# Exact solutions for a system of nonlinear plasma fluid equations

M. G. Prahović, R. D. Hazeltine, and P. J. Morrison
*Department of Physics and Institute for Fusion Studies, The University of Texas at Austin, Austin, Texas 78712*

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A method is presented for constructing exact solutions to a system of nonlinear plasma fluid equations that combines the physics of reduced magnetohydrodynamics and the electrostatic drift-wave description of the Charney–Hasegawa–Mima equation. The system has nonlinearities that take the form of Poisson brackets involving the fluid field variables. The method relies on modifying a class of simple equilibrium solutions, but no approximations are made. A distinguishing feature is that the original nonlinear problem is reduced to the solution of two linear partial differential equations, one fourth order and the other first order. The first-order equation has Hamiltonian characteristics and is easily integrated, supplying information about the propagation of solutions.

## I. INTRODUCTION AND OVERVIEW

In this paper the construction of exact analytic solutions for a system of nonlinear plasma fluid equations is discussed. The equations occur in a fluid model,[1] which combines the physics of reduced magnetohydrodynamics[2,3] (RMHD) and the Charney–Hasegawa–Mima (CHM) equation.[4] The combined model is of interest because RMHD is an important tool for the interpretation of experimental results and for the prediction and theoretical analysis of nonlinear plasma fluid behavior in tokamaks. (To date, most of this work has been done numerically.) In the context of plasma physics, the CHM equation has been used in the study of electrostatic fluctuations in hot, turbulent plasmas; it incorporates the physics of electrostatic drift waves, which is not described by RMHD.

The solutions admitted by this nonlinear system are physically interesting because they are fully electromagnetic, like many disturbances seen in tokamak experiments; and they can take the form of solitary waves, which can be long lived and very stable to perturbations. Hence they could describe plasma behavior that might be detected experimentally. The method by which a class of solutions is obtained here is also of intrinsic mathematical interest: the nonlinear system of governing partial differential equations (PDE's) is reduced to a linear system that is, in principle, exactly soluble by standard techniques. (If one wishes to be single-mindedly practical, the analytic solutions could also serve as a means of verifying the computer codes used for RMHD calculations.)

Here is an overview of what follows. In Sec. II the fluid equations are presented and their physical content is briefly discussed. Their nonlinear character is manifested by Poisson brackets involving the fluid field variables.

Section III is concerned with finding solutions to the fluid equations for the case of a perfectly conducting plasma. First, the construction of exact solutions for the equilibrium form of the equations is considered. A simple change of dependent variables is used to eliminate the Poisson brackets and reduce the problem to solving a single linear PDE. This provides a foundation and motivation for the more general problem of constructing exact solutions to the nonequilibrium equations. Solutions to the nonequilibrium equations based on the change of variables for the equilibrium case are seen to be similar to Alfvén waves. Next, by a slight modification of the change of dependent variables for the equilibrium case, the nonlinear, nonequilibrium equations are reduced to a pair of linear PDE's, one first order and the other fourth order. An algorithm for constructing solutions based on this reduction is presented. Finally, the first-order PDE is integrated by the method of characteristics. The characteristics are determined by a system of Hamiltonian ordinary differential equations (ODE's) that constrain the propagation of solutions in an interesting way. From a discussion of physical properties and some simple examples, these solutions are seen to be distinct from Alfvén waves, in general.

In Sec. IV a summary is presented. The limitations of our method, possible modifications to it, and areas for further work are discussed.

## II. FLUID EQUATIONS

### A. Geometry and coordinates

What follows is a description of the geometry and the coordinates used. First of all, the presumed geometry is toroidal, that of a tokamak with a circular cross section. However, the parameter beta for the plasma is assumed to be small—this excludes pressure-driven dynamics and magnetic curvature from the physics described by the fluid equations, thus making them applicable to cylindrical and slab geometries, also. Let us introduce a set of normalized coordinates:

$$x=(R-R_0)/a, \quad y=Z/a, \quad \text{and} \quad z=-\zeta. \qquad (1)$$

Here $(R,\zeta,Z)$ represent cylindrical coordinates centered on the symmetry axis of the tokamak: $R$ measures radial displacements away from the symmetry axis, $\zeta$ is the toroidal angle, and $Z$ measures vertical displacements above or below the horizontal symmetry plane of the tokamak. Here $R_0$ is

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the major radius of the tokamak. In the context of RMHD, $a$ is the tokamak's minor radius and is thus a scale characterizing fluid motions transverse to the magnetic field. If $a$ is taken to be of the order of the ion Larmor radius, it serves as a useful length scale for the description of electrostatic drift-wave physics in the context of the CHM equation. Hence $(x,y,z)$ is a right-handed set of local poloidal coordinates useful for describing plasma behavior on different length scales within the torus.

### B. Important physical quantities and their orderings

We next introduce the three normalized field variables that appear in the equations: $\phi,\psi$, and $\chi$. The quantity $\phi$ represents the electrostatic potential; $\psi$ represents the parallel component of the magnetic vector potential, or the poloidal magnetic flux; and $\chi$ represents a small perturbation of the plasma density. The unperturbed plasma density, denoted by $n_c$, is assumed to be constant in both space and time. The vacuum magnetic field is assumed to be purely toroidal and to dominate any magnetic fields due to the plasma. Thus $\psi$ represents the addition, due to the plasma, to the vacuum field.

The dimensionless ordering parameter is $\epsilon$, the inverse aspect ratio of the tokamak:

$$\epsilon \equiv a/R_0 \ll 1. \tag{2}$$

The electric and magnetic fields are ordered using $\epsilon$ to express the presumed dominance of the vacuum magnetic field: the scalar and vector potentials for the electromagnetic fields generated by the plasma are assumed $O(\epsilon)$ compared to that for the vacuum magnetic field. The plasma beta is $O(\epsilon^2)$, a "low beta" ordering. The plasma density is assumed to deviate from $n_c$ by a quantity $O(\epsilon)$. A normalized time coordinate $\tau$ is defined by

$$\tau \equiv \epsilon (t v_{\rm A}/a), \tag{3}$$

which is appropriate for the slow, shear-Alfvén fluid motions of interest. Here $t$ is the usual time coordinate; $v_{\rm A}$, a constant, is a measure of the Alfvén speed for the plasma. Thus all the important physical quantities are ordered in terms of $\epsilon$.

In terms of the $\epsilon$ orderings described above, the component of the fluid velocity perpendicular to the magnetic field is[1]

$$\mathbf{v}_\perp = \epsilon v_{\rm A} \hat{\mathbf{z}} \times \nabla_\perp \phi + O(\epsilon^2). \tag{4}$$

Here $\nabla_\perp$ is the poloidal component of the normalized gradient operator $a\nabla$:

$$\nabla_\perp \equiv \hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y}, \tag{5}$$

where $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are just the appropriate unit vectors. The first term on the right-hand side of (4) represents the usual $\mathbf{E}\times\mathbf{B}$ fluid drift, and the factor $\epsilon v_{\rm A}$ emphasizes that the fluid motions considered are very slow compared to the Alfvén speed.

### C. The reduced fluid equations

To obtain the reduced fluid equations for the combined system, the $\epsilon$ ordering scheme summarized above is incorporated into the appropriate exact, resistive MHD equations. To arrive at the approximate equations given below, the terms of lowest order in $\epsilon$ are kept. A complete derivation of the equations is available elsewhere.[1] The following short description is provided to make the physical content and the mathematical symbolism more transparent.

Before proceeding with the presentation of the fluid equations, we introduce two quantities that will appear quite often below. The first is

$$U \equiv \nabla_\perp^2 \phi, \tag{6}$$

the parallel component of the fluid vorticity. The second is

$$J \equiv \nabla_\perp^2 \psi, \tag{7}$$

the parallel component of the plasma current. To make the fluid equations more compact, it is also useful to introduce the Poisson bracket defined by

$$[f,g] \equiv \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} = \hat{\mathbf{z}}\cdot\nabla_\perp f \times \nabla_\perp g. \tag{8}$$

The first of the equations is the "shear-Alfvén law,"

$$\frac{\partial U}{\partial \tau} + [\phi, U] = -\left(\frac{\partial J}{\partial z} - [\psi, J]\right). \tag{9}$$

The left-hand side represents the convective time derivative of $U$: the second term, $[\phi,U]$, represents convection of $U$ due to the $\mathbf{E}\times\mathbf{B}$ fluid drift. Acting on $J$ on the right-hand side of (9) is the operator $\partial/\partial z - [\psi,\cdot]$, which is essentially $\mathbf{B}\cdot\nabla$. The physical content of the right-hand side of (9) is thus current-driven dynamics, such as kink modes.

The second of the equations comes from the parallel component of a modified Ohm's law:

$$\frac{\partial \psi}{\partial \tau} + \frac{\partial \phi}{\partial z} - [\psi,\phi] = \hat{\eta} J + \alpha\left(\frac{\partial \chi}{\partial z} - [\psi,\chi]\right). \tag{10}$$

Here the left-hand side represents the parallel component of the electric field. The quantity $\hat{\eta}$ is a normalized collisional resistivity. The last quantity in parentheses on the right-hand side represents pressure effects on parallel electron flow. In the combined model it is assumed that electrons almost exclusively carry the parallel current. The constant $\alpha$ is defined by

$$\alpha^2 = \rho_s^2/a^2, \tag{11}$$

where

$$\rho_s^2 = T_e/m_i \Omega_i^2. \tag{12}$$

Here $m_i$ is the ion mass, $T_e$ is the constant *electron* temperature in energy units, and $\Omega_i$ is the ion Larmor frequency:

$$\Omega_i = eB_T/m_i c, \tag{13}$$

where $B_T$ is a constant that measures the strength of the vacuum magnetic field. In the combined model, $\alpha$ represents the marriage of RMHD and electrostatic drift-wave physics: $\alpha = 0$ corresponds to RMHD; $\alpha \sim 1$ corresponds to an electromagnetic, resistive generalization of the electrostatic drift-wave physics described by the CHM equation.

The last equation we consider,

$$\frac{\partial \chi}{\partial \tau} + [\phi,\chi] + \frac{\partial J}{\partial z} - [\psi, J] = 0, \tag{14}$$

is derived from the equation for electron conservation and quasineutrality. The second term on the left side is just $\mathbf{E}\times\mathbf{B}$ convection of the plasma density. Electron parallel mobility is explicit in the last two terms: these come from the divergence of the parallel electron fluid velocity, which is essentially proportional to the parallel plasma current $J$.

## III. CONSTRUCTION OF EXACT SOLUTIONS

### A. Framework

Having introduced the fluid equations, we next discuss a method for arriving at exact solutions of them.

We denote the partial derivative of a quantity by a subscript, e.g., $\partial U/\partial\tau \equiv U_\tau$. Then, after rearranging the terms of (9) and (10) and subtracting (14) from (9), we can write

$$U_\tau + [\phi,U] + J_z + [J,\psi] = 0, \tag{15}$$

$$\psi_\tau + (\phi - \alpha\chi)_z + [\phi - \alpha\chi,\psi] = 0, \tag{16}$$

and

$$(U-\chi)_\tau + [\phi,U-\chi] = 0. \tag{17}$$

This is the nonlinear system we will study. Note that we are taking $\hat{\eta} = 0$ in (16); the resistivity of the plasma is neglected for all that follows.

To satisfy (17) we take

$$\chi = g(z) + U, \tag{18}$$

where $g$ is an arbitrary function of $z$. This is by no means the general solution to (17); it is simply a special case that satisfies (17) with little effort. Defining

$$\xi \equiv \phi - \alpha g(z), \tag{19}$$

and recasting (15) and (16) in terms of $\xi$ gives

$$U_\tau + [\xi,U] + J_z + [J,\psi] = 0 \tag{20}$$

and

$$\psi_\tau + (\xi - \alpha U)_z + [\xi - \alpha U,\psi] = 0, \tag{21}$$

where (18) has been used. We note in passing that from (19) and (6), the definition of $U$, we have

$$U = \nabla_\perp^2 \xi, \tag{22}$$

a relation that will be used often in what follows.

Now we have to find solutions to (20) and (21). Let us first consider the simpler case of axisymmetric equilibrium.

### B. Axisymmetric equilibrium

Under the assumption of axisymmetric equilibrium, $\partial/\partial\tau \equiv 0$ and $\partial/\partial z \equiv 0$, (20) and (21) reduce to

$$[\xi,\nabla_\perp^2\xi] - [\psi,\nabla_\perp^2\psi] = 0 \tag{23}$$

and

$$[\xi,\psi] - \alpha[U,\psi] = 0. \tag{24}$$

(Here $J \equiv \nabla_\perp^2\psi$ and $U = \nabla_\perp^2\xi$ were used.) We take

$$\psi = \gamma\xi, \tag{25}$$

where $\gamma$ is an arbitrary constant. Then (23) and (24) reduce to

$$(1-\gamma^2)[\xi,\nabla_\perp^2\xi] = 0 \tag{26}$$

and

$$\alpha\gamma[\xi,U] = 0. \tag{27}$$

In the same spirit as (25), we take

$$U = \delta\xi, \tag{28}$$

where $\delta$ is an arbitrary constant. This choice has the virtue of satisfying both (26) and (27) with little effort. In addition, it imposes the constraint that

$$\nabla_\perp^2\xi = \delta\xi. \tag{29}$$

This equation determines the shape of this equilibrium solution with $|\delta|^{-1/2}$ setting the scale for the poloidal variation of $\xi$ and the field variables $\phi$, $\psi$, and $\chi$, which depend on $\xi$ through (18), (19), (22), and (25). Thus finding some solutions of the *nonlinear* PDE's (23) and (24) has been reduced to solving the *linear* PDE (29): the troublesome nonlinear Poisson brackets have been eliminated with the *Ansätze* $\psi = \gamma\xi$ and $U = \delta\xi$.

This class of solutions for axisymmetric equilibrium has an interesting physical interpretation. For the low beta case being considered, the magnetic field in the tokamak takes the form[1]

$$\mathbf{B} = [B_T/(1+\epsilon x)]\hat{\mathbf{z}} - \epsilon B_T \hat{\mathbf{z}}\times\nabla_\perp\psi + O(\epsilon^2). \tag{30}$$

The second term on the right-hand side represents the poloidal magnetic field, $\mathbf{B}_p$. Operating on the relation $\psi = \gamma\xi$ with $\epsilon v_A \hat{\mathbf{z}}\times\nabla_\perp$ gives[5]

$$\epsilon v_A \hat{\mathbf{z}}\times\nabla_\perp\psi = \gamma(\epsilon v_A \hat{\mathbf{z}}\times\nabla_\perp\phi). \tag{31}$$

Comparing this with (30) and the relation (4) for $\mathbf{v}_\perp$, one can see that the left side is proportional to $\mathbf{B}_p$ and the right side is essentially proportional to $\mathbf{v}_\perp$. Thus (31) can be rewritten more suggestively as

$$\mathbf{v}_\perp = (-v_A/\gamma B_T)\mathbf{B}_p + O(\epsilon^2). \tag{32}$$

This result is similar to the fluid velocity for a nonlinear Alfvén wave found by Walén.[6] However, in our case (32) does not describe a propagating wave but rather a stationary equilibrium flow.

### C. Allowing for $\tau$ and $z$ dependence

Next we complicate the previous discussion somewhat with the addition of $\tau$ and $z$ dependence to $\xi$. As for the case of axisymmetric equilibrium, we continue to take $\psi = \gamma\xi$ and $U = \delta\xi$ and use these relations in (20) and (21) to arrive at the linear equations

$$\xi_\tau + \gamma\xi_z = 0 \tag{33}$$

and

$$\gamma\xi_\tau + (1-\alpha\delta)\xi_z = 0. \tag{34}$$

These two first-order PDE's in $\xi$ will be consistent with each other if we take

$$1 - \alpha\delta = \gamma^2. \tag{35}$$

From (33) one can see that the solution for $\xi$ must be of the form $\xi = \xi(x,y,z-\gamma\tau)$, which corresponds to a structure propagating toroidally.

One can see that the structure is a generalization of the Alfvén wave by repeating here the arguments that led to (32) from the relation $\psi = \gamma\xi$ in Sec. III B. The result is the same and (32) still applies with a small change: in Sec. III B the constant $\gamma$ has no obvious interpretation, whereas here it is the toroidal propagation velocity of the wave. The disturbance in the poloidal magnetic field $\mathbf{B}_p$ propagates along the direction of the toroidal vacuum field with the following features: if $\gamma > 0$ then the perpendicular fluid velocity $\mathbf{v}_\perp$ and $\mathbf{B}_p$ are antiparallel and the wave propagates in the positive $z$ direction; if $\gamma < 0$ then $\mathbf{v}_\perp$ and $\mathbf{B}_p$ are parallel and the wave propagates in the negative $z$ direction. These features are the same as for the Alfvén wave solutions of Walén, except that (in the dimensionless coordinates) Alfvén waves propagate with speed unity, whereas from (35) the Alfvén-like solution found above propagates with speed $|\gamma| = \sqrt{1 - \alpha\delta}$. The origin of the Alfvén-like solution is the $\alpha$-dependent term on the right-hand side of the parallel component of Ohm's law (10). As mentioned before, this term accounts for the effects of pressure on parallel electron flow, and it allows for a nonzero parallel component of the electric field, $E_\parallel$, in the absence of resistivity. The $\alpha \to 0$ limit of (10) reduces to $E_\parallel = 0$ and $|\gamma| = 1$, which, of course, characterizes RMHD and Alfvén waves.

In addition to the first-order equation (33), $\xi$ must once again satisfy $\nabla_\perp^2 \xi = \delta\xi$ because of the *Ansatz* $U = \delta\xi$. As in Sec. III B, this second-order equation determines the shape of the solution: $|\delta|^{-1/2}$ again sets the scale for poloidal variations in $\xi$. Thus the compatibility relation (35) is a constraint on the toroidal propagation velocity $\gamma$ set by the physical parameter $\alpha$ and the poloidal scale $|\delta|^{-1/2}$.

Finally, note that even though there are now two equations to solve for $\xi$, they are linear and therefore much more tractable than the nonlinear equations (20) and (21).

### D. A further generalization

A more general class of solutions can be obtained with the *Ansätze*

$$\psi = \gamma\xi(x,y,z,\tau) + f(x,y,z,\tau) \tag{36}$$

and

$$U = \delta\xi(x,y,z,\tau) + h(x,y,z,\tau). \tag{37}$$

As before, $\gamma$ and $\delta$ are arbitrary constants; $f$ and $h$ are arbitrary functions. As in Secs. III B and III C, the physical content of the solutions resulting from (36) and (37) will be examined; but this is deferred until the end of Sec. III F, when the form of the solutions will be more explicit.

Let us proceed with the construction of solutions. First, note that

$$J = \gamma\delta\xi + \gamma h + \nabla_\perp^2 f \tag{38}$$

from the definition $J \equiv \nabla_\perp^2 \psi$, (36), and (37). Using (36)–(38) in the nonlinear equations (20) and (21), one obtains the following two equations linear in $\xi$:

$$\xi_\tau + \gamma\xi_z + \left[\xi, \frac{1-\gamma^2}{\delta} h + \gamma f - \frac{\gamma}{\delta}\nabla_\perp^2 f\right] = (1/\delta)\{[f,\gamma h + \nabla_\perp^2 f] - (\gamma h + \nabla_\perp^2 f)_z - h_\tau\} \tag{39}$$

and

$$\xi_\tau + \frac{1-\alpha\delta}{\gamma}\xi_z + \left[\xi, \frac{1-\alpha\delta}{\gamma} f + \alpha h\right] = (1/\gamma)\,(\alpha h_z - [f,\alpha h] - f_\tau)\,. \tag{40}$$

Note that at this stage these equations are as general as (20) and (21), where $\chi$ has been eliminated in favor of $U$ with the use of $\chi = g(z) + U$, (18)—$f$ and $h$ are still arbitrary at this point.

Next we require that these two equations be redundant. This is by no means a necessary constraint, and, in fact, we have found a class of solutions where (39) and (40) are not redundant. However, as will be made apparent below, requiring redundancy reduces (39) and (40) to a truly linear PDE for $\xi$, which still leads to interesting solutions.

After some manipulation the conditions for redundancy are found to be

$$\gamma = (1 - \alpha\delta)/\gamma, \tag{41}$$

$$\nabla_\perp^2 f = -(\delta/\gamma)p(z,\tau), \tag{42}$$

and

$$\left[f, \frac{1}{\gamma} h\right] - \frac{1}{\gamma}[h - \delta p(z,\tau)]_z + \left(\frac{\delta}{\gamma} f - h\right)_\tau = 0. \tag{43}$$

Here $p$ is an arbitrary function of $z$ and $\tau$. Once again (41) will be a constraint on the toroidal propagation velocity $\gamma$; it is the same as (35) in Sec. III B. The application of these conditions reduces (39) and (40) to the single equation

$$\xi_\tau + \gamma\xi_z + [\xi, \gamma f + \alpha h] = (\alpha\delta/\gamma)p_z - (\gamma f + \alpha h)_\tau, \tag{44}$$

a linear, first-order PDE in $\xi$. From $U = \nabla_\perp^2 \xi$ and $U = \delta\xi + h$, we obtain the additional relation

$$\nabla_\perp^2 \xi = \delta\xi + h. \tag{45}$$

Again, one can interpret $|\delta|^{-1/2}$ as setting the scale for poloidal variations in $\xi$. Thus the two nonlinear equations in $\xi$, (20) and (21), have been transformed into the two linear equations (44) and (45); these linear equations in $\xi$ are supplemented by the three redundancy conditions, (41)–(43).

Next we must ensure that (44) and (45) are compatible with one another. For convenience we introduce the operator $\mathscr{L}$ defined by

$$\mathscr{L} \equiv -\gamma\frac{\partial}{\partial z} + [\gamma f + \alpha h, \cdot\,]. \tag{46}$$

Operating on (44) with $\nabla_\perp^2$ and making use of (45) and (46), one obtains the compatibility condition

$$\delta\xi_\tau + h_\tau - \nabla_\perp^2 \mathscr{L}\xi = -\nabla_\perp^2(\gamma f + \alpha h)_\tau. \tag{47}$$

Introducing the commutator $(\nabla_\perp^2, \mathscr{L})$, defined by

$$(\nabla_\perp^2, \mathscr{L}) \equiv \nabla_\perp^2 \mathscr{L} - \mathscr{L}\nabla_\perp^2, \tag{48}$$

and using (45), one can rewrite (47) in the more interesting form,

$$\delta\xi_\tau + h_\tau - \mathscr{L}(\delta\xi + h) - (\nabla_\perp^2, \mathscr{L})\xi = -\nabla_\perp^2(\gamma f + \alpha h)_\tau. \tag{49}$$

In the interest of simplicity, we impose the constraint that

$$(\nabla_1^2,\mathscr{L})\xi\equiv 0, \tag{50}$$

for every "well-behaved" function $\xi$. For compactness we define

$$H\equiv\gamma f+\alpha h. \tag{51}$$

Then

$$\mathscr{L}\equiv-\gamma\frac{\partial}{\partial z}+[H,\cdot]. \tag{52}$$

The $-\gamma(\partial/\partial z)$ term of $\mathscr{L}$ clearly commutes with $\nabla_1^2$; thus the crucial issue is to see what

$$(\nabla_1^2,[H,\cdot])\xi=\nabla_1^2[H,\xi]-[H,\nabla_1^2\xi]=0 \tag{53}$$

requires of $H$. After some manipulation, (53) reduces to

$$-(H_{xxy}+H_{yyy})\xi_x+(H_{xxx}+H_{xyy})\xi_y+2(H_{xx}-H_{yy})\xi_{xy}-2H_{xy}\xi_{xx}+2H_{xy}\xi_{yy}=0. \tag{54}$$

Since $\xi$ is arbitrary, each coefficient of a partial derivative of $\xi$ must vanish independently of the others, resulting in four PDE's that $H$ must satisfy:

$$(H_{xx}+H_{yy})_y=(\nabla_1^2H)_y=0, \tag{55}$$

$$(H_{xx}+H_{yy})_x=(\nabla_1^2H)_x=0, \tag{56}$$

$$H_{xx}-H_{yy}=0, \tag{57}$$

and

$$H_{xy}=0. \tag{58}$$

These require

$$H=\tfrac{1}{2}a(z,\tau)(x^2+y^2)+b(z,\tau)y+c(z,\tau)x+d(z,\tau). \tag{59}$$

Here $a,b,c$, and $d$ are arbitrary functions of $z$ and $\tau$. Note that $f$ can now be eliminated in favor of $h$ and $H$ through the relation

$$f=(1/\gamma)(H-\alpha h), \tag{60}$$

which follows directly from (51).

With (50) and (51), the first-order equation (44), and the explicit definition of $\mathscr{L}$ as given by (46), we can reexpress (49) as

$$[(\alpha\delta^2/\gamma)p+\gamma h]_z+(h-\delta H+\nabla_1^2H)_\tau=\gamma[f,h]. \tag{61}$$

This relation can be further simplified with the application of (60) and the redundancy conditions (41) and (43). The result is

$$\nabla_1^2H_\tau=-(\delta/\gamma)p_z, \tag{62}$$

a much more compact form for the compatibility condition.

There are two more PDE's to consider in addition to those for $\xi$. Using (60) and (62) in the redundancy relations (42) and (43), we obtain the following pair of equations for $h$:

$$\alpha\nabla_1^2h=\nabla_1^2H+\delta p \tag{63}$$

and

$$h_\tau-\mathscr{L}h=-(\gamma^2\nabla_1^2H-\delta H)_\tau. \tag{64}$$

The compatibility of these equations is treated in much the same way as for the $\xi$ equations: taking $\nabla_1^2$ of both sides of (64) and making use of (50), (62), and (63) in the result, one obtains the condition

$$\nabla_1^2H_z=-(\delta/\gamma)p_\tau. \tag{65}$$

It is more enlightening to rewrite (62) and (65) together, using

$$\nabla_1^2H=2a(z,\tau), \tag{66}$$

which follows from the definition of $H$. Doing this, one obtains the complementary relations

$$2a_\tau=-(\delta/\gamma)p_z \tag{67}$$

and

$$2a_z=-(\delta/\gamma)p_\tau. \tag{68}$$

From these equations it is easily found that

$$a=a_1(z+\tau)+a_2(z-\tau) \tag{69}$$

and

$$p=-(2\gamma/\delta)[a_1(z+\tau)-a_2(z-\tau)]+\kappa, \tag{70}$$

where $\kappa$ is an arbitrary constant and $a_1$ and $a_2$ are arbitrary functions. Thus consideration of the compatibility of the PDE's for $h$ and $\xi$ has yielded information about the structure of $p$ and $a$ and the relationship between them.

We next distill our four PDE's for $\xi$ and $h$ to two essential equations for $\xi$ alone. We first collect the four equations for $\xi$ and $h$. Recall from (45) that the second-order equation for $\xi$ can be expressed as

$$h=\nabla_1^2\xi-\delta\xi. \tag{71}$$

Using (46), (51), and (67), we can rewrite the first-order equation for $\xi$, (44), as

$$\xi_\tau-\mathscr{L}\xi=-(2\alpha a+H)_\tau. \tag{72}$$

With (66) and (70) the second-order relation for $h$, (63), becomes

$$\alpha\nabla_1^2h=2(1-\gamma)a_1+2(1+\gamma)a_2+\delta\kappa. \tag{73}$$

The first-order relation (64) becomes

$$h_\tau-\mathscr{L}h=-(2\gamma^2a-\delta H)_\tau, \tag{74}$$

with the use of (66).

Now we can eliminate $h$ from (73) using (71) to obtain

$$\alpha\nabla_1^2(\nabla_1^2\xi-\delta\xi)=2(1-\gamma)a_1+2(1+\gamma)a_2+\delta\kappa, \tag{75}$$

a fourth-order relation for $\xi$. With the help of (71) and (72), it is easy to show that (74), in fact, reduces to an identity. We also note that it is not difficult to directly ascertain that (72) and (75) are compatible with each other as they stand; no further constraints are needed to ensure their compatibility.

At this point we have only to integrate the linear equations (72) and (75) for $\xi$ to obtain a complete, explicit solution for our original system of nonlinear fluid equations. The shape, the spatial variation of $\xi$ is determined by (75); in Sec. III F, the integration of the first-order PDE (72) is considered in some detail to determine how solutions propagate. The following algorithm summarizes the results of this section.

## E. An algorithm for constructing solutions

(1) Choose values for the constants $\delta$ and $\alpha$ and thus determine $\gamma$ from the relation

$$\gamma=\sigma\sqrt{1-\alpha\delta}, \tag{76}$$

which follows from (41). Here $\sigma=\pm 1$. In Sec. III F, it will be seen that $\gamma$ is the toroidal propagation velocity of solutions. As mentioned after (45), $|\delta|^{-1/2}$ sets a scale for the poloidal variation of $\xi$. The parameter $\alpha$ represents the coupling of RMHD and CHM-equation physics, and it can take values from 0 to $\sim 1$; recall (11)–(13) and the discussion following them.

(2) Specify

$$H\equiv\tfrac{1}{2}[a_1(z+\tau)+a_2(z-\tau)](x^2+y^2) + b(z,\tau)y+c(z,\tau)x+d(z,\tau) \tag{77}$$

by choosing the functions $a_1$, $a_2$, $b$, $c$, and $d$. This form for $H$ is obtained after incorporating into (59) the information we obtained about the function $a$ from (67) and (68). The role that $H$ plays in determining the propagation of solutions and its physical significance will be discussed in Sec. III F. [In fact, (81) below shows that $H$ contributes directly to the poloidal flux $\psi$.]

(3) Choose a value for the constant $\kappa$ and then find $\xi$ by integrating

$$\xi_\tau+\sigma\sqrt{1-\alpha\delta}\,\xi_z+[\xi,H]=-(2\alpha[a_1+a_2]+H)_\tau \tag{78}$$

and

$$\alpha\nabla_\perp^2(\nabla_\perp^2\xi-\delta\xi)=2(1-\sigma\sqrt{1-\alpha\delta})a_1 + 2(1+\sigma\sqrt{1-\alpha\delta})a_2+\delta\kappa. \tag{79}$$

Equations (78) and (79) follow from (72), (75), and (76). Equation (79) determines the shape of solutions. In Sec. III F the integration of (78) is carried out explicitly; there it will be seen that (78) determines the propagation of solutions.

(4) Choose the function $g(z)$. The solutions for the field variables $\phi$, $\psi$, and $\chi$ readily follow: from (19),

$$\phi=\xi+\alpha g; \tag{80}$$

from (36), (60), and (71),

$$\psi=(1/\sigma\sqrt{1-\alpha\delta})(\xi-\alpha\nabla_\perp^2\xi+H); \tag{81}$$

and from (18) and (22),

$$\chi=g+\nabla_\perp^2\xi. \tag{82}$$

Other physical quantities of interest are the vorticity,

$$U=\nabla_\perp^2\xi, \tag{22}$$

and the parallel current,

$$J=\sigma\sqrt{1-\alpha\delta}\nabla_\perp^2\xi+2(a_1-a_2)-(\delta/\sigma\sqrt{1-\alpha\delta})\kappa, \tag{83}$$

which follows from the definition $J\equiv\nabla_\perp^2\psi$, (81), and (79).

Next the integration of the first-order equation (78) is considered.

## F. Integration of the first-order equation for $\xi$

We integrate the first-order PDE for $\xi$ (78) by the method of characteristics. The characteristics are determined by integrating the following system of ODE's associated with (78):

$$\frac{dx}{d\tau}=H_y=a(z,\tau)y+b(z,\tau), \tag{84}$$

$$\frac{dy}{d\tau}=-H_x=-a(z,\tau)x-c(z,\tau), \tag{85}$$

$$\frac{dz}{d\tau}=\gamma, \tag{86}$$

and

$$\frac{d\xi}{d\tau}=-(2\alpha a+H)_\tau. \tag{87}$$

Relation (76) has been used to make (86) more compact for the sake of the work to follow; $a$ is given by (69). A dynamical picture showing that these characteristic equations determine how solutions propagate is given toward the end of this section; for now we proceed with their integration.

We can readily integrate (86) to find

$$z=z_0+\gamma\tau, \tag{88}$$

where $z_0$ is a constant of integration.

We can use (88) to replace $z$ wherever it occurs in (84), (85), and (87) to facilitate the integration of these equations. For the moment let us focus upon (84) and (85). From (84) we obtain

$$\frac{dx}{d\tau}=H_y=a(z_0+\gamma\tau,\tau)y+b(z_0+\gamma\tau,\tau); \tag{89}$$

from (85),

$$\frac{dy}{d\tau}=-H_x=-a(z_0+\gamma\tau,\tau)x-c(z_0+\gamma\tau,\tau). \tag{90}$$

Note that this pair of equations is Hamiltonian in structure, with $H$ playing the role of the Hamiltonian function that governs the dynamics of $x$ and $y$. This Hamiltonian structure has an interesting dynamical consequence for the propagation of solutions.

Writing the *homogeneous* form of (89) and (90) in terms of matrices, we have

$$\frac{d}{d\tau}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0 & a(z_0+\gamma\tau,\tau)\\ -a(z_0+\gamma\tau,\tau) & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}. \tag{91}$$

It is interesting to note that if we define a position vector $\mathbf{r}\equiv x\hat{\mathbf{x}}+y\hat{\mathbf{y}}$ then (91) can be written in the form

$$\frac{d\mathbf{r}}{d\tau}=\mathbf{r}\times\hat{\mathbf{z}}a(z_0+\gamma\tau,\tau), \tag{92}$$

which describes a gyration in the $(x,y)$ plane with a *time-dependent* frequency $a(z_0+\gamma\tau,\tau)$. Consequently, it is not surprising that a fundamental matrix for (91) is

$$\boldsymbol{\mu}(\tau;z_0)=\begin{bmatrix}\cos\left(\int_0^\tau a(z_0+\gamma\tau',\tau')d\tau'\right) & \sin\left(\int_0^\tau a(z_0+\gamma\tau',\tau')d\tau'\right)\\ -\sin\left(\int_0^\tau a(z_0+\gamma\tau',\tau')d\tau'\right) & \cos\left(\int_0^\tau a(z_0+\gamma\tau',\tau')d\tau'\right)\end{bmatrix}, \tag{93}$$

which reduces to the identity matrix at $\tau=0$.

With the fundamental matrix $\boldsymbol{\mu}$ at our disposal, we can write the solution to (89) and (90) as

$$\mathbf{r}=\boldsymbol{\mu}(\tau;z_0)\mathbf{r}(0)+\boldsymbol{\mu}(\tau;z_0)\int_0^\tau \boldsymbol{\mu}^{-1}(\tau';z_0)\mathbf{f}(\tau';z_0+\gamma\tau')d\tau'. \tag{94}$$

Here

$$\mathbf{r}=\begin{bmatrix}x\\ y\end{bmatrix}; \tag{95}$$

$$\mathbf{r}(0)=\begin{bmatrix}x_0\\ y_0\end{bmatrix}, \tag{96}$$

where $x_0$ and $y_0$ are arbitrary constants of integration; and

$$\mathbf{f}(\tau;z_0+\gamma\tau)=\begin{bmatrix}b(z_0+\gamma\tau,\tau)\\ -c(z_0+\gamma\tau,\tau)\end{bmatrix}. \tag{97}$$

To construct the most general solution of the homogeneous form of (78), we need three functionally independent first integrals. One of these first integrals is

$$z_0(x,y,z,t)=z-\gamma\tau, \tag{98}$$

which follows from (88). Two additional first integrals are

$$\begin{bmatrix}x_0(x,y,z,\tau)\\ y_0(x,y,z,\tau)\end{bmatrix}=\boldsymbol{\mu}^{-1}(\tau;z-\gamma\tau)\begin{bmatrix}x\\ y\end{bmatrix}-\int_0^\tau \boldsymbol{\mu}^{-1}(\tau';z-\gamma\tau)\mathbf{f}(\tau';z-\gamma[\tau-\tau'])d\tau', \tag{99}$$

which follow from (94) upon replacing $z_0$ with $z-\gamma\tau$. The general solution to the homogeneous form of (78) is thus $\xi_h(x_0,y_0,z_0)$, an arbitrary function of the first integrals.

Here one can see the significance of the characteristic equations (84)–(86) for $x$, $y$, and $z$. The function $\xi_h(x_0,y_0,z_0)$ represents a structure propagating in a rather complicated way. For concreteness, suppose that at $\tau=0$, $\xi_h$ represents a function that is defined over some finite volume of space, $V_0$, and that at $\tau=0$ some point in $V_0$ has the coordinates $(x_i,y_i,z_i)$. The value of $\xi_h$ at this point is $\xi_h(x_i,y_i,z_i)\equiv\xi_h(i)$, a constant. The relations $x_0(x,y,z,\tau)=x_i$, $y_0(x,y,z,\tau)=y_i$, and $z_0(x,y,z,\tau)=z_i$ define a curve parametrized by $\tau$—the curve is a solution of (84)–(86) for the initial conditions $(x_i,y_i,z_i)$. Along this curve the value of $\xi_h$ remains $\xi_h(i)$. Thus each point in $V_0$ serves as an initial condition for a curve along which $\xi_h$ is constant, and thus the characteristic equations (84)–(86) determine a flow in $(x,y,z)$ space that maps each point of $V_0$ and its corresponding value of $\xi_h$ into another volume at time $\tau$, $V_\tau$. As time varies, the flow determined by (84)–(86) carries the volume $V_\tau$ along with it and may deform the shape of $V_\tau$ in very complicated ways. This is just a Lagrangian description of the propagation of the solution $\xi_h(x_0,y_0,z_0)$; $\xi_h(x_0,y_0,z_0)$ itself supplies the corresponding Eulerian description.

Some qualitative information on the propagation of $\xi_h$ follows from the Hamiltonian structure of (89) and (90). These equations describe motion in the poloidal plane $z=z_0+\gamma\tau$—this plane is moving in the $z$ (toroidal) direction with speed $\gamma$. Equations (89) and (90) provide a Lagrangian description of a flow confined to this poloidal plane: their solution, given by (94), maps the initial condition $(x_0,y_0)$ into the point $[x,(\tau),y(\tau)]$. Let $S_0$ be an area in the plane at time $\tau=0$; then each point of this area will be mapped into another area $S_\tau$ at time $\tau$. Now the Hamiltonian structure of (89) and (90) suggests that one interpret $x$ as a coordinate and $y$ as the momentum conjugate to $x$, so that the poloidal $(x,y)$ plane is also a phase space. Thus by Liouville's theorem the flow in the poloidal plane preserves area: $S_0$ and $S_\tau$ span the same area although, in general, they may not share the same shape because the flow might deform it.

Consequently, this also guarantees that the three-dimensional flow mapping the volume $V_0$ to $V_\tau$ in $(x,y,z)$ space preserves volume: as time $\tau$ varies, any poloidal cross section of $V_\tau$ may undergo area-preserving deformations as described in the paragraph above; $V_\tau$ does not suffer any elongations or compressions in the $z$ direction because the motion in that direction is a simple rigid translation at constant speed $\gamma$.

Finally, we obtain the general solution $\xi$ for the inhomogeneous equation (78). Equation (87) determines how $\xi$ will vary with $\tau$ along a characteristic curve determined by (84)–(86); $\xi$ is not generally a constant along such a curve because the right-hand side of (87) is not necessarily zero. Substituting the solutions for $x$, $y$, and $z$ found in (88) and (94) into the right side of (87) leaves $d\xi/d\tau$ equal to a function of the initial conditions $(x_0,y_0,z_0)$ and $\tau$ along a characteristic curve. Integrating the resulting expression with respect to $\tau$ along such a curve gives

$$\xi=\xi_h(x_0,y_0,z_0)-\int_0^\tau(2\alpha a+H)_\tau[x(x_0,y_0,\tau'),y(x_0,y_0,\tau'),z_0+\gamma\tau',\tau']d\tau'. \tag{100}$$

The first term on the right side is simply a constant of integration, an arbitrary function of the initial conditions. Equation (100) is the Lagrangian form of the general solution for (78). One recognizes the first term on the right side as the general solution for the homogeneous form of (78) and the second term as a "particular integral" that satisfies (78).

We can now comment on physical characteristics of the more general class of solutions discussed in Sec. III D, and we can compare them with the solutions found in Sec. III C. First, (81) is rewritten using the relations (80) and (82) for $\phi$ and $\chi$ to obtain

$$\psi = (1/\gamma)(\phi - \alpha\chi + H), \tag{101}$$

an explicit relation between $\psi$ and the other fields $\phi$ and $\chi$. As was done in Secs. III B and III C, $\epsilon v_A \hat{\mathbf{z}}\times\nabla_\perp$ is applied to both sides of (101) to obtain the following relation for $\mathbf{B}_p$:

$$-(\gamma v_A/B_T)\mathbf{B}_p = \mathbf{v}_\perp + \epsilon v_A \hat{\mathbf{z}}\times\nabla_\perp(-\alpha\chi + H) + O(\epsilon^2). \tag{102}$$

Here $\mathbf{B}_p = -\epsilon B_T \hat{\mathbf{z}}\times\nabla_\perp\psi$, from (30), and (4) for $\mathbf{v}_\perp$ have been used. Clearly for this class of solutions, $\mathbf{B}_p$ is not necessarily proportional to $\mathbf{v}_\perp$. It follows from the modified Ohm's law (10) that the $\alpha\chi$ term characterizes pressure effects on parallel electron flow. The $H$-dependent term does not vanish in the RMHD, $\alpha\to 0$ limit. It is, in fact, $H$ that allows for poloidal motion of the solutions—it is the Hamiltonian function that governs the poloidal dynamics, as was pointed out in the discussion of the characteristic equations in Sec. III F. Thus the solutions characterized by (102) are quite different from the Alfvén-like ($\alpha\neq 0$) and Alfvén ($\alpha = 0$) waves discussed in Sec. III C because they do not simply propagate toroidally (in the $z$ direction) at constant speed and because $\mathbf{B}_p$ is not necessarily proportional to $\mathbf{v}_\perp$, even for the RMHD limit.

Another comment about $H$: from (101) and (102) one sees that $H$ makes a contribution to $\psi$ and $\mathbf{B}_p$. In fact, from (84) and (85) the poloidal velocity of a particle moving along a characteristic curve is $-\hat{\mathbf{z}}\times\nabla_\perp H$. Thus $H$ has the dual distinction of determining poloidal propagation characteristics and determining a part of the poloidal magnetic field as well.

## G. Examples

Here we consider some special cases of (100) obtained by specializing $a$, $b$, $c$, and $d$ in $H$ of (77). Our choices for these four functions will determine the structure of the first integrals $x_0$ and $y_0$ through (99). For the cases we consider, their structure will be easy to discern and will give some insight into the behavior of $\xi$. How $\mathbf{B}_p$ will propagate in each case is pointed out to make the discussion more physically concrete. To conclude, a physical interpretation for the terms of $H$ and the role they play in determining how solutions propagate are discussed as well.

### 1. Case (i)

The first case we consider is a rather drastic simplification of the general result (99): *we take a, b, c, and d all to be zero, getting rid of H entirely.* Then we are simply left with

$$x_0 = x \quad \text{and} \quad y_0 = y. \tag{103}$$

Thus, in this case, the general solution for $\xi$ is of the form

$$\xi = \xi(x,y,z-\gamma\tau), \tag{104}$$

which corresponds to a structure propagating toroidally with speed $\gamma$.

In this case (81) reduces to

$$\psi = (1/\gamma)(\xi - \alpha\nabla_\perp^2\xi)(x,y,z,-\gamma\tau). \tag{105}$$

The arguments in parentheses stress that $\psi$ moves in exactly the same way as $\xi$: surfaces of constant poloidal flux simply propagate in the $z$ direction with constant velocity $\gamma$. Applying $\mathbf{B}_p = -\epsilon B_T \hat{\mathbf{z}}\times\nabla_\perp\psi$ to (105) shows that the disturbance $\mathbf{B}_p$ also propagates in the same way: if we follow a point moving along a characteristic curve, $\mathbf{B}_p$ at the point will be a constant vector. However, from (105) and the arguments given at the end of Sec. III F, the solution is not necessarily an Alfvén-like wave because, in general, $\mathbf{B}_p$ will not be proportional to $\mathbf{v}_\perp$ for this case.

### 2. Case (ii)

*Next let us take a to be a nonzero constant, still keeping b, c, and d zero.* Then

$$x_0 = x\cos a\tau - y\sin a\tau \tag{106}$$

and

$$y_0 = x\sin a\tau + y\cos a\tau. \tag{107}$$

If we introduce the poloidal polar coordinates $r$ and $\theta$ such that

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta, \tag{108}$$

then (106) and (107) may be written as

$$x_0 = r\cos(\theta + a\tau) \quad \text{and} \quad y_0 = r\sin(\theta + a\tau). \tag{109}$$

Consequently, the general solution for $\xi$ can take the form

$$\xi = \xi(r,\theta + a\tau, z - \gamma\tau). \tag{110}$$

Thus $\xi$ represents a structure that rotates poloidally with speed $a$, in addition to propagating toroidally.

Now consider the behavior of $\mathbf{B}_p$ for this case. With regards to propagation there are two distinct pieces of $\psi$: $\xi - \alpha\nabla_\perp^2\xi$, which propagates according to the characteristic equations, and $H = \frac{1}{2}a(x^2 + y^2)$, which is static. Thus $-\epsilon B_T\hat{\mathbf{z}}\times\nabla_\perp(1/\gamma)H$ is a static component of $\mathbf{B}_p$, whereas $-\epsilon B_T\hat{\mathbf{z}}\times\nabla_\perp(1/\gamma)(\xi - \alpha\nabla_\perp^2\xi)$ is the part that will propagate in the way described below.

For convenience, define

$$G(x_0,y_0,z_0) \equiv (1/\gamma)(\xi - \alpha\nabla_\perp^2\xi), \tag{111}$$

where the Cartesian forms (106) and (107) are used for the first integrals $x_0$ and $y_0$. Then

$$\begin{bmatrix} (-\hat{\mathbf{z}}\times\nabla_\perp G)_x \\ (-\hat{\mathbf{z}}\times\nabla_\perp G)_y \end{bmatrix} = \begin{bmatrix} \cos a\tau & \sin a\tau \\ -\sin a\tau & \cos a\tau \end{bmatrix} \begin{bmatrix} G_2 \\ -G_1 \end{bmatrix}. \tag{112}$$

Here the subscripts $x$ and $y$ on the left side denote the $x$ and $y$ components of $-\hat{\mathbf{z}}\times\nabla_\perp G$, and the subscripts 1 and 2 on the right side denote the partial derivatives of $G(x_0,y_0,z_0)$ with respect to its first and second arguments. Thus $G_2$ and $G_1$ are functions of the first integrals, and consequently they are constant along the characteristic curves, defined by

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos a\tau & \sin a\tau \\ -\sin a\tau & \cos a\tau \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \tag{113}$$

and

$$z = z_0 + \gamma\tau, \tag{114}$$

where $x_0$, $y_0$, and $z_0$ are now constants, the initial conditions for the curve. It is clear that the curve represented by (113) and (114) is a helix. If $a > 0$, then the helix is left handed; if $a < 0$, the helix is right handed. For definiteness in what follows, it is assumed that $a > 0$.

The behavior of the vector $-\hat{\mathbf{z}}\times\nabla_\perp G$ along a characteristic curve can be described as follows. First, picture a point moving along the curve as a function of time according to (113) and (114). Now picture a set of poloidal $x'$ and $y'$ axes with their origin at this point; they move along with the point so that the $x'$ axis is always parallel to the fixed $x$ axis and the $y'$ axis is always parallel to the fixed $y$ axis. Then from (112) one sees that relative to these primed axes, the vector $-\hat{\mathbf{z}}\times\nabla_\perp G$ at the moving point will rotate clockwise with frequency $a$. The vector maintains a fixed magnitude because it is moving along a characteristic curve and thus the components $G_2$ and $-G_1$ in (112) are constants.

Now note that

$$-\hat{\mathbf{z}}\times\nabla_\perp H = -\hat{\mathbf{z}}\times a(x\hat{\mathbf{x}} + y\hat{\mathbf{y}}) = a(y\hat{\mathbf{x}} - x\hat{\mathbf{y}}) \tag{115}$$

defines a vector field whose flow lines are circles centered about the origin in the poloidal $(x,y)$ plane. The vectors tangent to a given circle point in the clockwise sense, and they are all of the same magnitude. For a point moving along a characteristic helix, the vector $-\hat{\mathbf{z}}\times\nabla_\perp H$ always points in the clockwise direction and has a constant magnitude because as the point moves, it is always at a fixed radial distance $\sqrt{x_0^2 + y_0^2}$ from the $z$ axis in the poloidal plane. Thus the behavior of the total poloidal field,

$$\mathbf{B}_p = -\epsilon B_T \hat{\mathbf{z}}\times\nabla_\perp [G + (1/\gamma)H], \tag{116}$$

along a characteristic helix is the same as for $-\hat{\mathbf{z}}\times\nabla_\perp G$ alone: relative to the $x'$ and $y'$ axes, $\mathbf{B}_p$ has a fixed magnitude while it rotates with frequency $a$.

### 3. Case (iii)

*We take $a$, $b$, $c$, and $d$ all to be nonzero constants.* Then we find that

$$x_0 = \left(x + \frac{c}{a}\right)\cos a\tau - \left(y + \frac{b}{a}\right)\sin a\tau - \frac{c}{a} \tag{117}$$

and

$$y_0 = \left(x + \frac{c}{a}\right)\sin a\tau + \left(y + \frac{b}{a}\right)\cos a\tau - \frac{b}{a}. \tag{118}$$

We can drop the constants at the end of each of these relations to obtain another perfectly good pair of first integrals,

$$\hat{x}_0 = [x + (c/a)]\cos a\tau - [y + (b/a)]\sin a\tau \tag{119}$$

and

$$\hat{y}_0 = [x + (c/a)]\sin a\tau + [y + (b/a)]\cos a\tau. \tag{120}$$

Thus we see that as for case (ii) above, the general solution for $\xi$ represents a structure exhibiting poloidal rotation with speed $a$, except that now the rotation occurs about the origin with $(x,y)$ coordinates $(-c/a, -b/a)$. To make this more explicit, we introduce the polar coordinates $r_1$ and $\theta_1$, such that

$$x + (c/a) = r_1\cos\theta_1 \quad\text{and}\quad y + (b/a) = r_1\sin\theta_1. \tag{121}$$

Then the general solution for $\xi$ will take the form

$$\xi = \xi(r_1, \theta_1 + a\tau, z - \gamma\tau). \tag{122}$$

For this case the behavior of $\mathbf{B}_p$ is exactly the same as for case (ii), except that now the characteristic helices have their symmetry axis passing through $(-c/a, -b/a)$ rather than the origin in the poloidal plane.

### 4. Case (iv)

*If we take $a = 0$ and $b$, $c$, and $d$ nonzero constants*, then we obtain the first integrals

$$x_0 = x - b\tau \quad\text{and}\quad y_0 = y + c\tau. \tag{123}$$

Therefore the general solution for $\xi$ takes the form

$$\xi = \xi(x - b\tau, y + c\tau, z - \gamma\tau), \tag{124}$$

representing a structure that propagates rectilinearly through space with the velocity $b\hat{\mathbf{x}} - c\hat{\mathbf{y}} + \gamma\hat{\mathbf{z}}$. Thus if we follow a point moving along a characteristic curve, $\mathbf{B}_p$ at the point will be a constant vector.

For the special cases considered above, a physical interpretation can be given to each term of $H$. We examine the form of $H$ given in case (iii),

$$H = \tfrac{1}{2}a(x^2 + y^2) + by + cx + d, \tag{125}$$

because it has all the terms considered in the various cases. Consider the terms quadratic in $x$ and $y$:

$$-\hat{\mathbf{z}}\times\nabla_\perp [\tfrac{1}{2}a(x^2 + y^2)] = a(y\hat{\mathbf{x}} - x\hat{\mathbf{y}}); \tag{126}$$

$$\nabla_\perp^2 [\tfrac{1}{2}a(x^2 + y^2)] = 2a. \tag{127}$$

These terms thus contribute the static field (126) discussed in case (ii) to $\mathbf{B}_p$; (127) shows that the source of this contribution is a static, uniform toroidal current density. Now consider the remaining terms in $H$:

$$-\hat{\mathbf{z}}\times\nabla_\perp (by + cx + d) = b\hat{\mathbf{x}} - c\hat{\mathbf{y}}; \tag{128}$$

$$\nabla_\perp^2 (by + cx + d) = 0. \tag{129}$$

These relations show that (128) is a static vacuum field contribution to $\mathbf{B}_p$ in cases (iii) and (iv). All the above comments, appropriately generalized, apply when $a$, $b$, $c$, and $d$ depend on $z$ and $\tau$ as well.

The simple cases we have considered above are sufficient to show one how the parameters $a$, $b$, $c$, and $d$ determine the structure of the general solution for $\xi$, even when one generalizes to the case where $a$, $b$, $c$, and $d$ depend on $z$ and $\tau$. The parameter $a$ determines the speed of rotation about some origin in the poloidal, $(x,y)$ plane; $b$ and $c$ determine the center or origin for that poloidal rotation. In case (iv) with only $a = 0$, the finite shift becomes a rectilinear propagation with speed and direction determined by $b$, $c$, and $\gamma$.

Note that for the cases we have considered, with $a$, $b$, $c$, and $d$ all constants, the first-order equation for $\xi$ given by (78) is homogeneous: the source term $-(2\alpha a + H)_\tau = 0$. Thus the remarks of the preceding paragraph—appropriately generalized for $a$, $b$, $c$, and $d$ depending on $z$ and $\tau$—

only apply to the homogeneous part $\xi_h(x_0,y_0,z_0)$ of the general solution for $\xi$ given in (100). Note also that $d$ plays no role in determining the structure of the homogeneous solution—$d$ only appears in the source term $-(2\alpha a+H)_\tau$ of (78). Thus $d$ only plays a role in determining the particular integral for (78).

The next step is to integrate the fourth-order equation (79) to completely determine $\xi$. We prefer to defer this to a future publication.

## IV. SUMMARY AND DISCUSSION

We have constructed exact analytic solutions to a system of nonlinear plasma fluid equations that combine RMHD and CHM drift dynamics. (The resistivity of the plasma was neglected: $\hat{\eta}=0$.) In Sec. III A, with the *Ansatz* (18) for the relation between the density perturbation $\chi$ and the vorticity $U$, the problem of finding exact solutions was reduced to the integration of the two nonlinear equations (20) and (21). Their integration was further reduced to the integration of linear equations for the cases discussed in Sec. III; these cases are summarized below.

In Sec. III B, a class of axisymmetric equilibrium solutions describing stationary flows was found with the *Ansatz* $\psi=\gamma\xi$ and $U=\delta\xi$. The spatial behavior of these solutions is constrained by the linear, second-order PDE (29). The relation for the poloidal flux $\psi=\gamma\xi$ requires that $\mathbf{B}_p$ be proportional to $\mathbf{v}_\perp$, as expressed by (32).

In Sec. III C, Alfvén-like wave solutions were found by relaxing the constraints of axisymmetry and equilibrium. Their shape is again constrained by the second-order equation (29), but their propagation is determined by the linear, first-order PDE (33). These solutions behave like Alfvén waves (with $\mathbf{B}_p$ proportional to $\mathbf{v}_\perp$), except for the toroidal propagation velocity $\gamma$, constrained by (35).

Next, in Sec. III D, the solutions found in Sec. III C were generalized by adding arbitrary functions to the forms $\psi=\gamma\xi$ and $U=\delta\xi$ to obtain (36) and (37). The integration of (20) and (21) was again reduced to solving two linear equations—the fourth-order PDE (79) that determines the spatial behavior of solutions and the first-order PDE (78) that determines how they propagate. In Sec. III F we integrated the first-order equation explicitly by solving the associated characteristic ODE's to obtain a Lagrangian description of the propagation of solutions. From this and the physical implications of relation (81) for $\psi$, it was found that these solutions are distinct from the Alfvén-like solutions found in Sec. III C—they do not, in general, simply propagate in the toroidal direction at constant velocity $\gamma$, and $\mathbf{B}_p$ is not necessarily proportional to $\mathbf{v}_\perp$. These traits were made more explicit in Sec. III G, where some simple, special-case solutions to the first-order PDE and the behavior of $\mathbf{B}_p$ along the corresponding characteristic curves were considered in detail.

We conclude with a discussion of the limitations of our method for constructing solutions and where it might be modified to obtain classes of solutions distinct from the ones thus far presented.

First of all, one should note that the nonlinearities in the fluid equations—(15)–(17) in Sec. III A—exhibit a special structure: they exclusively take the form of the *Poisson brackets* defined by (8) involving the field variables $\phi$, $\psi$, $\chi$, and $U$, and $J$. The key feature of our method is the elimination of these nonlinear Poisson brackets, leaving only linear equations to integrate. Thus this approach is by no means a general method of constructing solutions for any given nonlinear system of equations.

The starting point for the construction of the more general solutions in Sec. III D is the *Ansätze* (18), (36), (37), which reduce the nonlinear system (15)–(17) to two equations linear in $\xi$, (39) and (40). However, the way we have chosen to proceed after this starting point is not unique; there are, at least, two principal points in the development that can be modified. First, recall that having (39) and (40) be redundant is a matter of choice: it is possible to construct a class of solutions distinct from the one hitherto discussed without this condition. Next, note that after imposing the redundancy of (39) and (40), having the commutator vanish in (50) is also a matter of choice. Relaxing this constraint will result in another distinct class of solutions.

As mentioned before, the *Ansätze* (36) and (37) are generalizations of the forms $\psi=\gamma\xi$ and $U=\delta\xi$, which were used to solve the system (15)–(17) under the assumption of axisymmetric equilibrium in Sec. III B. This particular class of equilibrium solutions, which serves as the origin of our construction, is an especially simple case of the equilibria possible for the nonlinear system.[5] To serve as the starting point for the construction of other propagating solutions, perhaps even more interesting equilibrium solutions could be generalized by adding arbitrary functions that relax the constraints of spatial symmetry and equilibrium.

Finally, we note that it might also be interesting to investigate how our means of constructing solutions fits into the framework of symmetry (Lie group) methods for the integration of systems of PDE's. Perhaps the explicit application of these methods to the nonlinear system we have considered would also yield physically interesting solutions.

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