Poisson bracket for the Vlasov equation on a symplectic leaf

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The degeneracy in the Lie–Poisson bracket, associated with the Hamiltonian structure of the Vlasov equation, is removed by restriction to a given symplectic leaf. The restricted equation of motion is written in terms of a generating function and it manifestly preserves the Casimir constraints of the system. A nondegenerate Poisson bracket in terms of the generating function is presented.

It is by now well known that many nondissipative continuous systems possess a Hamiltonian structure, which when viewed in terms of Eulerian variables has a noncanonical form. Examples from plasma physics include ideal magnetohydrodynamics (MHD) [1], the Vlasov equation [2], the two-fluid equations [3], and the BBGKY hierarchy [4]. A common feature of all these systems is that they possess Casimir invariants due to the degeneracy of their Poisson structure. These invariants foliate the phase space into submanifolds, called the *symplectic leaves*, which are invariant under dynamics. (A symplectic leaf is the phase space of an ordinary Hamiltonian system.) Thus it is of interest to study the evolution equations restricted to a single leaf. For instance, points on a leaf are dynamically accessible, subject only to energy and momentum constraints: this can be important when one uses statistical mechanical techniques. Another reason is that the restricted equation with its Casimir constraints removed is naturally variational, while the original equation can only be made so with the help of some ad hoc tricks [5]; this fact is important in deriving energy principles and performing stability analyses [6]. Recently Crawford and Hislop found a restriction for particular equilibria of the Vlasov equation in one dimension [7]. Here we generalize their result to arbitrary equilibria in three dimensions, and also derive explicit expressions for the Poisson bracket for the Vlasov equation on a symplectic leaf. First, we briefly review the Hamiltonian structure of the Vlasov equation in order to establish our notation.

The Vlasov equation is usually written as a partial differential equation on the particle phase space z:

$$\frac{\partial f}{\partial t}+[f, H]=0\,, \tag{1}$$

where $f(z, t)$ is the particle distribution function, $H(z, t)$ is the single particle Hamiltonian, and $[\ ,\ ]$ is the Poisson bracket. In terms of the canonical variables $z=(\boldsymbol{q}, \boldsymbol{p})$ the Poisson bracket takes the familiar form

$$[F, G]=\frac{\partial F}{\partial \boldsymbol{q}}\cdot\frac{\partial G}{\partial \boldsymbol{p}}-\frac{\partial F}{\partial \boldsymbol{p}}\cdot\frac{\partial G}{\partial \boldsymbol{q}}\,. \tag{2}$$

On the other hand, if one considers the physical observables $\mathscr{F}[f]$, which are functionals of the distribution function, one can show that their evolution obeys a Hamiltonian equation [2]

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$$\frac{\mathrm{d}\mathscr{F}}{\mathrm{d}t}=\{\mathscr{F},\mathscr{H}\}\,, \tag{3}$$

which is equivalent to the Vlasov equation (1). Here $\mathscr{H}$ is the Hamiltonian functional which satisfies $\delta\mathscr{H}/\delta f=H$, and $\{\ ,\ \}$ is a Lie–Poisson bracket, defined by

$$\{\mathscr{F}_1,\mathscr{F}_2\}[f]=\int \mathrm{d}\Gamma f\left[\frac{\delta\mathscr{F}_1}{\delta f},\frac{\delta\mathscr{F}_2}{\delta f}\right], \tag{4}$$

where $\mathrm{d}\Gamma$ denotes the Liouville measure on the particle phase space, e.g., $\mathrm{d}\Gamma=\mathrm{d}^3\boldsymbol{q}\,\mathrm{d}^3\boldsymbol{p}$ in canonical coordinates.

A striking feature of the Lie–Poisson bracket (4) is its infinite degeneracy: if we consider observables of the form

$$\mathscr{C}[f]=\int \mathrm{d}\Gamma\, C(f)\,, \tag{5}$$

where $C(f)$ is an arbitrary smooth function, then it is obvious that $\mathscr{C}$ commutes with any functional of f. Therefore such observables are conserved for any Hamiltonian $\mathscr{H}$:

$$\frac{\mathrm{d}\mathscr{C}}{\mathrm{d}t}=\{\mathscr{C},\mathscr{H}\}=\int \mathrm{d}\Gamma f[C'(f),H]=0\,. \tag{6}$$

These conserved quantities are known as the *Casimirs*. They define a foliation of the space of distribution functions into invariant submanifolds, which are symplectic by the Kirillov–Kostant–Souriau theorem [8]. Each of these submanifolds or symplectic leaves can be characterized as a group orbit. The characterization is determined as follows: given an initial distribution $f_0(z)$, there is a unique symplectic leaf that passes through it. Let Λ denote a canonical transformation of the particle phase space, then the points f having the form $f=f_0\circ\Lambda$ are on the same leaf as f_0. Thus we say that the group of canonical transformations generates the leaf that passes through f_0. It has a subgroup, called the *isotropy group* of f_0, for which $f=f_0$. Therefore in order for the group action and the leaf to have one-to-one correspondence, we must "mod out" this isotropy subgroup. Following ref. [7], we can represent a group element that is connected to the identity by a Lie series: $\Lambda=\mathrm{e}^{L_W}$, where $L_W\equiv[W,\ ]$ and W is a generating function. Thus a point on the leaf near f_0 can be written as

$$f=\mathrm{e}^{L_W}f_0\,. \tag{7}$$

A function that commutes with f_0 corresponds to an element of the isotropy subgroup. We call the set of all such functions the *isotropy kernel*. In order to use the generating function W as a local coordinate system on the leaf, we must keep it outside the isotropy kernel, i.e., $L_Wf_0\neq0$. We remark that not all elements of the group of canonical transformations can be represented by exponential maps, e.g., there are transformations that are not connected to the identity. But for initial value problems, where $f=f_0$ at $t=0$, $\Lambda(0)$ must be the identity. Representations of the form of (7) are thus sufficiently accurate to describe Vlasov dynamics.

Now we consider dynamics on the leaf. Our goal is to replace the Vlasov equation (1) by an equation for W, where now W is dependent on time. Our method is based on the following operator identity [9]:

$$\frac{\partial}{\partial t}\mathrm{e}^{L_W}=L_{\alpha(L_W)\partial_tW}\mathrm{e}^{L_W}\,, \tag{8}$$

where

$$\alpha(z)\equiv\int_0^1 \mathrm{d}\theta\,\mathrm{e}^{\theta z}=\frac{\mathrm{e}^z-1}{z} \tag{9}$$

is an entire function. We can derive (8) from the simpler identity

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\mathrm{e}^{\theta L_W}=\mathrm{e}^{\theta L_W}L_W\,. \tag{10}$$

Differentiating (10) with respect to time yields

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\partial}{\partial t}\mathrm{e}^{\theta L_W}\right)=\left(\frac{\partial}{\partial t}\mathrm{e}^{\theta L_W}\right)L_W+\mathrm{e}^{\theta L_W}L_{\partial_tW}\,, \tag{11}$$

which can be rearranged into

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left[\left(\frac{\partial}{\partial t}\mathrm{e}^{\theta L_W}\right)\mathrm{e}^{-\theta L_W}\right]=\mathrm{e}^{\theta L_W}L_{\partial_tW}\,\mathrm{e}^{-\theta L_W}=L_{\mathrm{e}^{\theta L_W}(\partial_tW)}\,. \tag{12}$$

The last equality follows from the fact that the Poisson bracket is not changed by a canonical transformation:

$$e^{\theta L_W}[F,G]=[e^{\theta L_W}F,e^{\theta L_W}G]\,. \tag{13}$$

Upon integrating (12) with respect to θ from 0 to 1, we obtain eq. (8). Also note that this identity remains true if ∂_t is replaced by a general variation δ, a fact that will be used later in deriving the leaf Poisson bracket.

Applying (8) to (7) yields

$$\frac{\partial f}{\partial t}=[\alpha(L_W)\,\partial_t W,f]\,, \tag{14}$$

which shows that $\partial f/\partial t$ is tangent to the leaf. (Generally a vector δf tangent to the leaf at point f has the form $[f,G]$, where G can be any smooth function. See refs. [7,6].) Substituting (14) into the Vlasov equation (1) and again using eq. (13) yields

$$e^{L_W}[e^{-L_W}(\alpha(L_W)\,\partial_t W-H),f_0]=0\,. \tag{15}$$

We see that the first factor in the square brackets must commute with f_0; denote this factor by C, an arbitrary function which satisfies $\{C,f_0\}=0$. In the general case when f_0 has no special symmetry, we have $C=C(f_0)$. Thus we arrive at the Vlasov equation in terms of W:

$$\frac{\partial W}{\partial t}=\beta(L_W)\,H+\beta(-L_W)\,C\,, \tag{16}$$

where $\beta(z)\equiv 1/\alpha(z)$, or more explicitly:

$$\beta(z)\equiv\frac{z}{e^z-1}=\sum_{n=0}^{\infty}B_n\frac{z^n}{n!}\,, \tag{17}$$

where B_n are the Bernoulli numbers. Clearly $\beta(z)$ is analytic near the real axis. The role of the function C in (16) is to keep W away from the isotropy kernel: it should be chosen so that $[\partial_t W,f_0]\neq 0$.

For the one-dimensional case considered in ref. [7], $f_0=f_0(v)$ was assumed to be an equilibrium, and $H=\frac{1}{2}v^2-\phi(x)$ (we have set $e=m=1$). So C can be any function of v. Letting $C=C_0+C_1+C_2+C_3+...$, and upon expanding (16) in a power series, yields

$$\begin{aligned}\frac{\partial W}{\partial t}&=(\tfrac{1}{2}v^2+C_0)\\&\quad+(C_1-\phi-\tfrac{1}{2}L_W(\tfrac{1}{2}v^2-C_0))\\&\quad+(C_2+\tfrac{1}{2}L_W(C_1+\phi))\\&\quad+(C_3+\tfrac{1}{2}L_WC_2+\tfrac{1}{12}L_W^2(C_1-\phi)+...\,.\end{aligned} \tag{18}$$

Choosing C_0, C_1, etc. to remove the x-independent part of the right-hand side order by order, we find

$$\begin{aligned}&C_0=-\tfrac{1}{2}v^2,\quad C_1=0\,,\\&C_2=-\tfrac{1}{2}\langle L_W\phi\rangle,\quad C_3=\tfrac{1}{12}\langle L_W^2\phi\rangle,\quad ...\end{aligned} \tag{19}$$

where $\langle\ \rangle$ stands for x-averaging. Therefore

$$\begin{aligned}\frac{\partial W}{\partial t}&=-\phi-\tfrac{1}{2}L_Wv^2+\tfrac{1}{2}(L_W\phi-\langle L_W\phi\rangle)\\&\quad-(\tfrac{1}{4}L_W\langle L_W\phi\rangle+\tfrac{1}{12}(L_W^2\phi-\langle L_W^2\phi\rangle))+...\,.\end{aligned} \tag{20}$$

This result is the same as that of eq. (31) of ref. [7] (except for a few misprints therein).

Now let us turn to the Poisson bracket for the leaf equation (16). By (7) we can regard any functional of f, $\mathscr{F}[f]$, also as a functional of W: $\hat{\mathscr{F}}[W]=\mathscr{F}[f]$. Using (8) (see the comment at the end of that paragraph) we obtain, similar to (14),

$$\delta f=[\alpha(L_W)\,\delta W,f]\,; \tag{21}$$

then by the chain rule, i.e., upon equating $\delta\hat{\mathscr{F}}[W;\delta W]=\delta\mathscr{F}[f;\delta f]$, we find

$$\frac{\delta\hat{\mathscr{F}}}{\delta W}=\alpha(L_W)\,L_{f_0}\,e^{-L_W}\left(\frac{\delta\mathscr{F}}{\delta f}\right). \tag{22}$$

Note that for a Casimir $\mathscr{C}$ we have $\delta\hat{\mathscr{C}}/\delta W=0$. Since f_0 is known, we can solve this equation for $\delta\mathscr{F}/\delta f$. Formally we denote the inverse of L_{f_0} by $L_{f_0}^{-1}$; in practice we need to solve the equations of motion with f_0 acting as Hamiltonian. There is an arbitrary function in the solution which commutes with f_0, coinciding with the function C in eq. (16). Thus

$$\frac{\delta\mathscr{F}}{\delta f}=e^{L_W}L_{f_0}^{-1}\beta(L_W)\,\frac{\delta\hat{\mathscr{F}}}{\delta W}+e^{L_W}C\,. \tag{23}$$

Note that $[C,f_0]=0$ implies $[e^{L_W}C,f]=0$, meaning that the second term in the above equation is the component transverse to the leaf, so it does not contribute to the Lie–Poisson bracket (4). Upon substituting (23) into (4) we obtain

$$\begin{aligned}&\{\hat{\mathscr{F}}_1,\hat{\mathscr{F}}_2\}_{f_0}[W]\\&\quad=\int d\Gamma\,\frac{\delta\hat{\mathscr{F}}_1}{\delta W}\,(\beta(-L_W)\,L_{f_0}^{-1}\beta(L_W))\,\frac{\delta\hat{\mathscr{F}}_2}{\delta W}\,.\end{aligned} \tag{24}$$

This Poisson bracket is non-degenerate in the sense that, if $\{\hat{\mathscr{F}}_1, \hat{\mathscr{F}}_2\}=0$ for all $\hat{\mathscr{F}}_1$, then we must have $\delta\hat{\mathscr{F}}_2/\delta W=0$. Any invariants of eq. (16) therefore come from the symmetries in the Hamiltonian.

The Lie series representation (7) has the advantage of being coordinate independent; in particular it does not require canonical variables. However, the formal power series can be cumbersome in practice. In the following we develop another version of our results by using a mixed-variable generating function, which requires canonical variables but can be easier to manipulate.

Let the canonical transformation A be generated by $S(\boldsymbol{q}, \boldsymbol{P}, t)$:

$$\boldsymbol{p}=\frac{\partial S}{\partial \boldsymbol{q}}, \quad \boldsymbol{Q}=\frac{\partial S}{\partial \boldsymbol{P}}, \tag{25}$$

where it is assumed that the Jacobian matrix of the transformation is non-degenerate:

$$\omega_{ij}(\boldsymbol{q}, \boldsymbol{P}, t) \equiv \frac{\partial^2 S}{\partial q_i\, \partial P_j}, \quad \det(\omega) \neq 0\,. \tag{26}$$

Here to be explicit we use the F_2-type generating function, but our method below can be adapted without difficulty to other types of generating functions. Locally one can always find a generating function that satisfies the non-degeneracy condition similar to (26) [10]. It is convenient to work in the mixed-variable space $(\boldsymbol{q}, \boldsymbol{P})$, denoted by subscript m. The distribution functions in various spaces are related to each other, through eq. (25), by

$$f(\boldsymbol{q}, \boldsymbol{p}, t) \equiv f_0(\boldsymbol{Q}, \boldsymbol{P}) \equiv f_m(\boldsymbol{q}, \boldsymbol{P}, t)\,. \tag{27}$$

The particle Poisson bracket (2) becomes

$$[F_m, G_m]=J_{ij}\left(\frac{\partial F_m}{\partial q_j}\frac{\partial G_m}{\partial P_i}-\frac{\partial F_m}{\partial P_i}\frac{\partial G_m}{\partial q_j}\right), \tag{28}$$

where the convention of summing over repeated indices is used, and $\mathbf{J}(\boldsymbol{q}, \boldsymbol{P}, t)$ is the inverse Jacobi matrix: $J_{ij}\omega_{jk}=\delta_{ik}$. Now we calculate $\partial f/\partial t$ with $(\boldsymbol{q}, \boldsymbol{p})$ held fixed. Differentiating (25) holding $(\boldsymbol{q}, \boldsymbol{p})$ fixed yields

$$\left(\frac{\partial \boldsymbol{P}}{\partial t}\right)_{(\boldsymbol{q},\boldsymbol{p})} = -\mathbf{J}\cdot\frac{\partial^2 S}{\partial \boldsymbol{q}\, \partial t},$$

$$\left(\frac{\partial \boldsymbol{Q}}{\partial t}\right)_{(\boldsymbol{q},\boldsymbol{p})} = \frac{\partial^2 S}{\partial \boldsymbol{P}\, \partial t}+\frac{\partial^2 S}{\partial \boldsymbol{P}\, \partial \boldsymbol{P}}\cdot\left(\frac{\partial \boldsymbol{P}}{\partial t}\right)_{(\boldsymbol{q},\boldsymbol{p})}. \tag{29}$$

Similarly differentiating (27) we obtain

$$\frac{\partial f_0}{\partial \boldsymbol{Q}}=\mathbf{J}\cdot\frac{\partial f_m}{\partial \boldsymbol{q}},$$

$$\frac{\partial f_0}{\partial \boldsymbol{P}}=\frac{\partial f_m}{\partial \boldsymbol{P}}-\frac{\partial^2 S}{\partial \boldsymbol{P}\,\partial \boldsymbol{P}}\cdot\frac{\partial f_0}{\partial \boldsymbol{Q}}. \tag{30}$$

Together they lead to the following equation which is an analog of (14):

$$\left(\frac{\partial f}{\partial t}\right)_{(\boldsymbol{q},\boldsymbol{p})} = \frac{\partial f_0}{q\boldsymbol{Q}}\cdot\left(\frac{\partial \boldsymbol{Q}}{\partial t}\right)_{(\boldsymbol{q},\boldsymbol{p})}+\frac{\partial f_0}{\partial \boldsymbol{P}}\cdot\left(\frac{\partial \boldsymbol{P}}{\partial t}\right)_{(\boldsymbol{q},\boldsymbol{p})} = \left[f_m, \frac{\partial S}{\partial t}\right]. \tag{31}$$

We remark again that this relation still holds if we replace the time derivative by a generic variation. On the other hand we have

$$[f, H]=[f_m, H_m]\,,$$

$$H_m(\boldsymbol{q}, \boldsymbol{P}, t) \equiv H\left(\boldsymbol{q}, \frac{\partial S}{\partial \boldsymbol{q}}, t\right). \tag{32}$$

Hence the Vlasov equation in the mixed-variable space reads

$$\left[f_m, \frac{\partial S}{\partial t}+H_m\right]=0\,. \tag{33}$$

The second factor in the square brackets must commute with f_m. Let $C(\boldsymbol{Q}, \boldsymbol{P})$ be an arbitrary function that commutes with $f_0(\boldsymbol{Q}, \boldsymbol{P})$, then we arrive at the equation of motion in terms of S:

$$\frac{\partial S}{\partial t}+H\left(\boldsymbol{q}, \frac{\partial S}{\partial \boldsymbol{q}}, t\right)=C\left(\frac{\partial S}{\partial \boldsymbol{P}}, \boldsymbol{P}\right), \tag{34}$$

which is equivalent to (16). This modified Hamilton–Jacobi equation was first introduced by Pfirsch and Morrison [11]. It can also be derived directly from an action principle [5].

Employing the same procedure as before we can derive the leaf Poisson bracket in terms of S. Here we only display the result:

$$\{\hat{\mathscr{F}}_1, \hat{\mathscr{F}}_2\}_{f_0}[S] = \int \mathrm{d}\Gamma_m \frac{\delta\hat{\mathscr{F}}_1}{\delta S} (L_{f_m}^{-1}) \frac{\delta\hat{\mathscr{F}}_2}{\delta S}, \tag{35}$$

where $\mathrm{d}\Gamma_m = \mathrm{d}^3\boldsymbol{q}\, \mathrm{d}^3\boldsymbol{P} \det(\omega)$ is the Liouville measure in the mixed-variable space.

In conclusion, we have derived the Vlasov equation on a symplectic leaf, where all points are now presumably dynamically accessible, subject only to energy and momentum constraints. We also found explicit expressions for the Poisson bracket for this equation, and showed it to be nondegenerate. For those readers familiar with geometrical aspects of the problem, the Poisson bracket on a leaf is the cosymplectic form of the Kirillov–Kostant–Souriau symplectic structure. Similar methods are expected to apply to other nondissipative models that describe fluids and plasmas.

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