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# A sufficient condition for the ideal instability of shear flow with parallel magnetic field

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A simple sufficient condition is given for the linear ideal instability of plane parallel equilibria with antisymmetric shear flow and symmetric or antisymmetric magnetic field. Application of this condition shows that plane Couette flow, which is stable in the absence of a magnetic field, can be driven unstable by a symmetric magnetic field. Also, although strong magnetic shear can stabilize shear flow with a hyperbolic tangent profile, there exists a range of magnetic shear that causes destabilization.

Shear flow is a very common phenomenon. It appears in such diverse areas as in astrophysical jets,[1] the magnetosphere,[2] and rotating plasmas. Recently, experiments[3] in the DIII-D tokamak show that there is a substantial increase in the perpendicular component of plasma flow velocity associated with the L-(low-) to H- (high-) confinement mode transition. Since shear flow contains a source of free energy, it can give rise to the Kelvin–Helmholtz (K–H) instability.[4] A necessary condition for K–H instability to occur is the Rayleigh inflection point condition. The physical role of the inflection point condition is explained by the conservation of vorticity:[5] In order to release the free energy contained in the shear flow, there needs to be a vorticity extremum, since only then does the restoring force against a perturbation vanish. Shear flows that are stable can become unstable when the magnetic field is included. The purpose of the present paper is to present a simple condition for such instability.

The presence of the magnetic field has a dual role for the instability of shear flow. The magnetic field exerts a tension on the fluid, which usually acts as a restoring force on a disturbance. So it is easy to imagine that the flow is completely stabilized if magnetic energy overpowers kinetic energy everywhere;[6] i.e., $A^2 > V^2$ in the whole region, where $A$ is the local Alfvén speed. This condition need only hold in some reference frame for stability to be established. It was also shown by using the semicircle theorem[7] that the flow is stable if $|A|_{\min} > (V_{\max} - V_{\min})/2$. References 8 and 9 have discussed the stabilizing effect of magnetic shear. On the other hand, sometimes the magnetic field can destabilize the shear flow, since it breaks the constraint of local conservation of vorticity and thus makes the shear flow free energy accessible. In this case, the existence of an inflection point is not necessary for instability. Kent[10] has shown that a stable symmetric flow can be driven unstable by a symmetric magnetic field if, on the boundary $A = 0$ and $V'V''-A'A'' > 0$, where prime denotes differentiation with respect to $y$. Stern[11] has also discussed the destabilizing effect of a piecewise continuous magnetic field on plane Couette flow. The actual role of the magnetic field depends on the specific profiles of both the flow and the magnetic field. Kent[6] has shown that a constant magnetic field stabilizes some, while it destabilizes other, monotonic flow profiles.

In the present letter, we consider a sufficient condition for instability, by assuming that the flow is antisymmetric and that the magnetic field has parity; i.e., it is either symmetric or antisymmetric. A technique[12] that is based on the use of symmetries and the Nyquist method is used to obtain a simple formula. Though the symmetries we assume may limit application to some practical problems, results obtained from these special profiles provide insight into the physics and will be helpful in the more realistic situations. In many circumstances, the shear flow can be approximated by antisymmetric profiles. An antisymmetric hyperbolic tangent profile has been used to model the edge flow in tokamaks.[8]

In order to focus on the shear flow driven K–H instability, we neglect dissipation. In many situations, this is justified since the dissipation diffusion time scale is much longer than the K–H time scale. The dynamics is assumed to be governed by ideal incompressible magnetohydrodynamics. For simplicity, we adopt slab geometry and assume an equilibrium with shear flow $\mathbf{V} = V(y)\hat{x}$ and parallel magnetic field $\mathbf{B} = B(y)\hat{x}$. Here, we assume that such a flow is confined between rigid walls located at $y = -l$ and $y = l$. Assuming that all the perturbed field components have the form $f(k,c,y)\exp ik(x-ct)$, the normal mode equation for the transverse displacement $w$ is[6]

$$\{[(V-c)^2 - A^2]w'\}' - k^2[(V-c)^2 - A^2]w = 0, \tag{1}$$

where prime denotes differentiation with respect to $y$, and $A(y)$, as noted above, is the local Alfvén velocity. Since the transverse displacement vanishes at the rigid walls, Eq. (1) has the boundary conditions $w(l) = w(-l) = 0$. Equation (1), together with the boundary conditions, gives the disper-

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a certain range of wave numbers $k$, the shear flow is unstable. Since Eq. (1) is regular for complex $c$, $c(k^2)$ in this case is an analytic function of $k$.

Here, we consider an extreme case with wave number $k=0$. If there exists an eigenvalue where $\mathrm{Im}(c) \equiv c_i \neq 0$ for $k=0$, then this is sufficient to say that the system is unstable. Strictly speaking, the growth rate $kc_i$ is zero when $k=0$, but analyticity of $c(k^2)$ ensures a finite growth rate near $k=0$. This argument has previously been used in Refs. 6 and 13.

Setting $k=0$ in Eq. (1), integrating, and applying the boundary conditions leads to

$$F(c) = \int_{-l}^{l} \frac{dy}{(V-c)^2 - A^2} = 0. \tag{2}$$

Without solving the above integral equation for the eigenvalue $c$, we can use Nyquist diagram method, in a manner similar to the Penrose criterion,[14] to determine whether or not there exist unstable modes. By a well-known theorem of complex analysis, the number of roots of an analytic function like $F$ in the upper half-plane $[\mathrm{Im}(c) > 0]$ is given by the number of times a polar plot of $F$ encircles the origin as $c$ traces out the curve as shown in Fig. 1. Path 3–1 has a distance $\epsilon$ from the real axis so that the singularity on the real axis is avoided. Thus $F(c)$ is an analytic function. However, in order not to miss any possible unstable modes, we take the limit $\epsilon \to 0$.

Along the path 1–2–3, $c = \mathrm{Re}^{i\theta}$ and in the limit $R \to \infty$, $F(c) \sim 2le^{2i\theta}/R^2$. The corresponding plot of $F$ is shown in Fig. 2. Since we assumed that the shear flow is antisymmetric and that the magnetic field is either symmetric or antisymmetric, we have along path 3–1 in Fig. 1

$$F(c_r + i\epsilon) = F^*(-c_r + i\epsilon),$$

where "*" means complex conjugate. Thus we have the following conclusions: (i) $\mathrm{Im}\, F(i\epsilon) = 0$, and (ii) if $\mathrm{Im}\, F(c_r + \epsilon) = 0$, $(c_r \neq 0)$, then $\mathrm{Im}\, F(-c_r + i\epsilon) = 0$, and $\mathrm{Re}\, F(c_r + i\epsilon) = \mathrm{Re}\, F(-c_r + i\epsilon)$. To determine the winding number [the number of times $F(c)$ encircles the origin], we can just count the points of crossing of the real axis. Denote crossing points by $n$; associated with such points are two quantities

$$\sigma_n = \begin{cases} 1, & \text{crossing of real axis with up direction,} \\ -1, & \text{crossing of real axis with down direction,} \end{cases}$$

and

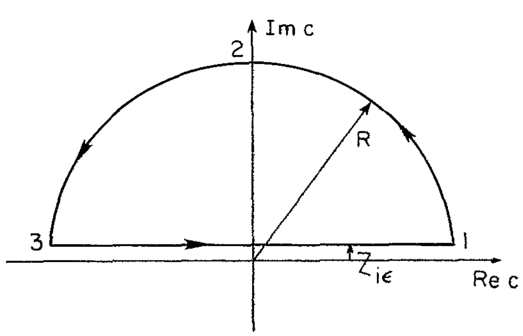


FIG. 1. Nyquist diagram in the $c$-plane.

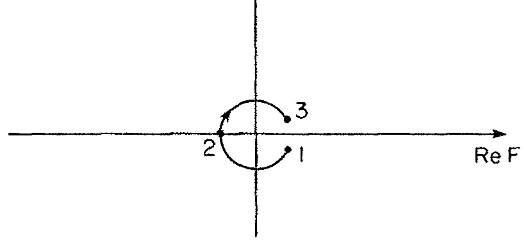


FIG. 2. Nyquist diagram in the $F$-plane.

$$r_n = \begin{cases} 1, & \mathrm{Re}\ F_n > 0, \\ 0, & \mathrm{Re}\ F_n = 0, \\ -1 & \mathrm{Re}\ F_n < 0. \end{cases}$$

Since the Nyquist diagram must be closed as $c$ traces the path of Fig. 1, this implies the following conclusions: (i) The total number of crossing points is even and $\Sigma_n \sigma_n = 0$; and (ii) for crossing points $i$ and $j$ with $r_i = r_j$ and $\sigma_i + \sigma_j = 0$, there is cancellation and thus no contribution to the winding number.

For the present problem, if $\mathrm{Re}\, F(0 + i\epsilon) > 0$, then the total number of crossing points with positive and negative $\mathrm{Re}\, F$ are both odd numbers, and we always have net crossing on each side of the real axis of $F(c)$. Now we consider, respectively, two possible cases.

*Case I*: In this case, we suppose there are no crossing points with $\mathrm{Re}\, F = 0$. Thus the net crossing with $\mathrm{Re}\, F > 0$ and $\mathrm{Re}\, F < 0$ must point in opposite directions. Hence, the Nyquist diagram encircles the origin at least once, and there exists at least one unstable mode.

*Case II*: In this case, there exist crossing points with $\mathrm{Re}\, F(\pm c_r + i\epsilon) = 0$, which implies that there exist marginal modes with $c = \pm c_r$. When this occurs, we can prove that the Nyquist diagram still indicates a nonzero winding number. In other words, it is impossible to have a Nyquist diagram with the net crossing for $\mathrm{Re}\, F > 0$ and the net crossing for $\mathrm{Re}\, F < 0$ pointing in the same direction. For the moment, suppose this is the case. The Nyquist diagram will be as shown in Fig. 3(a) and there exists no unstable mode. Now we change the $c$ contour a little bit, so that $\epsilon$ is very small but with finite value; instead of proceeding to the limit $\epsilon \to 0$. Since there exists no unstable mode, there are no crossing points with $\mathrm{Re}\, F = 0$ along the new contour. Furthermore, we still have $\mathrm{Re}\, F(0 + i\epsilon) > 0$, since $\epsilon$ is very small. Using the argument of case I, there exists an unstable mode, as the example shown in Fig. 3(b) indicates. This contradicts our original assumption and thus the proof is established.

From the above discussion, a sufficient condition for instability with antisymmetric shear flow and antisymmetric or symmetric magnetic fields is given by

$$F(0 + i\epsilon) = \int_{-l}^{l} \frac{dy}{(V - i\epsilon)^2 - A^2} > 0, \tag{3}$$

where the limit $\epsilon \to 0$ from above is assumed. For the case of antisymmetric shear flow with only one inflection point, the inflection point should be at $y = 0$. When $A = 0$ (i.e., with-

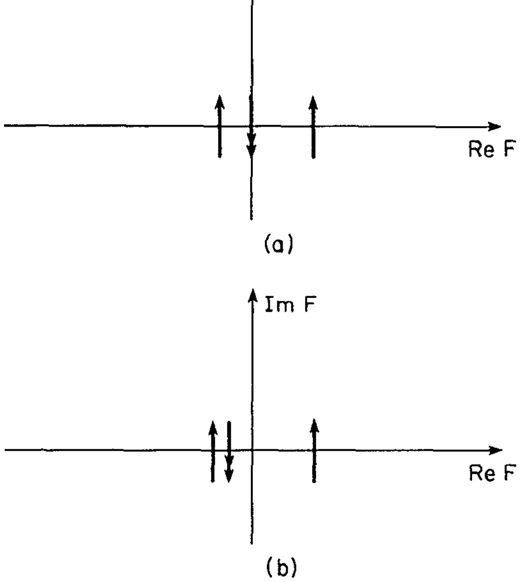


FIG. 3. A Nyquist diagram having crossing points with Re $F = 0$, and with the net crossing for Re $F > 0$ and Re $F < 0$ pointing in the same direction. (b) The Nyquist diagram of 3(a) with finite but small values of $\epsilon$, instead of $\epsilon = 0$.

out magnetic field), our sufficient condition Eq. (3) reduces to that obtained in Ref. 13, and this condition becomes sufficient and necessary for instability because of Lin's theorem.[5]

For a plane *Couette flow* profile $V(y) = by$, there is no vorticity extremum and thus this flow is K–H stable.[15] Stern[11] has shown that the Couette flow can be destabilized by a piecewise continuous magnetic field. Here, we add a symmetric magnetic field $A(y) = ay^2$ to the Couette flow equilibrium. The destabilizing effect of this symmetric magnetic field is easily demonstrated from our simple sufficient condition. Equation (4) gives

$$F(0+i\epsilon) = \frac{1}{V(l)b}\left(-2 + \frac{A(l)}{V(l)}\log\left|\frac{1+A(l)/V(l)}{1-A(l)/V(l)}\right|\right). \tag{4}$$

When the magnetic field at the boundaries is sufficiently strong; i.e., $A(l)/V(l) > f$, where $f \simeq 0.834$ is the value at which $F(0+i\epsilon) = 0$, $F(0+i\epsilon) > 0$ and there is instability.

For the second example, we consider a *hyperbolic tangent* shear flow $V(y) = V_0 \tanh(y/d_1)$. In the case without magnetic field, Eq. (3) is both sufficient and necessary for instability; it indicates that the hyperbolic tangent shear flow is unstable if, and only if, $l/d_1 > 2.39$. Now we add a magnetic shear $A(y) = A_0 y/d_2$. When the magnetic shear is strong enough so that $A_0/d_2 > V_0/d_1$, the shear flow will be always stable since the magnetic energy overpowers the kinetic energy everywhere; i.e., $A(y)^2 > V(y)^2$ for all $y$. We want to know what happens if the magnetic shear is not this strong. For simplicity of evaluating the integral in Eq. (3), we approximate the hyperbolic tangent profile by a piecewise continuous one as follows:

$$V(y) = \begin{cases} V_0, & y > d_1, \\ V_0 y/d_1, & |y| < d_1, \\ -V_0, & y < -d_1. \end{cases}$$

$$F(0+i\epsilon) = \frac{1}{d_1}\left(\frac{1}{V'A'}\log\frac{(V'+A'l/d_1)(V'-A')}{(V'+A')|V'-A'l/d_1|} - \frac{2}{V'^2 - A'^2}\right), \tag{5}$$

where $V' = V_0/d_1$, $A' = A_0/d_2$, and $V' > A'$, $l > d_1$ are assumed. In order to stabilize the unstable shear flow, it is necessary to have $F(0+i\epsilon) < 0$. When $A' > \bar{A}'$, where $\bar{A}'$ satisfies $\sqrt{d_1/l}\,V' > \bar{A}' > (d_1/l)V'$, the necessary condition is satisfied. However, it is interesting to notice that when $A' \sim (d_1/l)V'$; i.e., $A(l) \sim V(l)$, $F(0+i\epsilon)$ is always positive. A stable flow ($l/d_1 < 2.39$) can be driven unstable by the magnetic shear in this range. Thus magnetic shear does not always stabilize the K–H instability.

We conclude from the above two examples that the magnetic field in the midplane tends to stabilize the shear flow, while the magnetic field at the boundaries tends to destabilize the shear flow, especially when $A(l) \sim V(l)$. In the plane Couette flow example, $A(0)/V(0) = 0$, thus this flow is destabilized by the magnetic field at the boundaries. In the hyperbolic tangent flow example, the magnetic shear destabilizes the flow when $A' \sim (d_1/l)V'$; i.e., $A(l) \sim V(l)$. However, a large magnetic shear stabilizes the flow. In this case, the stabilizing effect of the magnetic field in the midplane overcomes the destabilizing effect of the magnetic field at the boundaries.

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