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Quantum mechanics as a generalization of Nambu dynamics to the Weyl–Wigner formalism

Iwo Bialynicki-Birula
Institute for Theoretical Physics, Polish Academy of Sciences, Lotników 32/46, 02-668 Warsaw, Poland

and

P.J. Morrison
Department of Physics and Institute for Fusion Studies, University of Texas at Austin, Austin, TX 78712, USA

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It is shown that Nambu dynamics can be generalized to any number of dimensions by replacing the O(3) algebra, a prominent feature of Nambu's formulation, by an arbitrary Lie algebra. For the infinite dimensional algebra of rotations in phase space one obtains quantum mechanics in the Weyl–Wigner representation from the generalized Nambu dynamics. Also, this formulation can be cast into a canonical Hamiltonian form by a natural choice of canonically conjugate variables.

1. Introduction

The purpose of this Letter is to provide a unified basis for various noncanonical Poisson brackets introduced in the past [1–8] and to give a new and important example of such a bracket. This unification is obtained by extending the concept of the triple bracket introduced by Nambu [9] in his generalized version of Hamiltonian dynamics. Our extension of the Nambu dynamics involves the replacement of the Lie algebra of the rotation group in three dimensions, that features prominently in the original formulation, by an arbitrary Lie algebra. When this algebra is chosen to be the (infinite dimensional) Lie algebra associated with the Weyl–Wigner representation, we obtain the phase-space formulation of quantum mechanics. The noncanonical bracket for the Wigner function that results from this formulation reduces in the classical limit to the well-known bracket for the classical distribution function.

Our extension of Nambu dynamics leads to a natural parametrization of the original noncanonical variables in terms of canonically conjugate pairs of new variables. With this parametrization, the noncanonical brackets are transformed into ordinary Poisson brackets. Thus, a phase-space variational principle can be constructed.

2. Lie algebraic basis for Nambu dynamics

We begin by considering a semi-simple Lie algebra with structure constants c^k_{ij} and metric tensor g_{ij} (see, for example, ref. [10]),

$$g_{ij} = -c^k_{il} c^l_{jk} , \tag{1}$$

that is used to raise and lower indices. We have introduced the minus sign here to make g_{ij} positive for the rotation group.

With Lie algebras we can associate dynamical systems whose states are described by the elements $w^i L_i$ of the Lie algebra. The w^i are to be viewed as phase-space coordinates of the system and L_i are the algebra generators. A natural Lie bracket can be constructed from the structure constants as follows:

$$\{A, B\}_c = w^k c_k^{ij} \frac{\partial A}{\partial w^i} \frac{\partial B}{\partial w^j} . \tag{2}$$

The Jacobi identity satisfied by the structure constants,

$$c_{ij}^m c_{mk}^l + c_{jk}^m c_{mi}^l + c_{ki}^m c_{mj}^l = 0 , \tag{3}$$

guarantees the Jacobi identity for the bracket $\{\ ,\ \}_c$,

$$\{A, \{B, C\}_c\}_c + \{B, \{C, A\}_c\}_c + \{C, \{A, B\}_c\}_c = 0 . \tag{4}$$

Such structures were discovered by Sophus Lie (1890) and are known to generate classical Eulerian dynamics of various continuous media [5]. Also, they have been studied in the context of Poisson manifolds [11,12].

The fact that the structure constants have three indices hints at the existence of a geometric bracket operation on three functions. It would be appealing if all three functions appeared on equal footing, which can be achieved by using the fully antisymmetric form of the structure constants [10],

$$c^{ijk} = g^{im} g^{jn} c_{mn}^k . \tag{5}$$

Thus we introduce the following *triple bracket*:

$$[A, B, C] = c^{ijk} \frac{\partial A}{\partial w^i} \frac{\partial B}{\partial w^j} \frac{\partial C}{\partial w^k} . \tag{6}$$

A simple relationship exists between $[A, B, C]$ and $\{A, B\}_c$ which is made manifest by inserting the Casimir of the Lie algebra,

$$S \equiv \tfrac{1}{2} g_{ij} w^i w^j , \tag{7}$$

into one of the slots of the triple bracket, i.e.

$$[A, B, S] = \{A, B\}_c . \tag{8}$$

Due to this relationship time evolution can be represented as follows:

$$\frac{dF}{dt} = [F, H, S] , \tag{9}$$

where F is an arbitrary dynamical variable. In this formulation the dynamics is determined by two generating functions, the Hamiltonian H and the Casimir which we shall call the *entropy*.

In the special case where the structure constants are those of the rotation group in three dimensions, $c^{ijk} = \epsilon_{ijk}$, our construction reduces to that given by Nambu [9],

$$[A, B, C] = \nabla A \cdot (\nabla B \times \nabla C) . \tag{10}$$

In order to describe the dynamics of a rigid rotator Nambu chose S to be the rotational kinetic energy and H to be the Casimir, the square of the total angular momentum. In this way he obtained Euler's equation from his triple bracket (10). It is interesting to note that for this case the roles of the Hamiltonian and the entropy can be interchanged as exemplified by Nambu's choice. However, this is a peculiarity of the rotation group for which any function S in eq. (8) produces a genuine Lie bracket. In general, the roles of S and H are distinct since S must be the Casimir in order for $\{A, B\}_c$ to satisfy the Jacobi identity, and thus S can be viewed as a kinematical object. On the other hand the Hamiltonian H is free to be chosen to generate the dynamics of the system of interest.

3. Triple bracket formulation of quantum mechanics

Now, we generalize the triple bracket formulation of the previous section, which is based on finite dimensional Lie algebra, to accommodate a particular infinite dimensional Lie algebra that we extract from the Weyl–Wigner representation [13,14] of quantum mechanics (for recent references see e.g. ref. [15]). We begin by defining the following family of operators:

$$\hat{E}(\boldsymbol{r}', \boldsymbol{p}') = \int d\Gamma \exp[i(\boldsymbol{r}' \cdot \boldsymbol{p} - \boldsymbol{p}' \cdot \boldsymbol{r})/\hbar] \times \exp[i(\boldsymbol{r} \cdot \hat{\boldsymbol{p}} - \boldsymbol{p} \cdot \hat{\boldsymbol{r}})/\hbar] , \tag{11}$$

where $d\Gamma \equiv d^n r\, d^n p/(2\pi\hbar)^n$, n is the number of dimensions, and the hat is used to indicate operators. The operators $\hat{E}(\boldsymbol{r}, \boldsymbol{p})$ can be viewed as a basis spanning the space of all quantum mechanical operators, a basis from which the Weyl–Wigner representation is derived. The Wigner function is obtained by projecting the density operator $\hat{\rho}$ onto the basis $\hat{E}$, i.e.

$$W(\boldsymbol{r}, \boldsymbol{p}) = \mathrm{Tr}\{\hat{\rho} \hat{E}(\boldsymbol{r}, \boldsymbol{p})\} , \tag{12}$$

$$\hat{\rho} = \int d\Gamma\, W(\boldsymbol{r}, \boldsymbol{p}) \hat{E}(\boldsymbol{r}, \boldsymbol{p}) . \tag{13}$$

In analogy with the finite dimensional case, Wigner functions play the role of coordinates for the Lie algebra spanned by the operators $\hat{E}$. In the simplest case of a pure state, we obtain Wigner's original formula,

$$W(\boldsymbol{r},\boldsymbol{p})=\mathrm{Tr}\{|\Psi\rangle\langle\Psi|\hat{E}(\boldsymbol{r},\boldsymbol{p})\}=\langle\Psi|\hat{E}(\boldsymbol{r},\boldsymbol{p})|\Psi\rangle$$

$$=\int \mathrm{d}^n s\,\exp(-\mathrm{i}\boldsymbol{s}\cdot\boldsymbol{p}/\hbar)\psi(\boldsymbol{r}+\boldsymbol{s}/2)\psi^*(\boldsymbol{r}-\boldsymbol{s}/2)\,. \tag{14}$$

Here we have chosen to work with a dimensionless Wigner function, which requires a normalization that differs from that of other authors (cf. ref. [15]).

The basis operators $\hat{E}(z)$ $(z\equiv(\boldsymbol{r},\boldsymbol{p}))$ obey the following commutator algebra:

$$(\mathrm{i}\hbar)^{-1}[\hat{E}(z_1),\hat{E}(z_2)]=$$

$$\int \mathrm{d}\Gamma_3\; C(z_1,z_2,z_3)\hat{E}(z_3)\,,$$

where the "structure kernel" $C(z_1, z_2, z_3)$ is given by

$$C(z_1,z_2,z_3)=\frac{2\times 4^n}{\hbar}$$

$$\times\sin\left(\frac{2}{\hbar}([z_1z_2]+[z_2z_3]+[z_3z_1])\right), \tag{16}$$

and

$$[z_iz_j]\equiv\boldsymbol{r}_i\cdot\boldsymbol{p}_j-\boldsymbol{p}_i\cdot\boldsymbol{r}_j\,. \tag{17}$$

We have chosen the name structure kernel since $C(z_1, z_2, z_3)$, like c^{ijk}, is antisymmetric under the interchange of its arguments,

$$C(z_1,z_2,z_3)=-C(z_2,z_1,z_3)$$

$$=C(z_2,z_3,z_1)\,, \tag{18}$$

and satisfies the Jacobi identity,

$$\int \mathrm{d}\Gamma\,[C(z_1,z_2,z)C(z,z_3,z')$$

$$+C(z_2,z_3,z)C(z,z_1,z')$$

$$+C(z_3,z_1,z)C(z,z_2,z')]$$

$$=0\,. \tag{19}$$

The structure kernel $C(z_1, z_2, z_3)$ defines a Lie algebra for phase-space functions as follows:

$$\{A,B\}_{\mathrm{M}}(z)=\int \mathrm{d}\Gamma_1\,\mathrm{d}\Gamma_2\,C(z,z_1,z_2)A(z_1)B(z_2)\,. \tag{20}$$

The subscript M is used since this bracket can be shown to be equivalent to

$$\{A,B\}_{\mathrm{M}}(\boldsymbol{r},\boldsymbol{p})$$

$$=\frac{2}{\hbar}A(\boldsymbol{r},\boldsymbol{p})\,\sin[\tfrac{1}{2}\hbar(\overleftarrow{\partial}_r\cdot\overrightarrow{\partial}_p-\overleftarrow{\partial}_p\cdot\overrightarrow{\partial}_r)]B(\boldsymbol{r},\boldsymbol{p})\,, \tag{21}$$

which is the usual form of the Moyal bracket [16]. This algebra generates a bracket on functionals which is an infinite dimensional analogue of the bracket $\{A, B\}_c$ defined by eq. (2),

$$\{\mathscr{A},\mathscr{B}\}_C=\int \mathrm{d}\Gamma_1\,\mathrm{d}\Gamma_2\,\mathrm{d}\Gamma_3\,W(z_1)C(z_1,z_2,z_3)$$

$$\times\frac{\delta\mathscr{A}}{\delta W(z_2)}\frac{\delta\mathscr{B}}{\delta W(z_3)}$$

$$=\int \mathrm{d}\Gamma\,W\left\{\frac{\delta\mathscr{A}}{\delta W},\frac{\delta\mathscr{B}}{\delta W}\right\}_{\mathrm{M}}. \tag{22}$$

Now, in complete analogy with section 2, we introduce the triple bracket,

$$[\mathscr{A},\mathscr{B},\mathscr{C}]=\int \mathrm{d}\Gamma_1\,\mathrm{d}\Gamma_2\,\mathrm{d}\Gamma_3\,C(z_1,z_2,z_3)$$

$$\times\frac{\delta\mathscr{A}}{\delta W(z_1)}\frac{\delta\mathscr{B}}{\delta W(z_2)}\frac{\delta\mathscr{C}}{\delta W(z_3)}, \tag{23}$$

where $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ are arbitrary functionals of W. Following Nambu we obtain the evolution equation for a functional $\mathscr{F}[W]$ in terms of our triple bracket in the form

$$\frac{\mathrm{d}\mathscr{F}}{\mathrm{d}t}=[\mathscr{F},\mathscr{H},\mathscr{S}]\,. \tag{24}$$

The natural choice for the Hamiltonian is $\mathscr{H}$, the energy of the system,

$$\mathscr{H}[W]=\int \mathrm{d}\Gamma\,W(\boldsymbol{r},\boldsymbol{p})H(\boldsymbol{r},\boldsymbol{p})\,, \tag{25}$$

where

$$H(\boldsymbol{r},\boldsymbol{p})=\frac{\boldsymbol{p}^2}{2m}+V(\boldsymbol{r})\,. \tag{26}$$

Similarly, there is a natural choice for the entropy $\mathscr{S}$, since the group of transformations generated by the structure kernel C has a continuous version of the Casimir used in section 2. Indeed, owing to the antisymmetry of C,

$$\{\mathscr{A}, \mathscr{S}\}_C = 0 \tag{27}$$

for all functionals $\mathscr{A}$ if

$$\mathscr{S} \equiv \tfrac{1}{2} \int \mathrm{d}\Gamma\, W^2(z)\,. \tag{28}$$

Eqs. (23)–(25) and (28) generate the usual (cf. e.g. refs. [15,16]) time evolution of the Wigner function, i.e.

$$\frac{\partial W}{\partial t} = -\{W, H\}_{\mathrm{M}}\,. \tag{29}$$

In the classical limit, when $\hbar \to 0$, the Moyal bracket reduces to the standard Poisson bracket and the bracket (22) for functionals of Wigner functions reduces to that for classical distribution functions. Such brackets were previously introduced [3,8] to cast the Vlasov equation into Hamiltonian form.

4. Transformation to a canonical Hamiltonian form

In this section we introduce a general method for expressing noncanonical brackets of the form (2) and (22) in terms of canonically conjugate variables. The method is a generalization of that used to express the Vlasov equation in canonical Hamiltonian form [4]. The new canonical variables, which may be viewed as generalized Clebsch potentials, are related to the original phase-space coordinates by the following expressions, for finite and infinite dimensional cases, respectively:

$$w^i = c_k^{ij} q^k p_j\,, \tag{30}$$

$$W(z) = \int \mathrm{d}\Gamma\, C(z, z_1, z_2) Q(z_1) P(z_2)\,. \tag{31}$$

Using the chain rule

$$\frac{\partial F}{\partial p_j} = \frac{\partial F}{\partial w^i} c_k^{ij} q^k\,, \quad \frac{\partial F}{\partial q^j} = \frac{\partial F}{\partial w^i} c_j^{ik} p_k\,, \tag{32}$$

and the Jacobi identity the following relationship between the two brackets is obtained:

$$\begin{aligned}
&\{A(w(q,p)), B(w(q,p))\}_{qp}(q,p) \\
&\quad \equiv \frac{\partial A}{\partial q^i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q^i} \\
&\quad = w^k c_k^{ij} \frac{\partial A}{\partial w^i}\frac{\partial B}{\partial w^j} \equiv \{A(w), B(w)\}_c(w)\,.
\end{aligned} \tag{33}$$

Similarly,

$$\begin{aligned}
&\{\mathscr{A}[W[Q,P]], \mathscr{B}[W[P,Q]]\}_{QP}[Q,P] \\
&\quad \equiv \int \mathrm{d}\Gamma \left(\frac{\delta\mathscr{A}}{\delta Q(z)}\frac{\delta\mathscr{B}}{\delta P(z)} - \frac{\delta\mathscr{A}}{\delta P(z)}\frac{\delta\mathscr{B}}{\delta Q(z)} \right) \\
&\quad = \int \mathrm{d}\Gamma_1\, \mathrm{d}\Gamma_2\, \mathrm{d}\Gamma_3\, W(z_1) C(z_1, z_2, z_3) \\
&\quad \times \frac{\delta\mathscr{A}}{\delta W(z_2)}\frac{\delta\mathscr{B}}{\delta W(z_3)} \\
&\quad \equiv \{\mathscr{A}[W], \mathscr{B}[W]\}_C[W]\,.
\end{aligned} \tag{34}$$

Since the canonical brackets generate the equations of motion in the usual way, we can also derive these equations form the phase-space action principle. In particular, for the Wigner function the action is

$$\begin{aligned}
S[Q,P] = \int \mathrm{d}t \Bigg(&\int \mathrm{d}\Gamma\, P(z,t)\partial_t Q(z,t) \\
&- \int \mathrm{d}\Gamma_1\, \mathrm{d}\Gamma_2\, \mathrm{d}\Gamma_3\, H(z_1) C(z_1, z_2, z_3) \\
&\times Q(z_2,t) P(z_3,t) \Bigg)\,,
\end{aligned} \tag{35}$$

where Q and P are varied independently. If Q and P satisfy the resulting Hamiltonian equations, the W constructed from eq. (31) obeys eq. (29).

5. Conclusion

We have generalized Nambu dynamics to include arbitrary Lie algebras, even of infinite dimension. An important example of this construction is quantum mechanics in the Weyl–Wigner representation. Hence, already quantized theories fit into this framework and the quantization attempted by Nambu appears to be unnecessary. We have shown how noncanonical brackets arise in a natural way from our triple bracket that are based on structure constants of Lie algebras. Further, we have given a general construction for introducing canonical variables with the corresponding phase-space action principle. In this way we have obtained the action principle for the Wigner function.

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