

Drift wave vortices in inhomogeneous plasmas

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The effects of density and temperature gradients on drift wave vortex dynamics are studied using a fully nonlinear model with the Boltzmann density distribution. The equation based on the full Boltzmann relation, in the short wavelength ($\sim \rho_s$) region, possesses no localized monopole solution, while in the longer wavelength [$\sim (\rho_s r_n)^{1/2}$] region the density profile governs the existence of monopolelike solutions. In the longer wavelength regime, however, the results of analysis show that due to the inhomogeneity of the plasma the monopoles cannot be localized sufficiently to avoid coupling to propagating drift waves. Thus, the monopole drift wave vortex is a long-lived coherent structure, but it is not precisely a stationary structure since the coupling results in a “flapping” tail. The flapping tail causes energy of the vortex to leak out, but the effect of the temperature gradient-induced nonlinearity is to reduce the leaking of this energy.

I. INTRODUCTION

It is well known that the dipole vortex is an exact solitary wave solution to the dissipationless Hasegawa–Mima¹ (H–M) drift wave equation, which in the fluid mechanics literature is referred to as the Rossby wave equation

$$(1 - \nabla^2) \frac{\partial \varphi}{\partial t} + v_d \frac{\partial \varphi}{\partial y} - [\varphi, \nabla^2 \varphi] = 0, \quad (1)$$

where

$$[\varphi, \nabla^2 \varphi] = \frac{\partial \varphi}{\partial x} \frac{\partial \nabla^2 \varphi}{\partial y} - \frac{\partial \varphi}{\partial y} \frac{\partial \nabla^2 \varphi}{\partial x}$$

is the Jacobian involving the electrostatic potential φ and the vorticity $\zeta = \nabla^2 \varphi$. The convective nonlinearity $[\varphi, \zeta]$ of Eq. (1), also known as the Poisson bracket or vector nonlinearity, facilitates the formation of dipole vortices.^{2,3} Recently, there have been numerous studies of this type of nonlinearity and the resulting vortex dynamics.^{3,4} In the present work, we consider a generalization of the H–M drift wave equation that includes a general density profile $n_0(x)$ and temperature profile $T_e(x)$.^{5–11}

Petviashvili⁵ first studied the problem with a temperature gradient and an exponential density profile. The model he proposed has recently been criticized by several authors^{7–11} because of inconsistency with Ertel’s theorem, the basic conservation law of potential vorticity. Some of these authors^{7,8} showed that monopole vortices exist only when the drift velocity in Eq. (1) is nonconstant, $v_d = v_d(x)$. More recently, Spatschek *et al.*¹² used a new high order and long wavelength scaling to develop a model that is similar to Petviashvili’s model. These authors argue for the existence of a monopole vortex solution in the very long wavelength region ($\sim \rho_s/\epsilon$), even with constant drift velocity v_d . Here ϵ is the usual drift theory expansion parameter $\epsilon = \rho_s/r_n$, where $\rho_s = c(m_i T_e)^{1/2}/eB$ is drift wave dispersion scale length, and r_n is the density gradient scale length. We also define $\eta_e = r_n/r_T$ as the ratio of the density-to-temperature gradient scale length.

It is well known that there exists a large class of vortex solutions for nonlinear drift waves. These solutions arise be-

cause of an arbitrary function $F(\varphi - ux)$ that appears in the equation for traveling wave solutions. Usually this function is chosen such that the solution vanishes as $r \rightarrow \infty$. The function F determines the relationship between the generalized potential vorticity, which to the first order is $\nabla^2 \varphi - \varphi/T(x) - \ln n_0(x)$, and the streamfunction, $\varphi - ux$, in a frame traveling with speed u . For the well-known dipole vortex solution, the choice of this relationship is piecewise linear with a jump in the slope $dF/d\varphi$. In spite of the presence of this jump the solution is consistent with the H–M equation.^{3–10} Alternatively one can demand F to be analytic. In this work we show for a model that includes an arbitrary temperature gradient and a constant density gradient, that this choice of F makes it impossible to have a binding effective potential. Here the density profile is strictly exponential, but the result is independent of the temperature profile. However, with a more complex background plasma, i.e., with nonconstant drift wave velocity $v_d(x)$, the monopole vortex is found in the $\rho_s/\epsilon^{1/2}$ wavelength region. Physically the nonconstant $v_d(x)$ adds a shear to the diamagnetic drift velocity. Since it is well known (Horton *et al.*¹³ and Sagdeev *et al.*¹⁴) that shear flow causes the formation of monopole vortices, it is perhaps to be expected that the shear flow from nonuniform $v_d(x)$ will create monopole solitary waves.

Also we show that the monopole vortices mentioned above in inhomogeneous plasmas are not strictly localized solitonlike monopoles. The effect of inhomogeneity in the background plasma on vortex dynamics is shown to give rise to an oscillating tail; the inhomogeneity forces a coupling of the vortex core to the tail of a radiative wake of drift waves, which thus causes radiative damping of the vortex core.

This work is organized as follows: in Sec. II, we derive the model equation for nonlinear drift waves in plasmas with density and temperature gradients. The steady-state traveling wave equation of the model is given and solutions of this equation are discussed in Sec. III. In Sec. IV we investigate the effect of inhomogeneity on the drift wave vortex and give analytical results. Section V describes the results of numerical investigation. Summary and conclusions are given in Sec. VI.

II. MODEL EQUATION

We consider a plasma in a uniform external magnetic field in the z direction, where the electrons move freely along the magnetic field. The dissipationless equation of motion and continuity equation for the ions are

$$\frac{d\mathbf{v}}{dt} = -\frac{e}{m_i} \nabla\Phi + \omega_{ci} \mathbf{v} \times \hat{\mathbf{z}}, \quad (2)$$

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) = 0, \quad (3)$$

where $d/dt = \partial/\partial t + \mathbf{v} \cdot \nabla$ and $\omega_{ci} = eB/m_i c$ is the ion cyclotron frequency. Combining these two equations, one can easily derive Ertel's theorem by neglecting the parallel compression $\nabla_{\parallel} v_{\parallel}$,

$$\frac{d}{dt} \left(\frac{\omega_{ci} + \omega_z}{n} \right) = 0, \quad (4)$$

where $\omega_z = \hat{\mathbf{z}} \cdot (\nabla \times \mathbf{v})$.

With the ordering,

$$\epsilon_i \equiv \frac{1}{\omega_{ci}} \frac{\partial}{\partial t} \sim \frac{\mathbf{v} \cdot \nabla}{\omega_{ci}} \ll 1,$$

to the lowest order in ϵ_i , we have

$$\mathbf{v} = (e/m_i \omega_{ci}) \hat{\mathbf{z}} \times \nabla\Phi, \\ \omega_z = \omega_{ci} (\rho_s^2/T_e) \nabla^2(e\Phi),$$

and

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \omega_{ci} \frac{\rho_s^2}{T_e} [e\Phi],$$

where $\rho_s = c_s/\omega_{ci}$ and $c_s = [T_e(x)/m_i]^{1/2}$. Now we define $T(x) = T_e(x)/T_0$ (where T_0 is a constant), $\varphi = (r_n/\rho_{s0})e\Phi/T_0$, and $\epsilon_n = \rho_{s0}/r_n$. Equation (4) becomes

$$\frac{\partial}{\partial t} \left(\frac{1 + \epsilon_n \nabla^2 \varphi}{n} \right) + \left[\varphi, \frac{1 + \epsilon_n \nabla^2 \varphi}{n} \right] = 0. \quad (5)$$

The space and time variables are normalized by ρ_{s0} and r_n/c_{s0} , respectively, where $r_n^{-1} = -d \ln n_0/dx$, and ρ_{s0} and c_{s0} are the ion Larmor radius and ion acoustic speed at the reference electron temperature T_0 , respectively. Equation (5) states that the potential vorticity is conserved on each fluid element moving with the velocity $\mathbf{v} = \hat{\mathbf{z}} \times \nabla\varphi(x, y, t)$.

To close Eq. (5), we assume that the particles satisfy the condition of quasineutrality with the electrons obeying the Boltzmann density distribution. Thus, the ion density is given by

$$n = n_0(x) \exp[e\Phi/T_e(x)] = n_0(x) \exp[\epsilon_n \varphi/T(x)], \quad (6)$$

where $n_0(x)$ and $T(x)$ are assumed to be analytic functions. Because of the fact that ϵ_n is a small parameter, for the numerical calculations in Sec. V the exponential in Eq. (6) is expanded to first order in ϵ_n . For certain strong monopole solutions with large negative φ , however, the expansion can lead to negative density states. Thus, whenever the expansion is performed we require that $\epsilon_n \varphi < T$ for all x .

III. TRAVELING WAVE EQUATION AND SOLUTIONS

Now we look for stationary traveling wave solutions of Eq. (5) of the form $\varphi = \varphi(x, y - ut)$. Such solutions travel with the velocity u in the y direction. Equations (5) and (6) together give the following:

$$-u \frac{\partial G}{\partial y} + [\varphi, G] = 0, \quad (7)$$

where

$$G = \frac{1 + \epsilon_n \nabla^2 \varphi}{n_0(x) \exp[\epsilon_n \varphi/T(x)]}.$$

Equation (7) gives the condition for stationarity as

$$[\varphi - ux, G] = 0, \quad (8)$$

which requires that

$$\frac{1 + \epsilon_n \nabla^2 \varphi}{n_0(x) \exp[\epsilon_n \varphi/T(x)]} = F(\varphi - ux), \quad (9)$$

where F is an arbitrary function of its argument.

If we follow Ref. 7 and choose F to be an analytic function determined by $n_0(x)$, then Eq. (9) becomes

$$\epsilon_n \nabla^2 \varphi = \frac{n_0(x)}{n_0(x - \varphi/u)} \exp\left(\frac{\epsilon_n \varphi}{T(x)}\right) - 1, \quad (10)$$

where

$$F(\varphi - ux) = 1/n_0[(ux - \varphi)/u]. \quad (11)$$

With this choice of F the right-hand side of Eq. (10) approaches zero as $|x| \rightarrow \infty$ and $\varphi \rightarrow 0$, which corresponds to the untrapped flow region. Although there exist other analytic choices for F that satisfy $\nabla^2 \varphi \rightarrow 0$ as $|x| \rightarrow \infty$ and $\varphi \rightarrow 0$, which may lead to localized solutions for reasonable density profiles $n_0(x)$, for the strictly exponential profile $n_0(x) = \exp(-\epsilon_n x)$ and the F of Eq. (11) bound solutions do not exist. In this case Eq. (10) becomes the following nonlinear Poisson equation for φ :

$$\epsilon_n \nabla^2 \varphi = \exp\left[\epsilon_n \left(\frac{1}{T(x)} - \frac{1}{u}\right) \varphi\right] - 1 \\ = -\frac{\partial}{\partial \varphi} \left(\varphi - \frac{\exp\{\epsilon_n [1/T(x) - 1/u] \varphi\}}{\epsilon_n [1/T(x) - 1/u]} \right). \quad (12)$$

The effective potential for Eq. (12) is then

$$V_{\text{exp}}(\varphi, x) = \varphi - \frac{\exp\{\epsilon_n [1/T(x) - 1/u] \varphi\}}{\epsilon_n [1/T(x) - 1/u]}. \quad (13)$$

We can easily see that this effective ("time"-dependent) potential $V_{\text{exp}}(\varphi, x)$, for any temperature profile $T(x)$, is simply not able to form a soliton well, and thus bind a local disturbance to form a solitary vortex (monopole or dipole). However, the fact that this choice for F does not produce a localized solution does not exclude the possibility that the dynamics of the plasmas in relaxing from a turbulent state finds another choice of $F(\varphi - ux)$ with a new structure that allows binding.

One example for a binding F structure is to follow the modon construction^{2,3} for the dipole vortex by choosing $F(\varphi - ux)$ to be piecewise linear. For the dipole vortex, F is

the same as Eq. (11) in the untrapped region $r > r_0$ and a different interior function $F_{\text{int}}(\varphi - ux)$ with differing slope $dF/d\varphi$ in the trapped region $r < r_0$. To see the relationship with the dipole vortex solution we consider the logarithm of Eq. (9)

$$\ln(1 + \epsilon_n \nabla^2 \varphi) = \ln F(\varphi - ux) + \ln n_0 + \epsilon_n \varphi / T(x). \quad (14)$$

With $n_0(x) = \exp(-\epsilon_n x)$, $F(\varphi - ux) = \exp[\epsilon_n(x - \varphi/u)]$ as defined by Eq. (11), and the ordering

$$u \sim v_{d0} \equiv v_d(x=0) = 1, \quad e\Phi/T_e \sim \rho_{s0}/r_{n0} \equiv \epsilon_n \sim \epsilon \ll 1, \quad (15)$$

Eq. (14) to lowest order in ϵ yields

$$\nabla^2 \varphi = \left(\frac{1}{T(x)} - \frac{v_{d0}}{u} \right) \varphi, \quad \text{for } r \gg r_0. \quad (16)$$

For the interior ($r < r_0$) solution we choose

$$\ln F_{\text{int}}(\varphi - ux) = -\epsilon_n(1 + p^2)(\varphi - ux), \quad (17)$$

so that, to the lowest order in ϵ , Eq. (14) reduces to

$$\begin{aligned} \nabla^2 \varphi + [1 + p^2 - 1/T(x)]\varphi \\ = [u(1 + p^2) - 1]x, \quad \text{for } r < r_0. \end{aligned} \quad (18)$$

Equation (18) for the interior vortex structure is driven by a source term that measures the mismatch between the vortex speed u and the linear wave speed $1/(1 + p^2)$. The constant $p^2 > 0$ is determined by continuity of φ , $\partial\varphi/\partial r$, and $\nabla^2\varphi$ at $r = r_0$. In the case where $T = \text{const}$ this choice for $F(\varphi - ux)$ produces dipole solutions³ of radius r_0 with an amplitude that increases as $v_{d0}r_0$. Thus, while we have shown that the simplest analytic choice of $F(\varphi - ux)$ forbids the existence of monopole vortices for the strictly exponential density profile, the plasma dynamics in relaxing from a turbulent state in which trapping takes place may naturally produce a $F(\varphi - ux)$ with nonanalyticity at the point of transition between the trapped and untrapped regions. An example from turbulent particle simulation that demonstrates an abrupt change in the slope of $F(\varphi - ux)$ as plotted versus vorticity is shown in Horton *et al.*¹³ However, the dipole vortices have been shown^{6,12} to be structurally unstable, in particular, a small term of KdV type nonlinearity can split them into monopolelike vortices, an effect to which we now turn.

In most magnetic confinement systems both $T_e(x)$ and $v_d(x)$ vary on the scale of r_n . We define the dimensionless parameters $\kappa_T \equiv \rho_{s0} d(1/T)/dx \sim \epsilon$ and $v'_{d0} \equiv \rho_{s0} dv_d/dx \sim \epsilon_n v_{d0}$. In the case of a nonconstant drift velocity $v_d(x)$, with or without a temperature gradient, the analytic choice of $F(\varphi - ux)$ in Eq. (11) can lead to the possibility of trapping and the formation of solitary vortices. Taking the logarithm of Eq. (10)

$$\begin{aligned} \ln(1 + \epsilon_n \nabla^2 \varphi) \\ = \ln n_0(x) - \ln n_0(x - \varphi/u) + \epsilon_n \varphi / T(x) \end{aligned} \quad (19)$$

and expanding the density profile as

$$\ln n_0(x) = -\epsilon_n [v_{d0}x + (v'_{d0}/2)x^2 + \dots] \quad (20)$$

yields

$$\ln(1 + \epsilon_n \nabla^2 \varphi) = \epsilon_n k^2(u, x)\varphi + \epsilon_n (v'_{d0}/2u^2)\varphi^2 + \dots, \quad (21)$$

where

$$k^2(u, x) \equiv 1/T(x) - v_d(x)/u \quad (22)$$

and

$$v_d(x) = 1 + v'_{d0}x. \quad (23)$$

Observe that $v_{d0} = 1$ follows from the choice of space-time units.

With the ordering

$$\rho_{s0}^2 \nabla^2 \sim e\Phi/T_e \sim \kappa_T \sim v'_{d0} \sim \rho_{s0}/r_{n0} \equiv \epsilon_n \sim \epsilon \ll 1$$

and

$$u \sim v_{d0} = 1, \quad (24)$$

and keeping only the terms of order ϵ^2 in Eq. (21), we get

$$\nabla^2 \varphi = k_0^2 \varphi + (v'_{d0}/2u^2)\varphi^2, \quad (25)$$

where we have expanded $k^2(u, x) = k_0^2 + \alpha x + \dots$ and considered $k_0^2 = (1 - v_{d0}/u) \sim \epsilon$ and $\alpha = (\kappa_T - v'_{d0}/u) \sim \epsilon^2$.

Notice that Eq. (25) has the same form as the Petviashvili equation for the steady state.⁵ But the crucial difference is that in the Petviashvili equation the nonlinear term arises due to the temperature gradient, while in the present case the nonlinear term is caused by the gradient of drift velocity. However, the gradient of the drift velocity also gives rise to a linear damping term $-(v'_{d0}/u)x\varphi$ on the right side of Eq. (25), which is the same order as the other terms. The important thing is that although the temperature gradient does not contribute a nonlinear term, its existence can balance the linear damping term caused by the gradient of drift velocity v'_{d0} . In this way the linear damping term is made smaller and exact monopole vortex solutions are possible when $\kappa_T = v'_{d0}/u$ or $\alpha = 0$.

The quadratic nonlinear Poisson Eq. (25) has monopole vortex solutions^{6,8} when $k_0^2 \equiv (1/T - v_d/u)_0 > 0$. The solutions can be approximately^{8,15} written as follows, for small r ,

$$\begin{aligned} \varphi(x, y, t) = -2.4k_0^2(2u^2/v'_{d0}) \\ \times \left\{ \cosh \left[\frac{3}{4}k_0 \sqrt{x^2 + (y - ut)^2} \right] \right\}^{-4/3}. \end{aligned} \quad (26)$$

From Eqs. (25) and (26) we see that the sign of the vortex amplitude no longer depends on the sign of velocity u as is the case for Petviashvili monopoles. Instead it depends on the sign of v'_{d0} . From Eq. (25), we recognize the symmetry relation

$$\varphi(-v'_{d0}; x, y) = -\varphi(v'_{d0}; x, y),$$

which we test in the simulations. The simulation results will be discussed in Sec. V.

We have also tested the new monopole vortex solution given here for solitonlike behavior upon collision. Preliminary results show that the overtaking collision is largely elastic. Details of these studies will be reported in a subsequent work.

The following amplitude-velocity relation can be obtained from Eq. (26):

$$u = \frac{1}{2}(v_{d0} \pm \sqrt{v_{d0}^2 + 0.83|v'_{d0}\varphi_m|}), \quad (27)$$

where φ_m is the amplitude of the monopole vortex given by Eq. (26). However, the monopole vortex solutions given here must propagate in the direction of the drift wave ($u > 0$) since the negative velocity monopoles have $k_0^2 = 1 + v_{d0}/|u| > 1$, which violates the ordering in Eq. (24) required for localization. In this short wavelength region, $r_0 \sim \rho_{s0}$, there are no localized monopole solutions as discussed in Sec. II. Therefore we can only take the positive sign in Eq. (27).

We can use the amplitude φ_m to estimate the maximum or minimum density of the vortex

$$\begin{aligned} n/n_0 &= \exp[\epsilon_n \varphi / T(x)] \approx \exp(\epsilon_n \varphi_m) \\ &= \exp[-(4.8\epsilon_n/v'_{d0})u(u - v_{d0})]. \end{aligned} \quad (28)$$

Deriving Eq. (25), and therefore Eq. (26) from Eq. (21) implies that we have expanded the Boltzmann density distribution of Eq. (6) to first order in ϵ_n and dropped all the higher-order terms. For consistency of the expansion,

$$\epsilon_n |\varphi_m| = 4.8\epsilon_n u(u - v_{d0})/|v'_{d0}| \leq 1.$$

We therefore get the interval condition for the velocity u

$$v_{d0} \leq u \leq \frac{1}{2}(v_{d0} + \sqrt{v_{d0}^2 + 0.83|v'_{d0}|/\epsilon_n}) \quad (29)$$

required for the validity of the expansion. Equation (29) shows that u must be the same order as v_{d0} . This implies that we consider $v'_{d0} \sim \epsilon_n$.

Using a multiple-scale method and a different ordering, namely taking

$$\rho_s^2 \nabla^2 \sim \frac{e\Phi}{T_c} \sim \frac{u}{c_s} \sim \kappa_T \sim k_0^2 \sim \epsilon^2 \quad \text{and} \quad \frac{1}{\omega_{ci}} \frac{\partial}{\partial t} \sim \epsilon^5, \quad (30)$$

Spatschek *et al.*¹² developed a model in which the steady-state equation has a form similar to Eq. (25), but with a

coefficient of the last nonlinear term in Eq. (25) proportional to $\kappa_T \equiv -(1/T)(\partial \ln T / \partial x)$. Although both the equations, our Eq. (25) and Eq. (5) in Ref. 12 have been shown to have monopole solutions,^{5,6,12,15} in reality the vortex wave function will extend into the region where

$$k^2(u, x) = k_0^2 + \alpha x + \dots < 0. \quad (31)$$

When this happens outgoing drift wave propagation occurs. The matching analysis in the following section shows that the amplitude of the outgoing wave is of order φ_{vortex} ($x = x_{\text{crit}}$) where x_{crit} is the point at which $k^2(u, x_{\text{crit}}) = 1/T - v_d/u = 0$. For large scale vortices with core size $r_0 \sim k_0^{-1} > \rho_{s0}/\epsilon^{2/3}$, in inhomogeneous plasma, the coupling to outgoing drift waves is a strong effect that eliminates the existence of the monopole vortex.

IV. RADIATION DAMPING OF SOLITARY DRIFT WAVE VORTEX IN INHOMOGENEOUS PLASMAS

In the small amplitude region exterior to the vortex core the wave field is given, from Eq. (21), by

$$\frac{\partial^2 \varphi_{k_y}(x)}{\partial x^2} + \left[-\left(\frac{1}{T(x)} - \frac{v_d(x)}{u} \right) - k_y^2 \right] \varphi_{k_y}(x) = 0, \quad (32)$$

with $\varphi_{k_y}(x)$ the Fourier transform in y of $\varphi(x, y, t)$ such that

$$\varphi(x, y, t) = \int dk_y \varphi_{k_y}(x) e^{ik_y(y - ut)}.$$

Equation (32) has turning points at $x = x_T$, where the x_T are determined by $v_d(x_T)/u - k_y^2 - 1/T(x_T) = 0$. For $k_y^2 \rightarrow 0$ the $x_T(k_y) \rightarrow x_{\text{crit}}$ defined by $k^2(u, x_{\text{crit}}) = 0$ in Eq. (31).

For $|x| > x_0$ in the exterior region, where x_0 is the length scale of the vortex core given by $k_0 x_0 \simeq 1$, the WKB solutions of Eq. (32) are

$$\varphi_{k_y}(x) = \begin{cases} A_{k_y} Q_{k_y}^{-1/4}(x, u) \exp\left(i \int_{x_T}^x Q_{k_y}^{1/2}(x', u) dx'\right), & Q_{k_y}(x, u) > 0, \\ A_{k_y} [-Q_{k_y}(x, u)]^{-1/4} \exp\left(-\int_{x_T}^x [-Q_{k_y}(x', u)]^{1/2} dx' - \frac{i\pi}{4}\right), & Q_{k_y}(x, u) < 0, \end{cases} \quad (33)$$

where $Q_{k_y}(x, u) = v_d(x)/u - 1/T(x) - k_y^2$ and the amplitude A_{k_y} will be determined by matching to the vortex solution. Equation (33) with $Q_{k_y} > 0$ describes the outgoing radiation wave in the high-density gradient and high-temperature region. In the region where $v_d(x)/u - 1/T(x) < k_y^2$, we again expand $v_d(x)/u - 1/T(x) \simeq -k_0^2 - \alpha x > 0$. Then $x_T \simeq -(k_0^2 + k_y^2)/\alpha$ for $k_0^2 > 0$ and

$$\begin{aligned} \int_{x_T}^x [-Q_{k_y}(x, u)]^{1/2} dx' \\ \simeq [x - (2/3|\alpha|)(k_0^2 + k_y^2)](k_0^2 + k_y^2)^{1/2}, \\ \text{for } |x_0| \ll |x| \ll |x_T|. \end{aligned}$$

The approximate exterior vortex solution can be obtained by neglecting the nonlinear term in Eq. (25),

$$\varphi(x, y, t) \simeq \varphi_m K_0(k_0 r), \quad (34)$$

where $r = [x^2 + (y - ut)^2]^{1/2}$ and $\varphi_m = -4.8k_0^2 u^2 / v'_{d0}$ is the amplitude of the vortex. The Fourier transform of Eq. (34) in y is¹⁶

$$\varphi_{k_y} = \frac{\varphi_m \exp(-|x|\sqrt{k_0^2 + k_y^2})}{2\sqrt{k_0^2 + k_y^2}}. \quad (35)$$

Matching Eqs. (35) and (33) with $Q_{k_y} < 0$, we can obtain the radiation amplitude formula

$$A_{k_y} \approx \frac{\varphi_m}{2(k_0^2 + k_y^2)^{1/4}} \exp\left(i\frac{\pi}{4} - \frac{2}{3|\alpha|} (k_0^2 + k_y^2)^{3/2}\right), \quad (36)$$

which is to be used in Eqs. (33). Therefore the monopole vortex solution of Eq. (25) is coupled to a radiation wave with significant amplitude unless the condition $|\alpha| \ll 2k_0^{3/2}/3 \sim \epsilon^{3/2}$ is satisfied.

We point out that for the case discussed in Ref. 12, where the stationary equation is similar to that of Eq. (25) but with the longer wavelength scaling $k_0 \sim \epsilon$ and $\alpha \sim \epsilon^3$, the monopole vortices should have oscillating tails with the substantial amplitude $\sim [(2/\epsilon)^{1/2} \varphi_m / 4] \exp(-2/3)$ as seen from Eq. (36). The numerical solutions given in their work do not show the oscillating tail since their equation [Eq. (8) in Ref. 12] does not explicitly contain the inhomogeneity. Their numerical results, therefore, are only for the very special case of $\kappa_T = v_{d0}'/u$, i.e., $\alpha = 0$. Therefore the solutions which the authors give are essentially those given earlier by Su *et al.*⁶ for the Petviashvili equation.

The decay rate of energy of the solitary drift wave can be computed by using the energy conservation law,

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \mathbf{S} = 0, \quad (37)$$

where

$$\mathcal{E}(x, y, t) = \frac{1}{2} [\varphi^2 / T(x) + (\nabla \varphi)^2], \quad (38)$$

and

$$\mathbf{S} = \left(\frac{v_d(x)}{2} \varphi^2 - \frac{\kappa_T}{3} \varphi^3 \right) \hat{y} - \varphi \nabla \frac{\partial \varphi}{\partial t} - \nabla^2 \varphi \left(\hat{z} \times \nabla \frac{\varphi^2}{2} \right) \quad (39)$$

are the local energy density and the energy flux, respectively. Integrating Eq. (38) over the space volume, with the help of Eqs. (25), (26), and (34), we obtain the energy for the monopole vortex

$$E_v = \int \mathcal{E} d^3x \approx \frac{8.2\pi L_c u^4 k_0^2}{(v_{d0}')^2} \left(\frac{4}{3} + k_0^2 \right), \quad (40)$$

where L_c is the length scale in z of the coherent vortex structure.

The outgoing wave propagation given by the matching of Eq. (35) to Eq. (33), leads to the decay of the vortex energy E_v . Integrating the energy balance Eq. (37) over the space volume $\int d^3x$, neglecting the last term of Eq. (39) proportional to φ^3 , since φ is exponentially small for $x \gg x_0$, and using Parseval's theorem and Eq. (33) to evaluate the outgoing wave energy flux, gives the decay rate of vortex energy

$$\begin{aligned} \frac{dE_v}{dt} &= \int d^3x \frac{\partial \mathcal{E}}{\partial t} \approx \int dy dz \left[\varphi \frac{\partial^2 \varphi}{\partial x \partial t} \right]_{x=-L}^{x=L} \\ &= -\frac{uL_c}{4\pi} \int_{-\infty}^{+\infty} dk_y i k_y \left[\varphi_{k_y}^*(x) \partial_x \varphi_{k_y}(x) \right. \\ &\quad \left. - \varphi_{k_y}(x) \partial_x \varphi_{k_y}^*(x) \right]_{x=-L}^{x=L} \\ &= -\frac{uL_c}{\pi} \int_0^{\infty} dk_y k_y |A_{k_y}|^2, \quad (41) \end{aligned}$$

which is independent of L for $L \gg x_0$.

Using Eq. (36) for A_{k_y} , we perform the k_y integral to obtain the vortex decay rate

$$\begin{aligned} \frac{dE_v}{dt} &= -\frac{uL_c \varphi_m^2}{4\pi} \int_0^{\infty} dk_y \frac{k_y}{\sqrt{k_0^2 + k_y^2}} \\ &\quad \times \exp\left(-\frac{4}{3|\alpha|} (k_0^2 + k_y^2)^{3/2}\right) \\ &= -\frac{B}{3b^{1/3}} \int_b^{\infty} dt \exp(-t) t^{-2/3} \\ &= -\frac{B}{3b^{1/3}} \Gamma\left(\frac{1}{3}, b\right), \quad (42) \end{aligned}$$

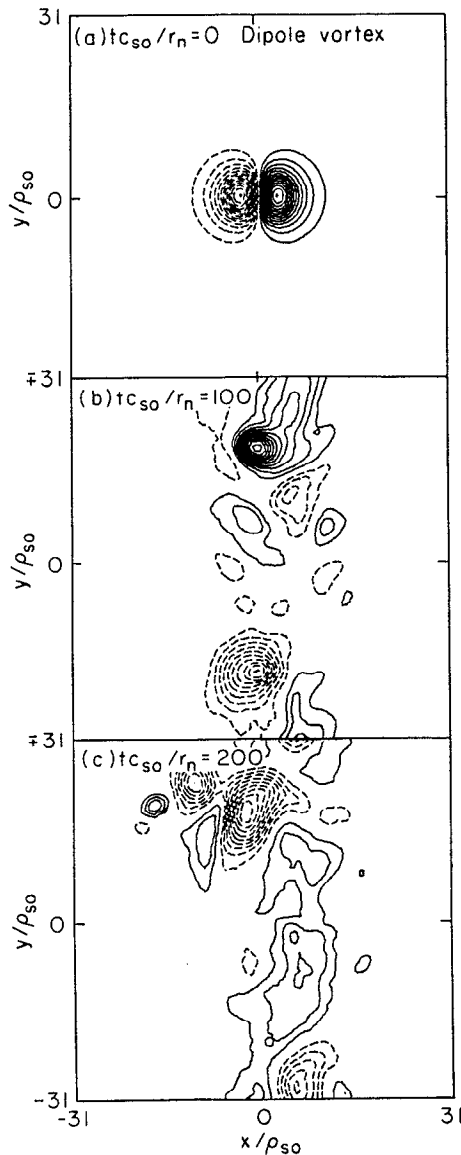


FIG. 1. (a) Contour plot of the electrostatic potential $\varphi(x, y, t)$ of Eq. (45). The dipole vortex (solution of the H-M equation) is taken as initial condition with $T(x) = \exp(-c_2 x)$, $c_2 = 0.046$, $u = 1.1$, $r_0 = 6.0$, and $v_{d0}' = 0.05$. The solid lines represent positive values of potential φ , and the dashed lines are negative values. (b) The dipole vortex is first split into two monopoles, cyclone ($\varphi < 0$) and anticyclone ($\varphi > 0$). (c) Only the cyclone is seen to survive.

where $B = uL_c \varphi_m^2 / 4\pi$, $b = 4k_0^3 / 3|\alpha|$, $t = b(1 + k_y^2 / k_0^2)^{3/2}$, and

$$\Gamma(\xi, \xi) = \int_{\xi}^{\infty} d\tau \exp(-\tau) \tau^{\xi-1}$$

is the incomplete gamma function.¹⁷ For large ξ , one can write

$$\Gamma(\xi, \xi) \approx \xi^{\xi-1} \exp(-\xi).$$

Therefore for small $\alpha \ll 4k_0^3 / 3$, i.e., large b , the decay rate of vortex energy is

$$\frac{dE_v}{dt} \approx -\frac{uL_c \varphi_m^2 |\alpha|}{16\pi k_0^3} \exp\left(-\frac{4k_0^3}{3|\alpha|}\right), \quad (43)$$

valid for $k_0^2 > 0$. Thus the damping is exponentially small provided $k_0^3 > 3\alpha/4$, or in terms of the vortex length scale, $\rho_{s0} \ll r_0 \sim k_0^{-1} \ll (4/3\alpha)^{1/3} \rho_{s0} \sim \rho_{s0} / \epsilon^{2/3}$. The damping time scale τ_L for the decay of the vortex can be estimated by

$$\tau_L = \frac{E_v}{|dE_v/dt|} \approx \frac{2.8\pi^2}{u|\alpha|} \left(\frac{4}{3} + k_0^2\right) \exp\left(\frac{4k_0^3}{3|\alpha|}\right). \quad (44)$$

The theory for the effect of the inhomogeneity in Eqs. (32)–(44) is based on the observation⁴ that for strong vortices (ones with $\Omega_E / \omega_k \gg 1$ where Ω_E is the rotation frequency in the core of vortex and $\omega_k \approx k_y u$), the inhomogeneity causes a leakage of wave energy from the vortex core, but does not strongly alter the interior solution. At some stronger level of inhomogeneity the core of the vortex is changed, and nonperturbative solutions exhibiting the effect of inhomogeneity are required.

V. NUMERICAL VORTEX SOLUTION IN INHOMOGENEOUS PLASMA

In order to facilitate the numerical solutions of Eq. (5), we first expand the equation according to the ordering in Eq. (24) and consider $\epsilon_r \sim \epsilon$. Keeping only the terms of order ϵ and ϵ^2 , we derive from Eq. (5) the reduced dynamical equation

$$\left(\frac{1}{T(x)} - \nabla^2\right) \frac{\partial \varphi}{\partial t} + (v_{d0} + v'_{d0} x - \kappa_T \varphi) \frac{\partial \varphi}{\partial y} - [\varphi, \nabla^2 \varphi] = 0, \quad (45)$$

which is valid only when the condition of Eq. (29) is satisfied. In later work we will present numerical solutions of the full Eq. (5) and compare with those presented here.

Rewriting Eq. (45), we get the conservation of mass in the two-dimensional system,

$$\frac{\partial}{\partial t} \left(\frac{\varphi}{T(x)} \right) + \nabla \cdot \left[-\frac{\partial \nabla \varphi}{\partial t} + \left(v_d(x) \varphi - \frac{\kappa_T \varphi^2}{2} \right) \hat{y} + (\nabla \varphi \times \hat{z}) \nabla^2 \varphi \right] = 0. \quad (46)$$

Multiplying Eq. (45) by φ , we can derive the energy conservation law given by Eqs. (37)–(39) in the previous section. The constants are subsequently used to monitor the solutions.

To solve Eq. (45), we use a uniform grid over x and k_y in 85×85 xk_y space with 3655 complex $\varphi_{x,k_y}(t)$ modes.

Since the first term of Eq. (45) depends on x , transforming Eq. (45) into $k_x k_y$ space would lead to a difficult convolution integral in k_x for the linear problem. Leaving the equation in x space and using the second-order central difference formula for ∂_x^2 , gives a tridiagonal system to solve for each $\partial_t \varphi(x, k_y, t)$. We use the Ahlberg–Nilson–Walsh algorithm for cyclic tridiagonal systems¹⁸ to reduce the operator $[1/T(x) - \nabla^2]$ to a cyclic tridiagonal matrix and then compute $\partial \varphi / \partial t$ for each mode, in which process the nonlinear convolution terms in Eq. (45) are evaluated by first transforming φ in $k_x k_y$ space to get derivatives of φ , then transforming φ and its derivatives into xy space to calculate the convolutions and, after that, transforming the results back into xk_y space. Finally, we use high-order Runge–Kutta time stepping and the Fourier transformation $y \leftrightarrow k_y$ at each time step. The constants of motion defined in Eqs. (37) and (46) are used to monitor the accuracy of the code. The approximate monopole solution Eq. (26) is taken as the initial condition $\varphi(x, y, t = 0)$. The temperature profile is $T(x) = \exp(-c_2 x)$, which avoids the negative temperature problem that arises when expanding $T(x)$ as $1 - c_2 x$. For this profile $\kappa_T = c_2 \exp(c_2 x)$. Typical simulations average 15 min CPU time on the CRAY-II for $\Delta t = 100 r_n / c_s$, which is about 10 rotations of the vortex core.

In the first case we initialized with the dipole vortex (solution of the H–M equation). As seen in Fig. 1 the dipole first separates into two monopoles, a cyclone $\varphi < 0$ and an anticyclone $\varphi > 0$, which was first reported by Su *et al.*⁶ Finally, only one monopole survives, which is determined by the sign of v'_{d0} . If $v'_{d0} > 0$, only the cyclone survives, while if $v'_{d0} < 0$, only the anticyclone survives.

In the second case we used $v'_{d0} = -0.1v_{d0}$, $u = 1.1v_{d0}$, and $c_2 = -0.011$, which give $k_0^2 = (1 - v_{d0}/u) \approx 0.1$ and $\alpha \approx (c_2 - v'_{d0}/u) \approx 0.08$. Therefore α is the same order as k_0^3 . We observe the forming of an oscillating tail with significant amplitude and a strong damping process of the monopole vortex. Figure 2 shows the streamline of $\varphi(x, y, t) = \text{const}$ at times $tc_s/r_n = 0, 40$, and 60, and the projections of their cross sections in the x and y directions. The particle density is, from Eq. (28), $n/n_0 \approx \exp(\epsilon_n \varphi_m) \approx 1.7$.

In the third case $v'_{d0} = 0.05v_{d0}$, $u = 1.1v_{d0}$, and $c_2 = 0.046$, and thus $\alpha \approx 0.0006$ and $k_0^3 \approx 0.027$. We can see in Fig. 3 that the amplitude of the oscillating tail is so insignificant that the monopole vortex keeps its shape for a long time without much damping. The particle density in this case is $n/n_0 \approx 0.58$.

Although in the second and third cases above, we discussed only the anticyclone vortex ($\varphi > 0$) in the second case, and only the cyclone vortex ($\varphi < 0$) in the third case, the same results are obtained for the cyclone vortex in the second case and the anticyclone vortex in the third case, because Eq. (45) possesses the following symmetry relation:

$$\varphi(\kappa_T, v'_{d0}; x, y, t) = -\varphi(-\kappa_T, -v'_{d0}; -x, y, t).$$

In Fig. 4, we used the same parameters and initial condition as those in Fig. 3, but dropped the scalar nonlinear term with coefficient κ_T in Eq. (45). The monopole vortex in this case is seen to connect to a radiation wake.

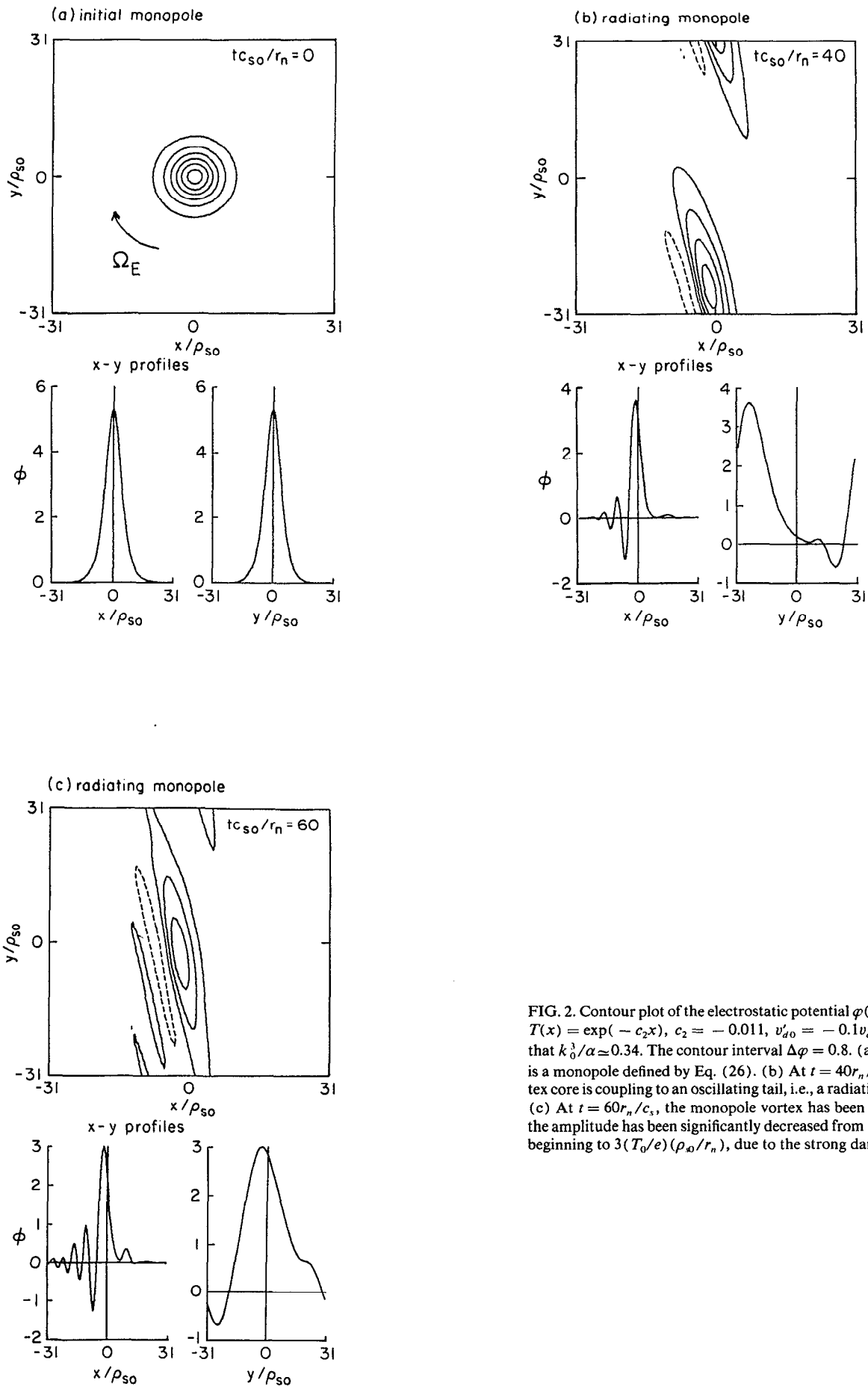


FIG. 2. Contour plot of the electrostatic potential $\phi(x,y,t)$ of Eq. (45) with $T(x) = \exp(-c_2 x)$, $c_2 = -0.011$, $v_{d0} = -0.1v_{d0}$, and $u = 1.1v_{d0}$ so that $k_0^3/\alpha \approx 0.34$. The contour interval $\Delta\phi = 0.8$. (a) The initial condition is a monopole defined by Eq. (26). (b) At $t = 40r_n/c_s$, the monopole vortex core is coupling to an oscillating tail, i.e., a radiative wake of drift waves. (c) At $t = 60r_n/c_s$, the monopole vortex has been strongly distorted and the amplitude has been significantly decreased from $5.28(T_0/e)(\rho_{s0}/r_n)$ at beginning to $3(T_0/e)(\rho_{s0}/r_n)$, due to the strong damping.

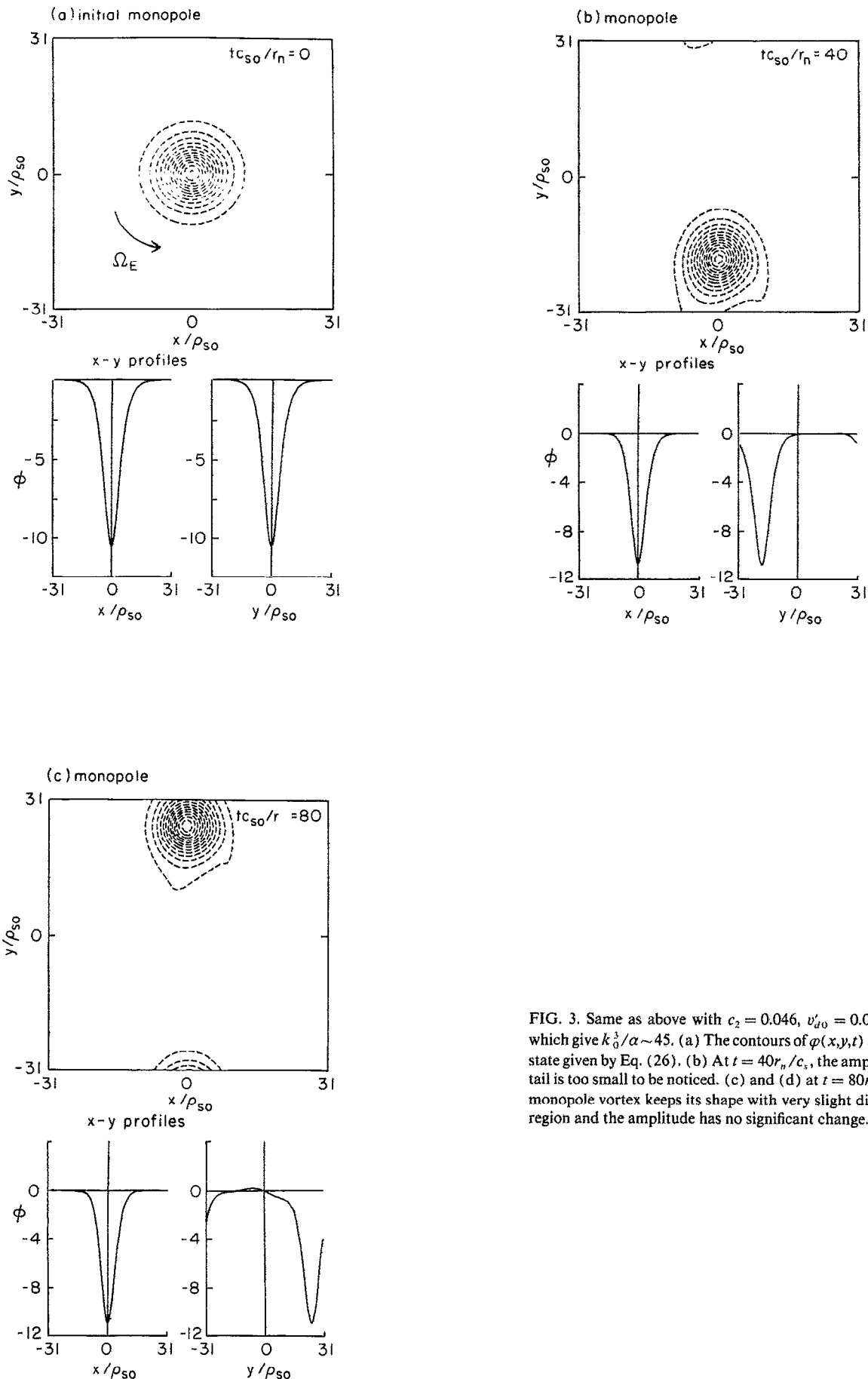


FIG. 3. Same as above with $c_2 = 0.046$, $v'_{d0} = 0.05v_{d0}$, and $u = 1.1v_{d0}$, which give $k_0^3/\alpha \sim 45$. (a) The contours of $\phi(x,y,t)$ at $t = 0$ show the initial state given by Eq. (26). (b) At $t = 40r_n/c_s$, the amplitude of the oscillating tail is too small to be noticed. (c) and (d) at $t = 80r_n/c_s$ and $100r_n/c_s$, the monopole vortex keeps its shape with very slight distortion in the exterior region and the amplitude has no significant change.

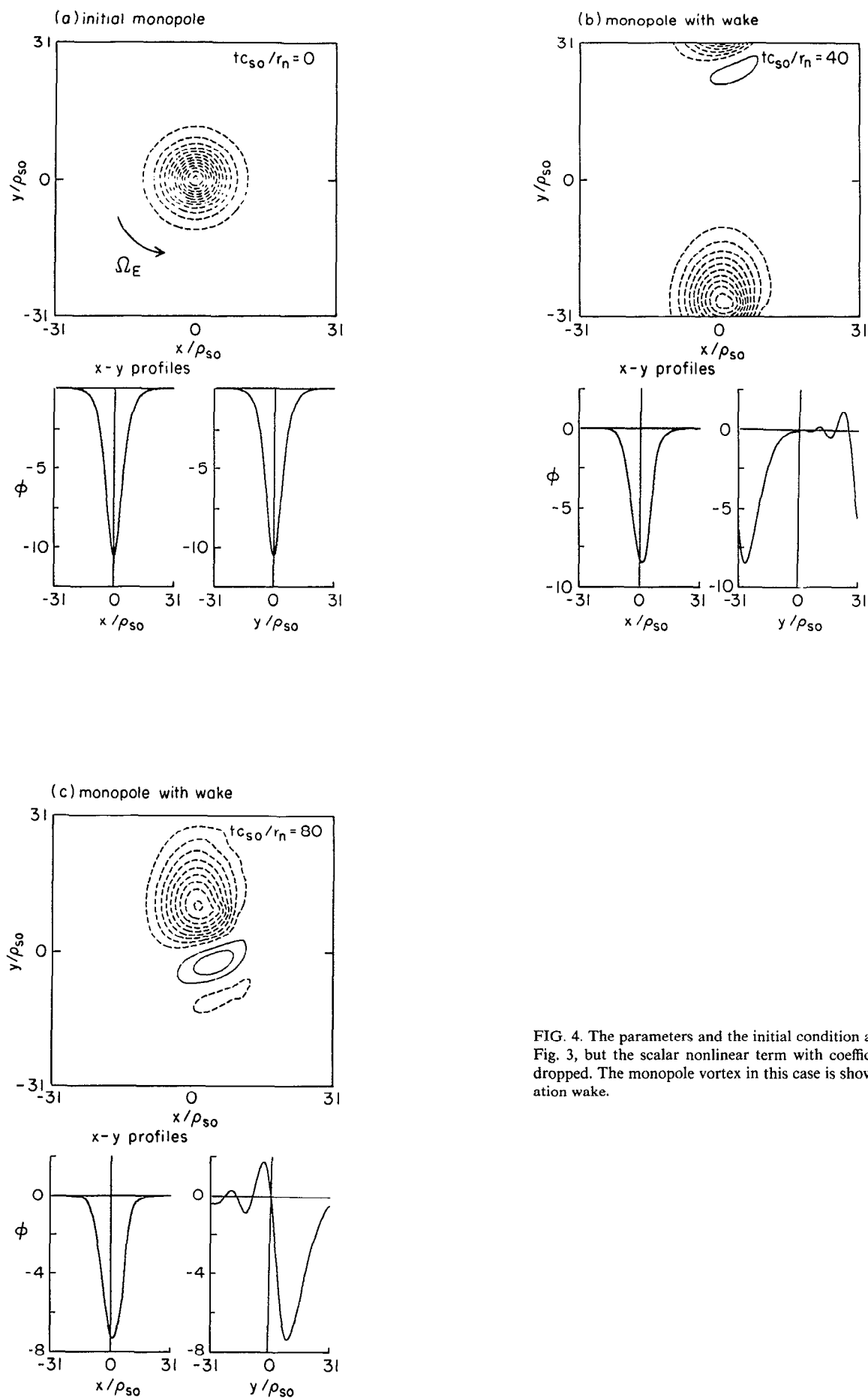


FIG. 4. The parameters and the initial condition are the same as those in Fig. 3, but the scalar nonlinear term with coefficient κ_T in Eq. (45) is dropped. The monopole vortex in this case is shown to connect to a radiation wake.

These numerical results support our argument that when the drift velocity depends on the space variable, monopolelike vortices can exist in the wavelength $\sim (\rho_s r_n)^{1/2}$ region. The simulations also verify that the stability of the vortices is affected by the combined effect of temperature and drift velocity gradients and that the vortices are stable or free of damping only when $\alpha \ll (4/3)k_0^3$.

VI. SUMMARY AND CONCLUSIONS

We have studied the effect of density and temperature gradients on drift wave vortices, both analytically and numerically. The results show that when the drift wave velocity v_d does not depend on the space variable, localized monopole vortices cannot be formed in the long wavelength $\sim (\rho_s r_n)^{1/2}$ region for the choice of F of Eq. (11), no matter what the temperature profile is. When the drift wave velocity is not constant, monopole vortex solutions can be found in the wavelength $\sim \rho_s/\epsilon^{1/2}$ region. However, the important result is that the monopole vortices are *not* the strictly localized monopoles found by many other authors.^{5,7,8,12} Although the gradient of the drift velocity is responsible for the formation of the monopole vortices, its existence also causes the formation of the oscillating tails or radiative wakes of drift waves, which connect to the cores of the vortices. The results show that as long as $\alpha \equiv (\kappa_T - v'_{d0}/u) \neq 0$, the wave energy of the vortices leaks out from the vortex cores through the oscillating tails. The effect of the temperature gradient-induced KdV nonlinearity is to reduce the energy leakage. The analytic and numerical results also show that only when the combined effect of the temperature and the drift velocity gradients satisfies $\alpha \equiv (\kappa_T - v'_{d0}/u) \ll (4/3)(1 - v_{d0}/u)^{3/2}$, will the leakage be small and negligible.

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