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SINGULAR EIGENFUNCTIONS FOR SHEARING FLUIDS

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We study the linear stability of a two-dimensional, inviscid shear flow in a channel (Rayleigh's problem[1]). At the point where the advection of a perturbation by the mean flow equals its phase speed, Rayleigh's equation contains a singular point. This is a local wave-mean flow resonance like the wave-particle resonances encountered in plasma physics, and generates a continuous eigenvalue spectrum.

We use techniques familiar in plasma physics[2] to construct singular eigensolutions for shearing fluid. These methods allow us to verify the existence and uniqueness of representative singular eigensolutions. By considering the initial-value problem, we establish that the continuum, together with any complex pairs of growing/decaying modes, form a complete basis set.

FORMULATION

Consider a channel, extending to $\pm\infty$ in the streamwise direction x, but bounded across the stream in y. Shearing equilibria of the two-dimensional Euler equations are then given by any velocity profile, $U(y)$. Here we consider monotonic cases, for which U is a single-valued function.

Perturbations to the equilibria can be characterized by their vorticity and a stream function,

$$\omega(y)\exp ik(x-ut), \qquad \psi(y)\exp ik(x-ut),$$

for streamwise wavenumber, k, and wave speed u. These quantities satisfy Rayleigh's equation,

$$(U-u)\,\omega = (U-u)(\psi'' - k^2\psi) = U''\psi. \quad (1)$$

The boundary conditions are that ψ vanish at the walls of the channel, which are located at $y = \pm 1$.

This second-order differential equation is singular at the point y_* for which $u = U(y)$. This is the *critical layer* of the mode. There are in general three types of eigensolutions: *(a)* The singular neutral modes of the continuum, which is intrinsically irregular at the critical layer. *(b)* Smooth, neutral modes for which $y_* = y_I$, where y_I is an inflexion point, $U''(y_I) = 0$. These we call *inflexion-point modes. (c)* Complex conjugate pairs which exist when flow profile contains an inflexion point, and correspond to growing/decaying discrete modes.

REGULARIZATION

In order to construct the singular eigensolutions, we follow van Kampen,[2] and rewrite (1) in the form,

$$\omega = \mathcal{P}\left(\frac{U''\psi}{U-u}\right) + \mathcal{C}\delta(y-y_*),$$

where $\mathcal{P}$ signifies "principal value," and $\mathcal{C}$ is (as yet) arbitrary. If we eliminate ω, then we find an inhomogeneous, integral equation,

$$\psi = \mathcal{P}\int_{-1}^{1}\mathcal{K}(y,y')\frac{U''(y')\psi(y')}{U(y')-u}dy' + \mathcal{C}\mathcal{K}(y,y_*), \quad (2)$$

where $\mathcal{K}(y,y')$ is the Green function of the Laplacian, $\psi'' - k^2\psi$. The kernel of this integral equation is singular at the critical layer, but the inhomogeneous term contains an arbitrary constant, $\mathcal{C}$. Furthermore, we have the freedom of a normalization condition. We select a normalization that regularizes the singular problem:

$$1 = \int_{-1}^{1}\omega(y)dy = \mathcal{C} + \mathcal{P}\int_{-1}^{1}\frac{U''\psi}{U-u}dy'. \quad (3)$$

With such a selection for $\mathcal{C}$, equation (2) becomes

$$\psi = \mathcal{K}(y;y_*) + \int_{-1}^{1}\mathcal{F}(y,y';y_*)\psi(y')dy', \quad (4)$$

where the *regular* kernel is

$$\mathcal{F}(y,y';y_*) = \left[\frac{\mathcal{K}(y,y') - \mathcal{K}(y,y_*)}{U(y) - U(y_*)}\right] U''(y').$$

Equation (4) is an inhomogeneous Fredholm equation of the second kind. We can call on Fredholm theory to establish that (unless there are homogeneous solutions which superficially complicate matters) there is a unique, particular solution for every y_* within the channel.

INITIAL-VALUE PROBLEM

A general superposition of singular eigensolutions is

$$\Omega(y) = \mathcal{P}U'' \int_{-1}^{1} \frac{\Lambda(y_*)\psi(y;y_*)}{U(y) - U(y_*)} dy_*$$

$$+\Lambda\left[1 - \mathcal{P}\int_{-1}^{1} \frac{U''(y')\psi(y';y)}{U(y') - U(y)} dy'\right]. \quad (5)$$

If we extend the coordinates y and y_* off the real axis, we can use methods of singular integral equation theory to invert this equation and write Λ in terms of Ω. Then,

$$\Lambda = \frac{1}{\epsilon_R^2 + \epsilon_I^2}\left[U''\mathcal{P}\int_{-1}^{1} \frac{\psi(y';y)\Omega(y')}{U(y') - U(y)} dy'\right.$$

$$\left. + \left(1 + \mathcal{P}\int_{-1}^{1} \frac{U''(y')\psi(y';y)}{U(y') - U(y)} dy'\right)\Omega\right] \quad (6)$$

where

$$\epsilon_R = 1 - \mathcal{P}\int_{-1}^{1} \frac{U''(y')\psi(y';y)}{U(y') - U(y)} dy'$$

and

$$\epsilon_I = -\pi\frac{\psi(y;y)U''}{U'}.$$

The inversion procedure works provided that there is no point in the complex plane for which $\epsilon_R = \epsilon_I = 0$. These relations are equivalent to the eigenvalue problem for the discrete, complex eigenvalues and the inflexion-point modes. However, by adding arbitrary multiples of the eigenfunctions of the complex modes into the superposition, and then suitably modifying the distribution Λ at the inflexion points, we can avoid any problems associated with the zeros of ϵ_R and ϵ_I. We can then represent an arbitrary initial condition in terms of the singular eigensolutions and complex pairs; in other words, we establish completeness.

FURTHER REMARKS

The solution of the initial-value problem indicates that we can represent an arbitrary initial condition in terms of superposed continuum solutions and complex modes. This allows us to consider the evolution over asymptotically large times, and make contact with the Laplace transform approach. In the long-time limit, integral spatial averages of singular-mode superpositions become vanishingly small as a result of phase mixing.

We can also pose the problem within a Hamiltonian framework, since the two-dimensional Euler equations have a Hamiltonian structure. In this setting, we find that the amplitudes of the singular eigenfunctions define a set of coordinates in terms of which the system takes a canonical, action-angle form. The canonization and diagonalization of the Hamiltonian fluid system extracts the true energy of a perturbation. Furthermore, it reveals a characteristic signature of the modes. Thus we can establish the existence of *positive* and *negative energy modes* in shearing fluid.

REFERENCES

1. Rayleigh, J. W. S. 1880. Proc. Lond. Math. Soc. **9**: 57-70.

2. Van Kampen, N.G. 1955. Physica **21**: 949-963.