Action principles for the Vlasov equation

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Five action principles for the Vlasov–Poisson and Vlasov–Maxwell equations, which differ by the variables incorporated to describe the distribution of particles in phase space, are presented. Three action principles previously known for the Vlasov–Maxwell equations are altered so as to produce the Vlasov–Poisson equation upon variation with respect to only the particle variables, and one action principle previously known for the Vlasov–Poisson equation is altered to produce the Vlasov–Maxwell equations upon variations with respect to particle and field variables independently. Also, a new action principle for both systems, which is called the leaf action, is presented. This new action has the desirable features of using only a single generating function as the dynamical variable for describing the particle distribution, and manifestly preserving invariants of the system known as Casimir invariants. The relationships between the various actions are described, and it is shown that the leaf action is a link between actions written in terms of Lagrangian and Eulerian variables.

I. INTRODUCTION

In the past, several action principles for the Vlasov–Maxwell equations have been presented. In 1958 Low[1] generalized the action principle for particles in a self-consistent electromagnetic field by using a Lagrangian displacement variable to describe a continuum of particles. Upon independent variations of this action with respect to the particle variable and the scalar and vector potentials, i.e., the four-potential, evolution equations equivalent to the Vlasov–Maxwell equations were obtained. Further discussion, extension, and application of this action can be found in Refs. 2–4. Inasmuch as the *Low action* is a continuum generalization of Hamilton's principle of the classical mechanics, which produces Lagrange's equations of motion, another action principle is based upon the *phase-space action principle*, which produces Hamilton's equations of motion. For this action the particle continuum is described by the Lagrangian displacement variable and its canonically conjugate momentum, while the fields are described by the four-potential, just as in the case of the Low action. The phase-space action has been applied in the context of self-consistent guiding-center and oscillation-center dynamics.[5–7] A third action, called the *Hamilton–Jacobi action*, was introduced in Refs. 8–10. This action describes the particle distribution by two functions, a mixed-variable generating function and a density function, while the fields are again described by the four-potential. Variations with respect to these two particle functions yield equations equivalent to the Vlasov equation, and variation with respect to the four-potential produces Maxwell's equations in the usual way.

Unlike the Low and phase-space actions that describe the particles in terms of Lagrangian variables, the Hamilton–Jacobi action can be viewed as a mixed Lagrangian–Eulerian variable description.[11] However, there is yet another way to describe the particle distribution in a purely Eulerian manner, by the introduction of two functions known as Clebsch potentials.[12–14] These two functions are canonically conjugate Hamiltonian variables and thus lead in a natural way to another action principle that we call the *Clebsch action*.

In this paper we present a new action principle for the Vlasov equation. We call this new action the *leaf action* because the theory is described in terms of a generating function whose dynamics manifestly preserves the so-called *Casimir invariants* of the Vlasov system (which are associated with conservation of phase-space volume), and because the constraint surfaces determined by these invariants are known as the *symplectic leaves*. In addition to preserving all the Casimir invariants, the leaf action has the novel and desirable feature of being variational with a single function as the dynamical variable. Generally speaking, a single function description of the particles has advantages for application of approximation methods employing trial functions, since one needs only guess one function.

For each of the action principles for the Vlasov–Maxwell equations mentioned above, there exists an alteration that results in an action principle for the Vlasov–Poisson equation, an alteration that to our knowledge is new. In these altered action principles the magnetic field is omitted and the electrostatic potential is determined self-consistently by solving the Poisson equation in terms of the Green's function. The electric field part of the action is then expressed in terms of the particle variables, resulting in a change in the particle Hamiltonian [e.g., the factor of $\frac{1}{2}$ in the last terms of Eqs. (11), (13), (19), and (20) below].

The natural question of how the five action principles are related arises. In the past only the usual Legendre transform relationship between the Low and phase-space actions was known. However, one expects that there exist transformations between all the various quantities used to describe the particle dynamics. Here we will discuss the relationships between the various actions for the Vlasov–Poisson equation; we will show that the leaf action provides a link between

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the actions written in terms of Lagrangian and Eulerian variables. Since these transformations involve only the particle variables, they apply to the actions for the Vlasov–Maxwell equations as well.

The paper is organized as follows. The remainder of the Introduction reviews the Hamiltonian structure of the Vlasov–Poisson equation, and the concepts of Casimir invariants and symplectic leaves. This background material facilitates the discussion of the leaf action. In Sec. II we describe the self-consistent Vlasov–Poisson action principles obtained by altering the Low, phase-space, Clebsch, and Hamilton–Jacobi actions. Section III is devoted to the discussion of the new leaf action in the context of the Vlasov–Poisson equation. In Sec. IV the relationships between the five actions is explored in detail. In Sec. V, we display the action principles for the Vlasov–Maxwell equations. The Low, phase-space, and Hamilton–Jacobi actions were previously given, but are included here for completeness. The Clebsch and leaf actions for the Vlasov–Maxwell equations are new. Finally, in Sec. VI we conclude.

It is well known by now that the Vlasov equation is a Hamiltonian system with the so-called Lie–Poisson bracket structure. Traditionally the Vlasov equation is written as a partial differential equation in the six-dimensional particle phase space [we use $\mathbf{z}\equiv(\mathbf{q},\mathbf{p})$ to denote the phase-space coordinates, and suppress the species label]:

$$\frac{\partial f}{\partial t}+[f,H]=0, \tag{1}$$

where $f(\mathbf{z},t)$ is the smooth Vlasov distribution function, $H(\mathbf{z},t)=\mathbf{p}^2/2m+e\phi(\mathbf{q},t)$ is the single particle Hamiltonian, and

$$[f,g]=\frac{\partial f}{\partial \mathbf{q}}\cdot\frac{\partial g}{\partial \mathbf{p}}-\frac{\partial f}{\partial \mathbf{p}}\cdot\frac{\partial g}{\partial \mathbf{q}} \tag{2}$$

is the canonical Poisson bracket. Now we can also consider the system in terms of the (infinite-dimensional) space of distributions, and the physical observables $\mathscr{F}[f]$, which are functionals on this space. It can be shown[13,15] that the Vlasov equation (1) is equivalent to the Hamiltonian equation

$$\frac{d\mathscr{F}}{dt}=\{\mathscr{F},\mathscr{H}\}, \tag{3}$$

where $\mathscr{H}$ is the system Hamiltonian whose functional derivative is H, i.e., $\delta\mathscr{H}/\delta f=H$, and $\{\cdot,\cdot\}$ is a Lie–Poisson bracket, defined by

$$\{\mathscr{F}_1,\mathscr{F}_2\}=\int d^6\mathbf{z}\, f\left[\frac{\delta\mathscr{F}_1}{\delta f},\frac{\delta\mathscr{F}_2}{\delta f}\right]. \tag{4}$$

Considering the distribution itself to be a functional, $f(\mathbf{z},t)=\int d^6z'\, f(\mathbf{z}',t)\delta(\mathbf{z}-\mathbf{z}')$, Eq. (4) produces Vlasov's equation directly as follows:

$$\frac{\partial f}{\partial t}=\{f,\mathscr{H}\}=-[f,H]. \tag{5}$$

For a derivation of this Lie–Poisson bracket from the canonical Hamiltonian formalism for the particle motion, see Refs. 12 and 16. The geometrical setting is developed in Ref. 17.

An important property of the Lie–Poisson bracket (4) is its infinite degeneracy, in the sense that there exists an infinite number of observables of the form

$$\mathscr{C}[f]=\int d^6\mathbf{z}\, C(f), \tag{6}$$

where $C(f)$ is an arbitrary smooth function, which commutes with all functionals of f:

$$\{\mathscr{C},\mathscr{F}\}=\int d^6\mathbf{z}\, f\left[C'(f),\frac{\delta\mathscr{F}}{\delta f}\right]$$
$$=\int d^6\mathbf{z}\frac{\delta\mathscr{F}}{\delta f}\,[f,C'(f)]=0. \tag{7}$$

In particular, $\mathscr{C}$ commutes with any Hamiltonian $\mathscr{H}$ and thus constitutes an infinite number of invariants of motion, one for each choice of $C(f)$. These invariants are known as the Casimirs; their level sets foliate the space of distributions into invariant subspaces, called the symplectic leaves, on which the dynamics is constrained. The physical meaning of the Casimir invariants has been discussed in Refs. 11 and 18–20. Roughly speaking, it is as follows. Imagine that we partition the particle phase space into small cells of equal volume, and to each cell attach a certain value of f. Then specifying all Casimir invariants (thus a symplectic leaf) is equivalent to specifying the number of cells that have a given value of f. The latter is conserved by Liouville's theorem.

It is sometimes desirable to restrict the Vlasov equation to a symplectic leaf, where presumably all points are dynamically accessible (i.e., subject only to *dynamical* constraints such as energy conservation). Crawford and Hislop[21] considered such a restriction for the one-dimensional electrostatic case. They used a Lie generating function $W(q,p,t)$ to represent all states close to an equilibrium $f_0(p)$:

$$f(q,p,t)=e^{[W,\cdot]}f_0. \tag{8}$$

The function W can be interpreted as a coordinate on the leaf [the subset $W=W(p,t)$ is excluded for it results in $f=f_0$]. Then they derived from the Vlasov equation (1), by an iterative scheme, the equation for W: $\partial W/\partial t=X_H(W)$, where X_H is a formal infinite series. In the present work we shall consider the restriction of the Vlasov equation to a symplectic leaf in the general case, by utilizing the power of action principles.

II. FOUR ACTION PRINCIPLES FOR THE VLASOV–POISSON EQUATION

The first action we discuss is the alteration of the well-known *Low action*.[1-4] It uses only the particle position, or Lagrangian displacement, $\mathbf{q}(\mathbf{z}_0,t)$ as the dynamical variable, where $\mathbf{z}_0\equiv(\mathbf{Q},\mathbf{P})$ labels the particles. For the Vlasov–Poisson equation we can treat the Poisson equation as a constraint because it does not contain any time derivative. The electrostatic potential $\phi(\mathbf{q},t)$ can be solved by the Green's function method

$$\phi(\mathbf{q},t)=e\int d^6\mathbf{z}_0'\, K\,[\mathbf{q}(\mathbf{z}_0,t)|\mathbf{q}(\mathbf{z}_0',t)]\, f_0(\mathbf{z}_0'), \tag{9}$$

where the sum over species is implied. Here $f_0(\mathbf{z}_0)$ is a given Vlasov distribution in the labeling space, and

$K(\mathbf{q}|\mathbf{q}')=K(\mathbf{q}'|\mathbf{q})$ is the Green's function for the Poisson equation,

$$\nabla^2 K(\mathbf{q}|\mathbf{q}') = -4\pi\delta^3(\mathbf{q}-\mathbf{q}'). \tag{10}$$

[In the case of infinite plasma $K(\mathbf{q}|\mathbf{q}')=e/|\mathbf{q}-\mathbf{q}'|$.] The Low action then reads

$$\mathscr{A}[\mathbf{q}] = \int dt \left[\int d^6\mathbf{z}_0 f_0(\mathbf{z}_0)\left(\frac{m}{2}\dot{\mathbf{q}}^2 - e\phi(\mathbf{q},t)\right) + \int d^3\mathbf{q}\frac{|\nabla\phi|^2}{8\pi}\right] = \int dt \int d^6\mathbf{z}_0\, f_0(\mathbf{z}_0)\left(\frac{m}{2}\dot{\mathbf{q}}^2 - \frac{e}{2}\phi(\mathbf{q},t)\right). \tag{11}$$

Here in the last equation ϕ is to be viewed as a shorthand for the expression given by Eq. (9); the factor $\frac{1}{2}$ in the second term arises from the partial cancellation between the $e\phi$ and $|\nabla\phi|^2$ terms of the first equation. Note that (11) is just the continuum version of Hamilton's principle. Variation yields Lagrange's equations of motion,

$$m\ddot{\mathbf{q}} = -e\nabla\phi, \tag{12}$$

which can be shown to be equivalent to the Vlasov equation by the standard manipulations.

Now consider the *phase-space action*, a close relative of the Low action. It is obtained from the latter by a Legendre transform $(\mathbf{q},\dot{\mathbf{q}})\mapsto(\mathbf{q},\mathbf{p})$, where $\mathbf{p}=m\dot{\mathbf{q}}$, and is given by

$$\mathscr{A}[\mathbf{q},\mathbf{p}] = \int dt \int d^6\mathbf{z}_0\, f_0(\mathbf{z}_0)\left(\mathbf{p}\cdot\dot{\mathbf{q}} - \frac{\mathbf{p}^2}{2m} - \frac{e}{2}\phi(\mathbf{q},t)\right), \tag{13}$$

where ϕ is again defined by Eq. (9). Variations are made with respect to $\mathbf{q}$ and $\mathbf{p}$ independently, which yield directly Hamilton's equations of motion,

$$\dot{\mathbf{q}} = \mathbf{p}/m, \quad \dot{\mathbf{p}} = -e\nabla\phi. \tag{14}$$

The action (13) possesses a geometrical character, viz., the integrand is a one-form in the particle phase space; this makes it amenable to the powerful Lie transform technique, which has been successfully exploited in the guiding-center[5,6] and oscillation-center theories.[7]

There is also a variant of the two actions just discussed that is worth mentioning. One can invert the coordinates $\mathbf{z}(\mathbf{z}_0,t)$ and reexpress the action in terms of the labeling fields $\mathbf{z}_0(\mathbf{z},t)$. This kind of action has proven useful in formulating variational fluid theories (see, e.g., Ref. 22).

The *Clebsch action*[12–14] is named after its counterpart in the fluid theories (see, e.g., Lamb[23]). Two potential functions $\alpha(\mathbf{z},t)$ and $\beta(\mathbf{z},t)$ are introduced for the particle distribution:

$$f=[\alpha,\beta] = \frac{\partial\alpha}{\partial\mathbf{q}}\cdot\frac{\partial\beta}{\partial\mathbf{p}} - \frac{\partial\alpha}{\partial\mathbf{p}}\cdot\frac{\partial\beta}{\partial\mathbf{q}}. \tag{15}$$

One can show that if α and β solve

$$\frac{\partial\alpha}{\partial t} + \frac{\mathbf{p}}{m}\cdot\frac{\partial\alpha}{\partial\mathbf{q}} - e\nabla\phi\cdot\frac{\partial\alpha}{\partial\mathbf{p}} = 0, \tag{16}$$

$$\frac{\partial\beta}{\partial t} + \frac{\mathbf{p}}{m}\cdot\frac{\partial\beta}{\partial\mathbf{q}} - e\nabla\phi\cdot\frac{\partial\beta}{\partial\mathbf{p}} = 0, \tag{17}$$

then f as given in (15) solves the Vlasov equation (1). Notice that in Eqs. (16) and (17) α and β are actually coupled through ϕ, which is now defined by

$$\phi(\mathbf{q},t) \equiv e\int d^6\mathbf{z}'\, K(\mathbf{q}|\mathbf{q}')f(\mathbf{z}',t). \tag{18}$$

Equations (16) and (17) arise upon variation of the following action:

$$\mathscr{A}[\alpha,\beta] = \int dt \int d^6\mathbf{z}\left(\alpha\frac{\partial\beta}{\partial t} - \frac{\mathbf{p}^2}{2m}[\alpha,\beta] - \frac{e}{2}\int d^6\mathbf{z}'[\alpha,\beta]K(\mathbf{q}|\mathbf{q}')[\alpha',\beta']\right), \tag{19}$$

where $\alpha'\equiv\alpha(\mathbf{z}',t)$ and $\beta'\equiv\beta(\mathbf{z}',t)$. This action possesses a notable feature that all other actions lack: the number of particles in a region V of the phase space, given by $\int_V d^6\mathbf{z}\, f(\mathbf{z},t)$, is determined by the value of α and β on the boundary of V. This suggests a potential application to problems that involve variable particle numbers such as beam injection.

The *Hamilton–Jacobi action* uses as its dynamical variables a mixed-variable generating function $S(\mathbf{q},\mathbf{P},t)$ for the particle orbits, and a density function $\varphi(\mathbf{q},\mathbf{P},t)$ that represents the number of particles on an orbit. It reads as

$$\mathscr{A}[\varphi,S] = -\int dt \int d^3\mathbf{q}\, d^3\mathbf{P}\, \varphi\left[\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial\mathbf{q}}\right)^2 + \frac{e}{2}\phi(\mathbf{q},t) - H_0\left(\frac{\partial S}{\partial\mathbf{P}},\mathbf{P},t\right)\right], \tag{20}$$

where $H_0(\mathbf{Q},\mathbf{P},t)$ is an arbitrary function and ϕ is given by Eq. (18) with $f(\mathbf{z},t)$ defined in terms of S and φ by

$$\varphi(\mathbf{q},\mathbf{P},t) = f\left(\mathbf{q},\frac{\partial S}{\partial\mathbf{q}},t\right)\left|\frac{\partial^2 S}{\partial\mathbf{q}\,\partial\mathbf{P}}\right|. \tag{21}$$

Variation with respect to φ immediately yields a modified Hamilton–Jacobi equation; variation with respect to S yields an equation that can be manipulated into the Vlasov equation for $f(\mathbf{z},t)$. The detailed calculations can be found in Ref. 10. The Hamilton–Jacobi action has been applied to the derivation of unambiguous energy expressions for kinetic guiding-center theory.[9–11]

III. THE LEAF ACTION

In this section we derive, from the phase-space action (13), the new *leaf action*. An arbitrarily given reference distribution $f_0(\mathbf{z}_0)$ determines a unique symplectic leaf. One may think of $\mathbf{z}_0$ as the initial particle position in the phase space,

$$\mathbf{q}(\mathbf{z}_0,0)=\mathbf{Q}, \quad \mathbf{p}(\mathbf{z}_0,0)=\mathbf{P}, \tag{22}$$

then $f_0(\mathbf{z}_0)$ would be the initial Vlasov distribution. But, one is by no means constrained to such an interpretation, which is sometimes too restrictive. Now if we let $(\mathbf{q},\mathbf{p})$ be generated by a single mixed-variable generating function $S(\mathbf{q},\mathbf{P},t)$:

$$\mathbf{p}=\frac{\partial S}{\partial\mathbf{q}}, \quad \mathbf{Q}=\frac{\partial S}{\partial\mathbf{P}}, \tag{23}$$

then as S varies, the Vlasov distribution $f(\mathbf{q},\mathbf{p},t)$, defined by

$$f\left(\mathbf{q},\frac{\partial S}{\partial\mathbf{q}},t\right) = f_0\left(\frac{\partial S}{\partial\mathbf{P}},\mathbf{P}\right), \tag{24}$$

will always stay on the same symplectic leaf prescribed by f_0. Thus by using S we have in effect restricted the variations to a single leaf. Here for explicitness we have chosen the F_2-type generating function, but one can also use any other type, and the whole calculation that follows will carry through with only minor modifications. In fact, it is known that a given type of generating function will, in general, develop caustic singularities, so in practice one may have to switch between the various types of generating functions. Locally, a generating function always exists, as shown, e.g., by Arnold.[24] In contrast to the initial value interpretation of Eq. (22), Eq. (23) suggests that we view the phase space as foliated by Lagrangian submanifolds, i.e., submanifolds labeled by constant $\mathbf{P}$, where $\mathbf{Q}$ serves as coordinate within the submanifold. [A Lagrangian submanifold is an n-dimensional subspace in the $2n$-dimensional phase space defined by the first of Eq. (23); for our problem $n=3$.] Since we want the generating function S to represent the actual dynamics, and not just a relabeling of the particles, a certain admissibility condition is required. The precise criterion will be given at the end of this section.

From Eq. (23) we have

$$\mathbf{p}\cdot\dot{\mathbf{q}} = \frac{\partial S}{\partial \mathbf{q}}\cdot\dot{\mathbf{q}} = \frac{dS}{dt} - \frac{\partial S}{\partial t}, \tag{25}$$

where dS/dt means total time derivative of S holding $\mathbf{z}_0$ fixed. Inserting this into Eq. (13), we see that the dS/dt term drops out; upon changing the integration variables from $(\mathbf{Q},\mathbf{P})$ to $(\mathbf{q},\mathbf{P})$ in Eq. (13) we obtain the desired leaf action,

$$\mathscr{A}[S] = -\int dt \int d^3\mathbf{q}\, d^3\mathbf{P} \left| \frac{\partial^2 S}{\partial \mathbf{q}\,\partial \mathbf{P}} \right| f_0\left(\frac{\partial S}{\partial \mathbf{P}},\mathbf{P}\right) \times \left[\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial \mathbf{q}}\right)^2 + \frac{e}{2}\,\phi(\mathbf{q},t) \right], \tag{26}$$

where ϕ is a functional of S, defined by

$$\phi(\mathbf{q},t) = \int d^3\mathbf{q}'\, d^6\mathbf{P}' \left| \frac{\partial^2 S}{\partial \mathbf{q}'\,\partial \mathbf{P}'} \right| f_0\left(\frac{\partial S}{\partial \mathbf{P}'},\mathbf{P}'\right) K(\mathbf{q}|\mathbf{q}'). \tag{27}$$

For the change of variables to be valid, we require the *van Vleck determinant* $\omega \equiv |\partial^2 S/\partial \mathbf{q}\,\partial \mathbf{P}|$ to be finite. As mentioned earlier this can always be achieved locally by a suitable choice of the generating function. The Jacobian matrix of this transformation,

$$\omega_{ij} = \frac{\partial^2 S}{\partial q_i\,\partial P_j} \tag{28}$$

comprises the symplectic two-form (the "Lagrange bracket") in the mixed-variable representation, because by Eq. (23) we have

$$\omega_{ij}\, dq_i \wedge dP_j = dq_i \wedge dp_i = dQ_j \wedge dP_j. \tag{29}$$

Here we have adopted the summation convention over repeated indices. Therefore its inverse $\mathbf{J}$, defined by $J^{ij}\omega_{jk} = \delta^i_k$, defines the Poisson bracket in the mixed-variable coordinates

$$[f,g]_{(\mathbf{q},\mathbf{P})} = J^{ij}\left(\frac{\partial f}{\partial q_j}\frac{\partial g}{\partial P_i} - \frac{\partial f}{\partial P_i}\frac{\partial g}{\partial q_j} \right). \tag{30}$$

[Note that the subscript $(\mathbf{q},\mathbf{P})$ only serves to indicate that we are using the mixed-variable coordinates; the Poisson bracket itself does not depend on which coordinates one uses.] A direct calculation can also verify the following identities:

$$\frac{\partial}{\partial q_j}(\omega J^{ij}) = 0 = \frac{\partial}{\partial P_i}(\omega J^{ij}). \tag{31}$$

It is straightforward to carry out the variation. After some algebra, which makes use of Eq. (31), we find

$$\frac{\delta \mathscr{A}}{\delta S} = \omega \left[f_0\left(\frac{\partial S}{\partial \mathbf{P}},\mathbf{P}\right), \frac{\partial S}{\partial t} + H\left(\mathbf{q},\frac{\partial S}{\partial \mathbf{q}},t\right) \right]_{(\mathbf{q},\mathbf{P})}. \tag{32}$$

Since $\omega \neq 0$, and the bracket $[\cdot,\cdot]_{(\mathbf{q},\mathbf{P})}$ is nondegenerate, a general solution ("first integral") of $\delta\mathscr{A}/\delta S = 0$ is

$$\frac{\partial S}{\partial t} + H\left(\mathbf{q},\frac{\partial S}{\partial \mathbf{q}},t\right) = C\left(\frac{\partial S}{\partial \mathbf{P}},\mathbf{P},t\right), \tag{33}$$

where $C(\mathbf{z}_0,t)$ is an arbitrary function that commutes with $f_0(\mathbf{z}_0)$. This generalized form of the Hamilton–Jacobi equation was introduced in Ref. 10. It has the following interpretation: if we regard S as the generating function for a canonical transformation from the phase space $\mathbf{z}$ to the labeling space $\mathbf{z}_0$, then Eq. (33) states that $C(\mathbf{z}_0,t)$ is the transformed Hamiltonian in the $\mathbf{z}_0$ space. Thus the requirement of f_0 commuting with C implies that f_0 is a solution of the Vlasov equation in the labeling space. Then the distribution $f(\mathbf{z},t)$ defined by Eq. (24) is also a solution of the Vlasov equation in the original phase space. To see this argument more clearly, let us differentiate Eq. (23) with respect to time while holding $\mathbf{z}$ constant; we obtain

$$\left(\frac{\partial \mathbf{P}}{\partial t}\right)_{(\mathbf{q},\mathbf{p})} = -\mathbf{J}\cdot\frac{\partial^2 S}{\partial \mathbf{q}\,\partial t}, \tag{34}$$

$$\left(\frac{\partial \mathbf{Q}}{\partial t}\right)_{(\mathbf{q},\mathbf{p})} = \frac{\partial^2 S}{\partial \mathbf{P}\,\partial t} + \frac{\partial^2 S}{\partial \mathbf{P}\,\partial \mathbf{P}}\cdot\left(\frac{\partial \mathbf{P}}{\partial t}\right)_{(\mathbf{q},\mathbf{p})}. \tag{35}$$

Also by the chain rule we have

$$\frac{\partial f_0}{\partial \mathbf{Q}} = \mathbf{J}\cdot\frac{\partial}{\partial \mathbf{q}} f_0\left(\frac{\partial S}{\partial \mathbf{P}},\mathbf{P}\right), \tag{36}$$

$$\frac{\partial f_0}{\partial \mathbf{P}} = \frac{\partial}{\partial \mathbf{P}} f_0\left(\frac{\partial S}{\partial \mathbf{P}},\mathbf{P}\right) - \frac{\partial^2 S}{\partial \mathbf{P}\,\partial \mathbf{P}}\cdot\frac{\partial f_0}{\partial \mathbf{Q}}. \tag{37}$$

These relationships, together with the definition of f, lead to

$$\left(\frac{\partial f}{\partial t}\right)_{(\mathbf{q},\mathbf{p})} = \frac{\partial f_0}{\partial \mathbf{Q}}\cdot\left(\frac{\partial \mathbf{Q}}{\partial t}\right)_{(\mathbf{q},\mathbf{p})} + \frac{\partial f_0}{\partial \mathbf{P}}\cdot\left(\frac{\partial \mathbf{P}}{\partial t}\right)_{(\mathbf{q},\mathbf{p})} = \left[f_0\left(\frac{\partial S}{\partial \mathbf{P}},\mathbf{P}\right), \frac{\partial S}{\partial t} \right]_{(\mathbf{q},\mathbf{P})}. \tag{38}$$

On the other hand, since the Poisson bracket does not depend on the specific representation, we have

$$\left[f_0\left(\frac{\partial S}{\partial \mathbf{P}},\mathbf{P}\right), H\left(\mathbf{q},\frac{\partial S}{\partial \mathbf{q}},t\right) \right]_{(\mathbf{q},\mathbf{P})} = [f,H]_{(\mathbf{q},\mathbf{p})}. \tag{39}$$

Substituting Eqs. (38) and (39) into (32), we see that when reexpressed in $\mathbf{z}$ space the latter is exactly the Vlasov equation (1). Equation (38) also provides a precise criterion on the admissibility of the generating functions—we must not allow $\partial S/\partial t$ to commute with f_0. In the one-dimensional electrostatic problem this requirement completely determines C (see Ref. 25).

IV. RELATIONSHIPS BETWEEN VARIOUS ACTIONS

So far in this paper we have discussed how the phase-space and the leaf actions can be derived from the Low action. The leaf action uses a single function as its variable, so a natural question that arises is whether there exists a canonical description. As we shall see momentarily, this leads us to the Hamilton–Jacobi action (20). The usual method of going to the canonical description, the Legendre transform, cannot be applied directly because Eq. (26) is linear in $\partial S/\partial t$. One of the standard remedies in this case is the Dirac constraint method.[26,27] The canonical momentum conjugate to S is still defined by the coefficient of $\partial S/\partial t$:

$$\Pi = -\omega f_0\Big(\frac{\partial S}{\partial \mathbf{P}},\mathbf{P}\Big), \tag{40}$$

but since this equation does not involve $\partial S/\partial t$, it can only be used as a constraint (the *primary Dirac constraint*). Introducing a Lagrange multiplier $\lambda(\mathbf{q},\mathbf{P},t)$ for this constraint we then obtain a three-variable action,

$$\mathscr{A}[\Pi,S,\lambda] = \int dt \int d^3\mathbf{q}\, d^3\mathbf{P}\Big\{\Pi\Big[\frac{\partial S}{\partial t} + \frac{1}{2m}\Big(\frac{\partial S}{\partial \mathbf{q}}\Big)^2 + \frac{e}{2}\,\phi(\mathbf{q},t)\Big] - \lambda\Big[\Pi + \omega f_0\Big(\frac{\partial S}{\partial \mathbf{P}},\mathbf{P}\Big)\Big]\Big\}. \tag{41}$$

Variation with respect to λ yields the constraint (40); variation with respect to Π yields

$$\frac{\partial S}{\partial t} + H\Big(\mathbf{q},\frac{\partial S}{\partial \mathbf{q}},t\Big) - \lambda = 0; \tag{42}$$

variation with respect to S yields a complicated equation that can be simplified, with help from Eq. (42), into

$$\frac{\partial \Pi}{\partial t} = -\omega J^{ij}\frac{\partial}{\partial q_j}\Big[f_0\Big(\frac{\partial S}{\partial \mathbf{P}},\mathbf{P}\Big)\frac{\partial^2 S}{\partial P_i\,\partial t}\Big] + \omega\Big[f_0\Big(\frac{\partial S}{\partial \mathbf{P}},\mathbf{P}\Big),\lambda\Big]_{(\mathbf{q},\mathbf{P})}. \tag{43}$$

For the constraint (40) to remain true during the system's evolution, its time derivative must also hold. Taking the time derivative of Eq. (40) and using Eqs. (42) and (43), we obtain the *secondary Dirac constraint*:

$$\Big[f_0\Big(\frac{\partial S}{\partial \mathbf{P}},\mathbf{P}\Big),\lambda\Big]_{(\mathbf{q},\mathbf{P})} = 0. \tag{44}$$

This equation can be solved explicitly, and the general solution for λ is just C, the same as that used in Eq. (33). This solution makes the secondary Dirac constraint compatible with the equations of motion without producing any tertiary constraints. So the Dirac constraint procedure terminates at this point, and Eqs. (40), (42), and (43) are shown to be consistent.

Replacing λ by C in Eq. (41) reduces it to a two-variable action,

$$\mathscr{A}[\Pi,S] = \int dt \int d^3\mathbf{q}\, d^3\mathbf{P}\,\Pi\Big[\frac{\partial S}{\partial t} + \frac{1}{2m}\Big(\frac{\partial S}{\partial \mathbf{q}}\Big)^2 + \frac{e}{2}\,\phi(\mathbf{q},t) - C\Big(\frac{\partial S}{\partial \mathbf{P}},\mathbf{P},t\Big)\Big]. \tag{45}$$

By changing the integration variables back to $(\mathbf{Q},\mathbf{P})$ we can show that the last term in Eq. (41) does not contain either S or Π, and can thus be omitted. This removes all references to f_0 from the action (45). Therefore we can relax the Dirac constraint (40) and treat it as an initial condition, because we have shown that if it holds at one time, then it will hold for all times. With this argument we delegate the specification of a leaf to the initial condition, and Π and S can be considered as free variables in the variation. The action (45) and the Hamilton–Jacobi action (20) are identical, if we equate Π with $-\varphi$ and C with H_0. The geometrical setting of Ref. 17 is of interest for this link between the space of S and the space of (Π,S).

It remains to establish a relationship between the Hamilton–Jacobi action (20) and the Clebsch action (19). A link can only be made when there exists a function $g(\mathbf{Q},\mathbf{P})$ that commutes with $H_0(\mathbf{Q},\mathbf{P},t)$, in which case we can introduce a function $\alpha(\mathbf{q},\mathbf{p},t)$ by

$$\alpha\Big(\mathbf{q},\frac{\partial S}{\partial \mathbf{q}},t\Big) = g\Big(\frac{\partial S}{\partial \mathbf{P}},\mathbf{P}\Big). \tag{46}$$

By a calculation similar to Eq. (38) we find

$$\Big(\frac{\partial \alpha}{\partial t}\Big)_{(\mathbf{q},\mathbf{p})} = \Big[\alpha\Big(\mathbf{q},\frac{\partial S}{\partial \mathbf{q}},t\Big),\frac{\partial S}{\partial t}\Big]_{(\mathbf{q},\mathbf{P})}. \tag{47}$$

So, for any function $\beta(\mathbf{q},\mathbf{p},t)$ we have

$$\int d^6\mathbf{z}\,\beta\,\frac{\partial \alpha}{\partial t} = -\int d^3\mathbf{q}\,d^3\mathbf{P}\,\omega \times\Big[\alpha\Big(\mathbf{q},\frac{\partial S}{\partial \mathbf{q}},t\Big),\beta\Big(\mathbf{q},\frac{\partial S}{\partial \mathbf{q}},t\Big)\Big]_{(\mathbf{q},\mathbf{P})}\frac{\partial S}{\partial t}. \tag{48}$$

Therefore, if we choose β to satisfy

$$\Big[\alpha\Big(\mathbf{q},\frac{\partial S}{\partial \mathbf{q}},t\Big),\beta\Big(\mathbf{q},\frac{\partial S}{\partial \mathbf{q}},t\Big)\Big]_{(\mathbf{q},\mathbf{P})} = \frac{\varphi}{\omega}, \tag{49}$$

then the right-hand side of Eq. (48) becomes the first term of the Hamilton–Jacobi action (20). Solving Eq. (49) for φ, inserting it into Eq. (20), and changing the integration variables to $(\mathbf{q},\mathbf{p})$, we then obtain the Clebsch action. (The H_0 term vanishes because by assumption H_0 and g commute.) Finally, we note that Eqs. (49) and (21) give the distribution function f consistent with Eq. (15).

V. ACTION PRINCIPLES FOR THE MAXWELL–VLASOV EQUATIONS

In this section we display the five actions for the Vlasov–Maxwell equations. These actions differ from the Vlasov–Poisson actions in that, besides being electromagnetic, the particle and field variables [represented by a four-potential $(\mathbf{A},\phi)$] are to be varied independently. Three of them, the Low, phase-space, and Hamilton–Jacobi actions, were previously known in this context. Variations of these actions with respect to the particle variables yields either the particle equations of motion or the Vlasov equation; variations with respect to the four-potential $(\mathbf{A},\phi)$ yields the Maxwell equations. We shall omit the detailed calculations here since they are either straightforward, or have been given in the references.

The field part of all the actions has the same standard form:

$$\mathscr{A}_F[\mathbf{A},\phi] = \frac{1}{8\pi}\int d^4x\left[\left(-\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi\right)^2 - (\nabla\times\mathbf{A})^2\right], \tag{50}$$

where $d^4x \equiv dt\, d^3\mathbf{x}$. The complete actions are listed below.

The Low action[1-4]

$$\mathscr{A}[\mathbf{q};\mathbf{A},\phi] = \int dt \int d^6\mathbf{z}_0 f_0(\mathbf{z}_0)\left(\frac{m}{2}\dot{\mathbf{q}}^2 + \frac{e}{c}\dot{\mathbf{q}}\cdot\mathbf{A}(\mathbf{q},t) - e\phi(\mathbf{q},t)\right) + \mathscr{A}_F; \tag{51}$$

the phase-space action,

$$\mathscr{A}[\mathbf{q},\mathbf{p};\mathbf{A},\phi] = \int dt \int d^6\mathbf{z}_0 f_0(\mathbf{z}_0)\left[\mathbf{p}\cdot\dot{\mathbf{q}} - \frac{1}{2m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}(\mathbf{q},t)\right)^2 - e\phi(\mathbf{q},t)\right] + \mathscr{A}_F; \tag{52}$$

the Clebsch action,

$$\mathscr{A}[\alpha,\beta;\mathbf{A},\phi] = \int dt \int d^6\mathbf{z}\left\{\alpha\frac{\partial\beta}{\partial t} - [\alpha,\beta]\left[\frac{1}{2m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2 + e\phi\right]\right\} + \mathscr{A}_F; \tag{53}$$

the Hamilton–Jacobi action[8]

$$\mathscr{A}[\varphi,S;\mathbf{A},\phi] = -\int dt \int d^3\mathbf{q}\, d^3\mathbf{P}\, \varphi \times\left[\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial \mathbf{q}} - \frac{e}{c}\mathbf{A}(\mathbf{q},t)\right)^2 + e\phi(\mathbf{q},t) - H_0\left(\frac{\partial S}{\partial \mathbf{P}},\mathbf{P},t\right)\right] + \mathscr{A}_F; \tag{54}$$

and the leaf action,

$$\mathscr{A}[S;\mathbf{A},\phi] = -\int dt \int d^3\mathbf{q}\, d^3\mathbf{P}\left|\frac{\partial^2 S}{\partial\mathbf{q}\,\partial\mathbf{P}}\right| f_0\left(\frac{\partial S}{\partial\mathbf{P}},\mathbf{P}\right) \times\left[\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial \mathbf{q}} - \frac{e}{c}\mathbf{A}(\mathbf{q},t)\right)^2 + e\phi(\mathbf{q},t)\right] + \mathscr{A}_F. \tag{55}$$

TABLE I. This table summarizes the action principles presented in the paper, by the variables used (columns) and the kinds of action (rows). The arrows indicate the logical relationships established in the paper.

Lagrangian		Lagrangian–Eulerian		Eulerian
Low action $\mathscr{A}[\mathbf{q}]$		Leaf action $\mathscr{A}[S]$		?
↓	↗	↓		
Phase-space action $\mathscr{A}[\mathbf{q},\mathbf{p}]$		Hamilton–Jacobi action $\mathscr{A}[S,\phi]$	→	Clebsch action $\mathscr{A}[\alpha,\beta]$

VI. CONCLUSION

We have presented a comprehensive discussion of action principles for the Vlasov–Poisson and Vlasov–Maxwell systems, and have introduced a totally new kind of action called the leaf action. The paper is compactly summarized in Table I. The actions fall into three groups, according to the type of variables used to describe the particle distribution. The top row of the table indicates the three variable types: Lagrangian, mixed Lagrangian–Eulerian, and Eulerian. Below these variable headings are two rows of actions. The second row is distinguished by the fact that the actions have two variables, which because of the form of the actions are canonically conjugate Hamiltonian variables. Thus this row can be called Hamiltonian. The first row contains actions that are written in terms of a single variable, analogous to Hamilton's principle of classical mechanics. Since Hamilton's principle gives rise to Lagrange's equations of motion, this row could be termed Lagrangian. The arrows indicate the logical relationships that we have established in the paper between the actions. The transformation between the Low and phase-space actions was accomplished by the Legendre transform; that between the phase-space and leaf actions was accomplished by the restriction discussed in Sec. III; that between the leaf and Hamilton–Jacobi actions was accomplished by the Dirac constraint method of Sec. IV; and finally the transformation between the Hamilton–Jacobi and Clebsch actions was accomplished by a variable change in Sec. IV. The reader may have noticed that there is a missing entry in the table, corresponding to an action that uses a single Eulerian function as the dynamical variable. We remark that the leaf action written in terms of a Lie generating function (rather than a mixed-variable generating function) can fill this position. More precisely, we can express the canonical transformation of Eq. (23) by a Lie transform: $\mathbf{z} = e^{\{\cdot,W\}}\mathbf{z}_0$, and then derive (also from the phase-space action) the leaf action in terms of W. The detailed calculation requires lengthy algebra that involves delicate manipulation of an infinite power series of Lie derivatives, and will be presented elsewhere. However, the W version of the Vlasov equation restricted to a symplectic leaf [i.e., Eq. (33)] has been discussed in Ref. 25.

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