$

The latter relation is more transparent when

$$\mathbf{V}\equiv(1/m)\left[\mathbf{P}-(e/c)\mathbf{A}^{*(0)}(\mathbf{x},q_4)\right] \tag{104}$$

is introduced. This implies that

$$\left.\frac{\partial S^{(1)}}{\partial \mathbf{x}}\right|_{\mathbf{P}}=\left.\frac{\partial S^{(1)}}{\partial \mathbf{x}}\right|_{\mathbf{V}}-\frac{e}{c}\left(\frac{\partial}{\partial \mathbf{x}}\mathbf{A}^{*(0)}\right)\cdot\frac{\partial S^{(1)}}{\partial \mathbf{P}}=\left.\frac{\partial S^{(1)}}{\partial \mathbf{x}}\right|_{\mathbf{V}}+\frac{e}{c}\left(\frac{\partial}{\partial \mathbf{x}}\mathbf{A}^{*(0)}\right)\cdot\boldsymbol{\xi}. \tag{105}$$

If, in addition, we use

$$\frac{e}{c}\frac{\partial \mathbf{A}^{*(0)}}{\partial q_4}=mg'\frac{q_4}{v_0}\mathbf{b}^{(0)}, \tag{106}$$

as follows from Eq. (64), we can replace Eq. (103) by

$$\left.\frac{\partial S^{(1)}}{\partial \mathbf{x}}\right|_{\mathbf{V}}=-\frac{e}{c}\boldsymbol{\xi}\times\mathbf{B}^{*(0)}+\xi_4 mg'\frac{q_4}{v_0}\mathbf{b}^{(0)}+\frac{e}{c}\mathbf{A}^{*(1)}. \tag{107}$$

The zeroth-order distribution function always selects $\mathbf{V}=P_4=0$. It is therefore reasonable to expand $S^{(1)}$ in powers of $\mathbf{V}$ and P_4. Since only first-order derivatives of $S^{(1)}$ occur explicit knowledge of $S^{(1)}$ up to first order in $\mathbf{V}$ and P_4 is sufficient:

$$S^{(1)}=\hat{S}^{(1)}(\mathbf{x},q_4)-\boldsymbol{\xi}\cdot m\mathbf{V}-\xi_4P_4+\text{higher-order terms}, \tag{108}$$

where the first-order terms are chosen so as to yield the relations (102) for $P_4=\mathbf{V}=0$. In addition, we obtain from Eq. (108) at $P_4=\mathbf{V}=0$

$$\left.\frac{\partial S^{(1)}}{\partial \mathbf{x}}\right|_{\mathbf{P}}=\frac{\partial \hat{S}^{(1)}}{\partial \mathbf{x}}+\frac{e}{c}\left(\frac{\partial}{\partial \mathbf{x}}\mathbf{A}^{*(0)}\right)\cdot\boldsymbol{\xi} \tag{109}$$

and

$$\left.\frac{\partial S^{(1)}}{\partial q_4}\right|_{\mathbf{P}}=\frac{\partial \hat{S}^{(1)}}{\partial q_4}+\frac{e}{c}\left(\frac{\partial}{\partial q_4}\mathbf{A}^{*(0)}\right)\cdot\boldsymbol{\xi}=\frac{\partial \hat{S}^{(1)}}{\partial q_4}+mg'\frac{q_4}{v_0}\mathbf{b}^{(0)}\cdot\boldsymbol{\xi}=0. \tag{110}$$

From Eq. (107) we find, again with Eq. (108), ξ_4 and the components of $\boldsymbol{\xi}$ perpendicular to $\mathbf{B}^{*(0)}$, $\boldsymbol{\xi}_{\perp *}$:

$$\xi_4=\frac{1}{mg'B_{\parallel}^{*(0)}}\mathbf{B}^{*(0)}\cdot\left(\frac{\partial \hat{S}^{(1)}}{\partial \mathbf{x}}-\frac{e}{c}\mathbf{A}^{*(1)}\right), \tag{111}$$

$$\boldsymbol{\xi}_{\perp *}=\frac{c}{eB^{*(0)2}}\left[\mathbf{b}^{*(0)}\cdot\left(\frac{\partial \hat{S}^{(1)}}{\partial \mathbf{x}}-\frac{e}{c}\mathbf{A}^{*(1)}\right)\mathbf{B}^{*(0)}\times\mathbf{b}^{(0)}-\mathbf{B}^{*(0)}\times\left(\frac{\partial \hat{S}^{(1)}}{\partial \mathbf{x}}-\frac{e}{c}\mathbf{A}^{*(1)}\right)\right]. \tag{112}$$

The full displacement vector $\boldsymbol{\xi}$ is then

$$\boldsymbol{\xi}=\boldsymbol{\xi}_{\perp *}+\lambda(\mathbf{x},q_4)\mathbf{B}^{*(0)}. \tag{113}$$

We find λ from condition (110):

$$\lambda=-\frac{1}{mg'B_{\parallel}^{*(0)}}\left(\frac{\partial \hat{S}^{(1)}}{\partial q_4}+mg'\frac{q_4}{v_0}\mathbf{b}^{(0)}\cdot\boldsymbol{\xi}_{\perp *}\right)=-\frac{1}{mg'B_{\parallel}^{*(0)}}\frac{\partial \hat{S}^{(1)}}{\partial q_4}+\frac{c}{eB^{*(0)2}B_{\parallel}^{*(0)}}\mathbf{b}^{(0)}\cdot\mathbf{B}^{*(0)}\times\left(\frac{\partial \hat{S}^{(1)}}{\partial \mathbf{x}}-\frac{e}{c}\mathbf{A}^{*(1)}\right). \tag{114}$$

The last quantity needed for $T_{\rho}^{(2)\mu}$ is $\dot{\mathbf{A}}^{(1)}$. It has to be in agreement with the constraints. Since these constraints must hold along the orbits, corresponding constraints for their time derivatives along the orbits must also be valid:

$$\frac{d}{dt}\frac{\partial S}{\partial \mathbf{x}}=\frac{e}{c}\frac{d}{dt}\mathbf{A}^*,\quad \frac{d}{dt}\frac{\partial S}{\partial q_4}=0\quad \text{at } t=\hat{t}. \tag{115}$$

These conditions can be viewed as being equations for the new quantities $(d/dt)\boldsymbol{\xi}$ and $(d/dt)\xi_4$. They could be solved for these quantities for any $\dot{\mathbf{A}}^{(1)}$. This is, however, not necessary, since $T_{\rho}^{(2)\mu}$ does not depend on $(d/dt)\boldsymbol{\xi}$ and/or $(d/dt)\xi_4$. We thus have the result that the following quantities can be freely chosen:

$$\mathbf{A}^{(1)},\dot{\mathbf{A}}^{(1)},\hat{S}^{(1)}(\mathbf{x},q_4,\mu) \tag{116a}$$

while $\phi^{(1)}$ is subjected to the constraint

$$\nabla\cdot\mathbf{E}^{(1)}=4\pi\rho^{(1)}. \tag{116b}$$

The μ dependence of $\hat{S}^{(1)}$ has been added for the reason given after Eq. (92).

VII. THE ENERGY-MOMENTUM TENSOR FOR THE LINEARIZED MAXWELL-KINETIC GUIDING CENTER THEORY

In this section we use the results of the previous section to derive general rules for obtaining in each special case the, rather complicated, energy-momentum tensor $T_\rho^{(2)\mu}$ for the Maxwell-kinetic guiding center theory. This amounts to tailoring Eqs. (46) and (47) to the case at hand. It follows from Eqs. (84), (102), (109), (110) and from the remark after Eq. (92) that all terms which contain $f_\nu^{(0)}$ undifferentiated, the following substitutions have to be made in Eq. (46):

$$\mathbf{P}\rightarrow(e_\nu/c)\mathbf{A}^{*(0)},\quad P_4\rightarrow 0, \tag{117a}$$

$$\frac{\partial S_\nu^{(1)}}{\partial\mathbf{x}}\rightarrow\frac{\partial\hat{S}_\nu^{(1)}}{\partial\mathbf{x}}+\frac{e_\nu}{c}\left(\frac{\partial}{\partial\mathbf{x}}\mathbf{A}^{*(0)}\right)\cdot\boldsymbol{\xi},\quad\frac{\partial S_\nu^{(1)}}{\partial q_4}\rightarrow 0, \tag{117b}$$

$$\frac{\partial S_\nu^{(1)}}{\partial\mathbf{P}}\rightarrow-\boldsymbol{\xi},\quad\frac{\partial S_\nu^{(1)}}{\partial P_4}\rightarrow-\xi_4, \tag{117c}$$

$$\int d\widetilde{P}f_\nu^{(0)}\cdots\rightarrow\int d\mu\,h_\nu^{(0)}\cdots, \tag{117d}$$

$\boldsymbol{\xi}$ being given by Eqs. (112)–(114) and ξ_4 by Eq. (111).

There is one term containing derivatives of $f_\nu^{(0)}$, namely,

$$T\equiv-\sum_\nu\int d\hat{q}\,d\widetilde{P}\left(\frac{\partial S_\nu^{(1)}}{\partial\widetilde{q}_\rho}-\frac{e_\nu}{c}A_\rho^{(1)}\right)\times\frac{\partial}{\partial\widetilde{q}_i}\left(f_\nu^{(0)}\frac{\partial S_\nu^{(1)}}{\partial\widetilde{P}_i}\right)\frac{\partial\mathscr{H}_\nu^{(0)}}{\partial\widetilde{P}_\lambda}. \tag{118}$$

This term can be written as

$$\begin{aligned}T=&-\sum_\nu\int d\hat{q}\frac{\partial}{\partial\widetilde{q}_i}\int d\widetilde{P}\left(\frac{\partial S_\nu^{(1)}}{\partial\widetilde{q}_\rho}-\frac{e_\nu}{c}A_\rho^{(1)}\right)f_\nu^{(0)}\frac{\partial S_\nu^{(1)}}{\partial\widetilde{P}_i}\frac{\partial\mathscr{H}_\nu^{(0)}}{\partial\widetilde{P}_\lambda}+\sum_\nu\int d\hat{q}\,d\widetilde{P}f_\nu^{(0)}\\&\times\frac{\partial S_\nu^{(1)}}{\partial\widetilde{P}_i}\frac{\partial}{\partial\widetilde{q}_i}\left[\left(\frac{\partial S_\nu^{(1)}}{\partial\widetilde{q}_\rho}-\frac{e_\nu}{c}A_\rho^{(1)}\right)\frac{\partial\mathscr{H}_\nu^{(0)}}{\partial\widetilde{P}_\lambda}\right]\\=&-\sum_\nu\int d\hat{q}\frac{\partial}{\partial\widetilde{q}_i}\left[\left(\frac{\partial S_\nu^{(1)}}{\partial\widetilde{q}_\rho}-\frac{e_\nu}{c}A_\rho^{(1)}\right)h_\nu^{(0)}\frac{\partial S_\nu^{(1)}}{\partial\widetilde{P}_i}\frac{\partial\mathscr{H}_\nu^{(0)}}{\partial\widetilde{P}_\lambda}\right]+\sum_\nu\int d\hat{q}\,h_\nu^{(0)}\frac{\partial S_\nu^{(1)}}{\partial\widetilde{P}_i}\frac{\partial}{\partial\widetilde{q}_i}\left[\left(\frac{\partial S_\nu^{(1)}}{\partial\widetilde{q}_\rho}-\frac{e_\nu}{c}A_\rho^{(1)}\right)\frac{\partial\mathscr{H}_\nu^{(0)}}{\partial\widetilde{P}_\lambda}\right],\end{aligned} \tag{119}$$

where this expression is understood again with the substitutions (117).

Whenever the quantity $\partial S_\nu^{(1)}/\partial t$ occurs, it is to be replaced according to Eq. (47) by

$$\frac{\partial S_\nu^{(1)}}{\partial t}-e_\nu A_{(0)}^{(1)}\rightarrow-[S_\nu^{(1)},H_\nu^{(0)}]+\frac{e_\nu}{c}\mathbf{v}_g^{(0)}\cdot\mathbf{A}^{(1)}-\left(\mathbf{E}^{(1)}\cdot\frac{\partial}{\partial\mathbf{E}^{(0)}}+\mathbf{B}^{(1)}\cdot\frac{\partial}{\partial\mathbf{B}^{(0)}}\right)H_\nu^{(0)}. \tag{120}$$

We note further that

$$\frac{\partial^2\mathscr{H}_\nu^{(0)}}{\partial\widetilde{P}_i\,\partial\widetilde{P}_k}=0, \tag{121}$$

and that in Eq. (32) one has

$$\frac{1}{2}F_{\mu\lambda,\gamma}^{(1)}F_{\sigma\rho,\tau}^{(1)}\frac{\partial^2\mathscr{H}_\nu^{(0)}}{\partial F_{\mu\lambda,\gamma}^{(0)}\,\partial F_{\sigma\rho,\tau}^{(0)}}\rightarrow 0, \tag{122}$$

because of the constraints built into $f_\nu^{(0)}$ and $F_{\mu\lambda,\gamma}$ involving only $\mathbf{v}_g$, V_4.

We give, in addition, a few helpful relations:

$$\begin{aligned}F_{\mu\rho}^{(1)}\frac{\partial}{\partial F_{\mu\lambda}^{(0)}}=&\frac{1}{2}\mathbf{E}^{(1)}\cdot\frac{\partial}{\partial\mathbf{E}^{(0)}}\delta_{\rho0}\delta_{\lambda0}-\frac{1}{2}\left(\mathbf{E}^{(1)}\times\frac{\partial}{\partial\mathbf{B}^{(0)}}\right)_\lambda\delta_{\rho0}(1-\delta_{\lambda0})+\frac{1}{2}\left(\mathbf{B}^{(1)}\times\frac{\partial}{\partial\mathbf{E}^{(0)}}\right)_\rho\delta_{\lambda0}(1-\delta_{\rho0})\\&+\frac{1}{2}\left(E_\rho^{(1)}\frac{\partial}{\partial E_\lambda^{(0)}}+\delta_{\rho\lambda}\mathbf{B}^{(1)}\cdot\frac{\partial}{\partial\mathbf{B}^{(0)}}-B_\lambda^{(1)}\frac{\partial}{\partial B_\rho^{(0)}}\right)(1-\delta_{\rho0})(1-\delta_{\lambda0}),\end{aligned} \tag{123}$$

$$F_{\mu\lambda}^{(1)}\frac{\partial}{\partial F_{\mu\lambda}^{(0)}}=\mathbf{E}^{(1)}\cdot\frac{\partial}{\partial\mathbf{E}^{(0)}}+\mathbf{B}^{(1)}\cdot\frac{\partial}{\partial\mathbf{B}^{(0)}}. \tag{124}$$

All derivatives like $\partial/\partial\mathbf{E}^{(0)}$ have the meaning

$$\frac{\partial}{\partial\mathbf{E}^{(0)}}=\frac{\partial}{\partial\mathbf{E}}\bigg|_{\mathbf{E}=\mathbf{E}^{(0)}}. \tag{125}$$

Of special interest are

$$\frac{\partial}{\partial E_i^{(0)}}\mathbf{v}_E^{(0)}=c\frac{(\mathbf{e}_i\times\mathbf{B}^{(0)})}{B^{(0)2}}, \tag{126}$$

$$\frac{\partial}{\partial B_i^{(0)}}\mathbf{v}_E^{(0)}=c\frac{(\mathbf{E}^{(0)}\times\mathbf{e}_i)}{B^{(0)2}}-\frac{2B_i^{(0)}}{B^{(0)2}}\mathbf{v}_E^{(0)}. \tag{127}$$

As an illustration, we derive the second-order energy for a perturbed homogeneous system with nonvanishing unperturbed magnetic field but vanishing unperturbed electric field. We restrict to a case that was of special interest in the Maxwell–Vlasov theory, namely, that no field perturbations are initially present, i.e., all initial perturbations are perturbations of the distribution functions with vanishing corresponding charge density. Thus

$$\mathbf{B}^{(0)}\neq 0, \quad \mathbf{E}^{(0)}\equiv 0, \quad F_{\mu\lambda}^{(1)}\equiv 0, \quad A_{\rho}^{(1)}\equiv 0. \tag{128}$$

Equation (46) for $T_0^{(2)0}$ then reduces to

$$T_0^{(2)0} = -\sum_\nu \int d\hat{q}\, d\widetilde{P}\, \frac{\partial S_\nu^{(1)}}{\partial t} \frac{\partial}{\partial \widetilde{q}_i}\left(f_\nu^{(0)} \frac{\partial S_\nu^{(1)}}{\partial \widetilde{P}_i}\right) + \sum_\nu \int d\hat{q}\, d\widetilde{P} f_\nu^{(0)} (\mathscr{H}_\nu^{(2)} - \mathscr{H}_\nu^{(0)(2)}), \tag{129}$$

and Eq. (120) to

$$\frac{\partial S_\nu^{(1)}}{\partial t} = -[S_\nu^{(1)}, H_\nu^{(0)}]. \tag{130}$$

Equations (32) and (33) yield

$$\mathscr{H}_\nu^{(2)} = \mathscr{H}_\nu^{(0)(2)} = 0. \tag{131}$$

Furthermore, one has from

Eqs. (64): $\mathbf{A}^{*(0)} = \mathbf{A}^{(0)} + (m_\nu c/e_\nu) v_0 g(q_4/v_0)\mathbf{b}^{(0)}$, (132a)

$$e_\nu \phi^{*(0)} = \mu B^{(0)} + (m_\nu/2) q_4^2; \tag{132b}$$

Eq. (70): $\mathbf{B}^{*(0)} = \mathbf{B}^{(0)}$; (132c)

Eq. (74): $\mathbf{v}_g^{(0)} = (q_4/g')\mathbf{b}^{(0)}$; (132d)

Eq. (75): $V_4 = 0$. (132e)

As a consequence, one obtains

$$\frac{\partial H_\nu^{(0)}}{\partial q_4} = 0 \tag{133}$$

and it holds that

$$\mathbf{b}^{(0)}\cdot \frac{\partial}{\partial \mathbf{x}} \mathbf{A}^{(0)} = 0, \quad \mathbf{A}^{(0)}\cdot \mathbf{b}^{(0)} = 0. \tag{134}$$

This leads to

$$[S_\nu^{(1)}, H_\nu^{(0)}] = \frac{q_4}{g'} \mathbf{b}^{(0)}\cdot \frac{\partial \hat{S}_\nu^{(1)}}{\partial \mathbf{x}}, \tag{135}$$

$$\boldsymbol{\xi} = -\frac{c}{e_\nu B^{(0)}}\left(\mathbf{b}^{(0)} \times \frac{\partial \hat{S}_\nu^{(1)}}{\partial \mathbf{x}}\right) - \frac{1}{m_\nu g'} \mathbf{b}^{(0)} \frac{\partial \hat{S}_\nu^{(1)}}{\partial q_4}, \tag{136}$$

$$\xi_4 = \frac{1}{m_\nu g'} \mathbf{b}^{(0)}\cdot \frac{\partial \hat{S}_\nu^{(1)}}{\partial \mathbf{x}}. \tag{137}$$

The second-order energy $F^{(2)}$ then becomes

$$F^{(2)} = \int d^3x\, T_0^{(2)0} = \sum_\nu \int d^3x\, dq_4\, d\mu\, h_\nu^{(0)} \left(\boldsymbol{\xi}\cdot \frac{\partial}{\partial \mathbf{x}} + \xi_4 \frac{\partial}{\partial q_4}\right) \times \left(\frac{q_4}{g'} \mathbf{b}^{(0)}\cdot \frac{\partial \hat{S}_\nu^{(1)}}{\partial \mathbf{x}}\right) \tag{138}$$

and, with Eqs. (87) and (90) as well as with Eqs. (136) and (137),

$$F^{(2)} = \sum_\nu \int d^3x\, dq_4\, d\mu\, \frac{B^{(0)}}{m_\nu} f_{g\nu}^{(0)} \times \left(-\frac{\partial \hat{S}_\nu^{(1)}}{\partial q_4} \mathbf{b}^{(0)}\cdot \frac{\partial}{\partial \mathbf{x}} + \mathbf{b}\cdot \frac{\partial \hat{S}_\nu^{(1)}}{\partial \mathbf{x}} \frac{\partial}{\partial q_4}\right) \times \left(\frac{q_4}{g'} \mathbf{b}^{(0)}\cdot \frac{\partial \hat{S}_\nu^{(1)}}{\partial \mathbf{x}}\right). \tag{139}$$

Introducing complex quantities by the rule

$$AB \to \tfrac{1}{2}\mathrm{Re}\, A^* B \tag{140}$$

and with

$$\hat{S}_\nu^{(1)} \sim e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{141}$$

one obtains

$$\begin{aligned} F^{(2)} &= V \sum_\nu \int dq_4\, d\mu\, \frac{B^{(0)}}{m_\nu} f_{g\nu}^{(0)}(q_4,\mu) \times (\mathbf{k}\cdot\mathbf{b}^{(0)})^2 \frac{\partial}{\partial q_4}\left(|\hat{S}_\nu^{(1)}|^2 \frac{q_4}{g'}\right) \\ &= -V \sum_\nu \int dq_4\, d\mu\, \frac{B^{(0)}}{m_\nu} (\mathbf{k}\cdot\mathbf{b}^{(0)})^2 |\hat{S}_\nu^{(1)}|^2 \times \frac{q_4}{g'} \frac{\partial}{\partial q_4} f_{g\nu}^{(0)}(q_4,\mu), \end{aligned} \tag{142}$$

where V is a normalization volume. We note that $F^{(2)}$ depends on $\hat{S}_\nu^{(1)}$ only via $|\hat{S}_\nu^{(1)}|^2$.

Since the first-order charge density $\rho^{(1)}$ is a q_4, μ integral over an expression that is linear in $\hat{S}_\nu^{(1)}$, one can satisfy the assumption $\rho^{(1)} = 0$ (made at the beginning of this example) by a proper distribution of positive and negative values of $\hat{S}_\nu^{(1)}$, on which $F^{(2)}$ does not depend.

Recalling that according to Eq. (68) q_4/g' is the component of the velocity parallel to $\mathbf{B}^{(0)}$, we see that expression (142) resembles the corresponding ones obtained within the framework of the Maxwell–Vlasov theory for homogeneous equilibria with $\mathbf{B}^{(0)} = 0$ and for infinitely strongly localized perturbations of general equilibria. The most important difference is seen in the following respective terms:

$$(\mathbf{k}\cdot\mathbf{v})\mathbf{k}\cdot \frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\bigg|_{\text{Vlasov theory}} \tag{143a}$$

and

$$(\mathbf{k}\cdot\mathbf{b}^{(0)})^2 \frac{q_4}{g'} \frac{\partial}{\partial q_4} f_{g\nu}^{(0)}\bigg|_{\text{kinetic guiding center theory}}. \tag{143b}$$

Whereas in the Maxwell–Vlasov theory any deviation of $f_\nu^{(0)}$ from being a monotonic function of $|\mathbf{v}|$ allows negative energy modes to exist, it is solely the $v_\parallel$ dependence of the distribution function in the kinetic guiding center theory that is decisive; the μ dependence does not matter. The condition for the existence of negative-energy modes, which in the Maxwell–Vlasov theory is

$$(\mathbf{k}\cdot\mathbf{v})\mathbf{k}\cdot \frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}} > 0, \quad \text{for some } \mathbf{k},\mathbf{v},\nu, \tag{144a}$$

is replaced in the Maxwell-kinetic guiding center theory by

$$(\mathbf{k}\cdot\mathbf{b}^{(0)})^2 \frac{q_4}{g'} \frac{\partial}{\partial q_4} f^{(0)}_{g\nu} > 0, \quad \text{for some } \mathbf{k}, q_4, \nu. \tag{144b}$$

The restricted class of initial conditions for which expression (142) is valid means, however, that the inequality (144b) is only a sufficient condition. We expect that in the kinetic guiding center theory, initially nonvanishing field perturbations will be important. We remark that the sufficient condition (144b) for the existence of negative energy modes, when applied to distribution functions $f^{(0)}_{g\nu}(q_4,\mu) = \bar{f}^{(0)}_{g\nu}(\epsilon,\mu)$ with $\epsilon = \mu B^{(0)} + \frac{1}{2} m_\nu (q_4/g')^2$, becomes

$$\left(\mathbf{k}\cdot\mathbf{b}^{(0)} \frac{q_4}{g'}\right)^2 \frac{1}{g'} \frac{\partial \bar{f}^{(0)}_{g\nu}(\epsilon,\mu)}{\partial \epsilon} > 0 \quad \text{for some } \mathbf{k}, \epsilon, \mu, \nu. \tag{145}$$

In this form it is similar to the well-known sufficient stability condition[11]

$$\frac{\partial \bar{f}^{(0)}_{g\nu}}{\partial \epsilon} < 0.$$

Condition (144b) or (145) does not, however, imply that the system is linearly unstable, but there is the possibility of nonlinear instability.[12]

VIII. SUMMARY

The introduction of a modified Hamilton–Jacobi formalism as a tool allows straightforward construction of the energy-momentum and angular momentum tensors for any kind of nonlinear or linearized Maxwell-collisionless kinetic theories, which may be different for different particle species in a plasma, without any restriction. Contrary to the original Hamilton–Jacobi theory, which consists of an equation for the mixed-variable generating function for a canonical transformation to variables with vanishing corresponding Hamiltonian, the modified Hamilton–Jacobi theory deals with a canonical transformation from the perturbed to the unperturbed system or, more generally, from the system considered to some reference system. The application to the Maxwell–Vlasov theory is possible without any further developments. The Maxwell-kinetic guiding center theory has on the particle side to do with a nonstandard Lagrangian system. This was handled within the formalism of Dirac's constraint theory. The constraints led in the nonlinear theory to a special form of the distribution function defined in an extended phase space. It contains the guiding center distribution function defined in $v_\parallel$, μ, $\mathbf{x}$ space, where μ is the magnetic moment. In the linearized theory the constraints introduce, in addition, a displacement vector in $v_\parallel$, $\mathbf{x}$ space similar to that in $\mathbf{x}$ space occurring in macroscopic theories. As an example of the Maxwell-kinetic guiding center theory the second-order energy for a perturbed homogeneous magnetized plasma is calculated with initially vanishing field perturbations. The expression is compared with a corresponding one of the Maxwell–Vlasov theory. As long as the possible existence of negative-energy modes follows solely from the $v_\parallel$ dependence of the unperturbed guiding center distribution function, the μ-dependence does not matter. The criterion found is the same as in the Maxwell–Vlasov theory for wave propagation parallel to $\mathbf{B}^{(0)}$. The condition is, of course, only a sufficient condition because of the class of initial perturbations considered. It is expected that in the kinetic guiding center theory initially nonvanishing field perturbations will be important.

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APPENDIX A: PROOF THAT $\hat{\varphi}_\nu$ GIVEN BY EQ. (13) SOLVES EQ. (9)

The proof is similar to those in Refs. 13 and 14 for the original Van Vleck determinant.

Let A_{ik} be the cofactor of $\partial^2 S_\nu/\partial q_i\, \partial P_k$ in the determinant

$$\hat{\varphi}_\nu = \left|\left| \frac{\partial^2 S_\nu}{\partial q_i\, \partial P_k} \right|\right|. \tag{A1}$$

With this definition (under the summation convention given in Sec. II)

$$A_{li} \frac{\partial^2 S_\nu}{\partial q_l\, \partial P_k} = A_{il} \frac{\partial^2 S_\nu}{\partial q_k\, \partial P_l} = \hat{\varphi}_\nu \delta_{ik} \tag{A2}$$

and

$$\delta\hat{\varphi}_\nu = A_{ik} \frac{\partial^2 \delta S_\nu}{\partial q_i\, \partial P_k} \tag{A3}$$

then hold. With these relations, Eq. (8), and with notation (12), we have

$$\begin{aligned}
\frac{\partial \hat{\varphi}_\nu}{\partial t} &= A_{ik} \frac{\partial^3 S_\nu}{\partial t\, \partial q_i\, \partial P_k} \\
&= A_{ik} \frac{\partial^2}{\partial q_i\, \partial P_k} \left[-H_\nu\left(\frac{\partial S_\nu}{\partial q_i}, q_i, t\right) \right. \\
&\quad \left. + H^{(0)}_\nu\left(P_i, \frac{\partial S_\nu}{\partial P_i}, t\right)\right] \\
&= A_{ik}\left[-\frac{\partial}{\partial q_i}\left(\frac{\partial H_\nu}{\partial p_l} \frac{\partial^2 S_\nu}{\partial q_l\, \partial P_k}\right) \right. \\
&\quad \left. + \frac{\partial}{\partial P_k}\left(\frac{\partial H^{(0)}_\nu}{\partial Q_l} \frac{\partial^2 S_\nu}{\partial P_l\, \partial q_i}\right)\right] \\
&= -\frac{\partial}{\partial q_i}\left(\frac{\partial H_\nu}{\partial p_l}\right)\delta_{il}\hat{\varphi}_\nu - \frac{\partial H_\nu}{\partial p_l} \frac{\partial}{\partial q_l}\hat{\varphi}_\nu \\
&\quad + \frac{\partial}{\partial P_k}\left(\frac{\partial H^{(0)}_\nu}{\partial Q_l}\right)\delta_{kl}\hat{\varphi}_\nu + \frac{\partial H^{(0)}_\nu}{\partial Q_l} \frac{\partial}{\partial P_l}\hat{\varphi}_\nu \\
&= -\frac{\partial}{\partial q_l}\left(\frac{\partial H_\nu}{\partial p_l}\hat{\varphi}_\nu\right) + \frac{\partial}{\partial P_l}\left(\frac{\partial H^{(0)}_\nu}{\partial Q_l}\hat{\varphi}_\nu\right),
\end{aligned} \tag{A4}$$

which proves the statement.

APPENDIX B: PROOF OF RELATIONS (15) AND (16)

When

$$\varphi_\nu = \hat{\varphi}_\nu \hat{f}_\nu(P_i, q_i, t) \tag{B1}$$

is inserted in Eq. (9), one obtains for $\hat{f}_\nu$ with the notation (12)

$$\frac{\partial \hat{f}_\nu}{\partial t} + \frac{\partial H_\nu}{\partial p_l}\frac{\partial \hat{f}_\nu}{\partial q_l} - \frac{\partial H_\nu^{(0)}}{\partial Q_i}\frac{\partial \hat{f}_\nu}{\partial P_i} = 0. \tag{B2}$$

For

$$\hat{f}_\nu(P_i,q_i,t) = f_\nu\left(\frac{\partial S_\nu}{\partial q_i},q_i,t\right), \tag{B3}$$

with a notation for f_ν corresponding to Eq. (12), the following relations hold:

$$\frac{\partial \hat{f}_\nu}{\partial t} = \frac{\partial f_\nu}{\partial t} + \frac{\partial f_\nu}{\partial p_i}\frac{\partial^2 S_\nu}{\partial q_i\,\partial t}$$

$$= \frac{\partial f_\nu}{\partial t} + \frac{\partial f_\nu}{\partial p_i}\left(-\frac{\partial H_\nu}{\partial q_i} - \frac{\partial H_\nu}{\partial p_l}\times\frac{\partial^2 S_\nu}{\partial q_l\,\partial q_i} + \frac{\partial H_\nu^{(0)}}{\partial Q_l}\frac{\partial^2 S_\nu}{\partial P_l\,\partial q_i}\right), \tag{B4}$$

$$\frac{\partial \hat{f}_\nu}{\partial q_i} = \frac{\partial f_\nu}{\partial q_i} + \frac{\partial f_\nu}{\partial p_l}\frac{\partial^2 S_\nu}{\partial q_l\,\partial q_i}, \tag{B5}$$

$$\frac{\partial \hat{f}_\nu}{\partial P_i} = \frac{\partial f_\nu}{\partial p_l}\frac{\partial^2 S_\nu}{\partial q_l\,\partial P_i}. \tag{B6}$$

Using Eqs. (B4)–(B6) in Eq. (B2) yields the equation

$$\frac{\partial f_\nu}{\partial t} + \frac{\partial H_\nu}{\partial p_i}\frac{\partial f_\nu}{\partial q_i} - \frac{\partial H_\nu}{\partial q_i}\frac{\partial f_\nu}{\partial P_i} = 0 \tag{B7}$$

for $f_\nu(p_i,q_i,t)$, which is Eq. (17). Relation (15) is thus proved.

For

$$\hat{f}_\nu(P_i,q_i,t) = f_\nu^{(0)}\left(P_i,\frac{\partial S_\nu}{\partial P_i},t\right) \tag{B8}$$

one has, with a notation for $f_\nu^{(0)}$ corresponding to Eq. (12),

$$\frac{\partial \hat{f}_\nu}{\partial t} = \frac{\partial f_\nu^{(0)}}{\partial t} + \frac{\partial f_\nu^{(0)}}{\partial Q_i}\frac{\partial^2 S_\nu}{\partial P_i\,\partial t}$$

$$= \frac{\partial f_\nu^{(0)}}{\partial t} + \frac{\partial f_\nu^{(0)}}{\partial Q_i}\left(-\frac{\partial H_\nu}{\partial p_l}\frac{\partial^2 S_\nu}{\partial q_l\,\partial P_i} + \frac{\partial H_\nu^{(0)}}{\partial P_i} + \frac{\partial H_\nu^{(0)}}{\partial Q_l}\frac{\partial^2 S_\nu}{\partial P_l\,\partial P_i}\right) \tag{B9}$$

$$\frac{\partial \hat{f}_\nu}{\partial q_i} = \frac{\partial f_\nu^{(0)}}{\partial Q_l}\frac{\partial^2 S_\nu}{\partial P_l\,\partial q_i} \tag{B10}$$

$$\frac{\partial \hat{f}_\nu}{\partial P_i} = \frac{\partial f_\nu^{(0)}}{\partial P_i} + \frac{\partial f_\nu^{(0)}}{\partial Q_l}\frac{\partial^2 S_\nu}{\partial P_l\,\partial P_i}. \tag{B11}$$

Inserting Eqs. (B9)–(B11) into Eq. (B2) yields

$$\frac{\partial f_\nu^{(0)}}{\partial t} + \frac{\partial H_\nu^{(0)}}{\partial P_i}\frac{\partial f_\nu^{(0)}}{\partial Q_i} - \frac{\partial H_\nu^{(0)}}{\partial Q_i}\frac{\partial f_\nu^{(0)}}{\partial P_i} = 0 \tag{B12}$$

for $f_\nu^{(0)}(P_i,Q_i,t)$, which is Eq. (18). Relation (16) is thus proved.

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