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On neutral plasma oscillations

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We examine the conditions for the existence of spectrally stable neutral modes in a Vlasov–Poisson plasma and show that for stable equilibria of systems that have unbounded spatial domain, the only possible neutral modes are those with phase velocities that correspond to stationary inflection points of the equilibrium distribution function. It is seen that these neutral modes can possess positive or negative free energy.

1. Introduction

In this note we examine the possibility of a Vlasov–Poisson plasma supporting neutral modes (i.e. undamped electrostatic plasma oscillations) and demonstrate the existence of a neutral mode possessing negative free energy. A neutral mode is a mode with a phase velocity, v_p, that corresponds to a stationary point of the equilibrium distribution function. In spite of the early work by Case [1] and others [2,3], a thorough treatment of neutral oscillations has apparently not been undertaken. Previous analysis of neutral modes has concentrated on modes where the phase velocity of the wave corresponds to a local minimum or maximum of the equilibrium distribution. We extend the discussion of neutral modes to the case where v_p corresponds to a stationary point of the distribution that is also an inflection point (which we will refer to as a stationary inflection point). We show that stationary inflection point modes are the only neutral modes that can exist in a stable, spatially unbounded plasma.

Neutral modes evidently suffer no Landau damping, a characteristic which makes neutral modes possessing negative free energy of interest from the standpoint of nonlinear instability. Free energy, $\delta^2 F$, is the energy difference between an equilibrium and a dynamically accessible perturbed state [4–6]. Modes that have negative free energy can lead to instability in two ways. Dissipation in a system where the spectrum contains negative energy modes removes energy from these modes resulting in increased amplitude. For example, in finite Larmor radius models dissipation, which has a stabilizing effect on the positive energy mode, leads to instability if the model admits negative energy modes. Negative energy modes can also result in instability through nonlinearities. The prototypical example is the Cherry oscillator [7]. This system is spectrally stable (i.e. linearly stable in terms of solutions of the form $e^{i\omega t}$) but a resonant nonlinear coupling between the negative and positive energy modes gives rise to solutions that diverge in finite time for arbitrarily small couplings. The nonlinearity transfers energy from the negative energy mode to the positive energy mode, causing the amplitude of each to increase catastrophically. Nonresonant nonlinear couplings can also lead to instability [8].

The remainder of this note is organized as follows. In section 2 we review the Nyquist method and use it to show that neutral oscillations at stationary inflection points are the only ones allowed for stable equilibria. In section 3 we derive the free energy of a neutral mode and show that neutral modes are the only modes where the dielectric energy is the correct linear plasma energy [6,9]. We conclude with a discussion in section 4.

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2. Neutral modes and stability

We consider a one-dimensional Vlasov–Poisson system with an equilibrium electron distribution, $f^0(v)$, and a fixed, neutralizing ionic background. We assume perturbations of the equilibrium distribution of the form

$$\delta f(x, v, t) = \int\int \mathrm{d}\omega\, \mathrm{d}k\, \mathrm{e}^{\mathrm{i}(kx-\omega t)} f_k(\omega, v) \quad (1)$$

and a corresponding electric field perturbation

$$\delta E(x, t) = \int\int \mathrm{d}\omega\, \mathrm{d}k\, \mathrm{e}^{\mathrm{i}(kx-\omega t)} E_k(\omega)\ . \quad (2)$$

After linearizing, (1) and (2) become

$$-\mathrm{i}(\omega-kv)f_k - \frac{q}{m} E_k f^{0\prime} = 0 \quad (3)$$

and

$$\mathrm{i}kE_k = -4\pi q \int \mathrm{d}v\, f_k . \quad (4)$$

Combining (3) and (4) leads to the plasma dispersion function

$$\epsilon(k, \omega) = 1 - \frac{\omega_\mathrm{p}^2}{k^2} \int \mathrm{d}v \frac{f_0'}{v-\omega/k}, \quad (5)$$

where $\omega_\mathrm{p} = 4\pi q^2 n_0/m$ is the electron plasma frequency, $f^0 = n_0 f_0$ and, in general, the contour is chosen to be that used by Landau in solving the initial value problem. This ensures ϵ is an analytic function of ω. The dispersion relation $\epsilon = 0$ provides the connection between k and ω. For neutral modes, $v_\mathrm{p} = \omega/k$ is a stationary point of f_0 and thus the integrand is regular; hence the contour can be chosen simply to be the real axis.

For an equilibrium to be spectrally stable, perturbations must not grow unbounded. This means there must be no solutions of $\epsilon(k, \omega) = 0$ with $\mathrm{Im}(\omega) > 0$, as such solutions would give rise to exponential growth. We will use the Nyquist method to show that, if the equilibrium distribution is spectrally stable, i.e. $\epsilon(k, \omega) = 0$ has no solutions with $\mathrm{Im}(\omega) > 0$, neutral modes with phase velocity corresponding to minima or maxima of f_0 cannot exist. In the following, for the sake of clarity, we will suppress k in the argument of ϵ.

Consider [10]

$$\oint_\mathrm{C} \frac{1}{\epsilon} \frac{\partial \epsilon}{\partial \omega} \mathrm{d}\omega\ . \quad (6)$$

For any reasonable initial distribution, the only poles in the integrand come from the zeros of ϵ, that is, from the roots of the dispersion relation; the residue at these poles is equal to the multiplicity of the corresponding root. For example, suppose that ϵ has an nth order root at $\omega = \omega_0$. Near ω_0,

$$\epsilon \approx \mathrm{const} \times (\omega-\omega_0)^n \quad (7)$$

and

$$\frac{1}{\epsilon}\frac{\partial \epsilon}{\partial \omega} \approx \frac{n}{\omega-\omega_0}, \quad (8)$$

which has a residue of n. Thus

$$N = \frac{1}{2\pi \mathrm{i}} \oint_\mathrm{C} \frac{1}{\epsilon}\frac{\partial \epsilon}{\partial \omega} \mathrm{d}\omega \quad (9)$$

is the sum of the multiplicities of roots of $\epsilon = 0$ enclosed by the contour C. If we choose C to consist of a semicircle enclosing the upper half of the ω-plane and the real axis, then N will be the number of unstable roots of the dispersion relation. Further, since

$$\frac{\partial \epsilon}{\partial \omega} = -\frac{\omega_\mathrm{p}^2}{k^3} \int \mathrm{d}v \frac{f_0'}{(v-\omega/k)^2} \quad (10)$$

we have

$$\left| \frac{1}{\epsilon}\frac{\partial \epsilon}{\partial \omega} \right| \to 0 \quad \text{as } |\omega| \to \infty \quad (11)$$

and the contribution to N from the circular part of the contour is zero. Therefore

$$N = \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{1}{\epsilon}\frac{\partial \epsilon}{\partial \omega} \mathrm{d}\omega = \frac{1}{2\pi \mathrm{i}} \log\left(\frac{\epsilon(\omega=\infty)}{\epsilon(\omega=-\infty)} \right). \quad (12)$$

Choosing the phase of ϵ so that $\epsilon(\omega=-\infty) = 1$, we then have $\epsilon(\omega=\infty) = \mathrm{e}^{2\mathrm{i}n\pi}$ and $N = n$. Thus the *number* of unstable solutions is given by the phase change in ϵ from $\omega = -\infty$ to $\omega = \infty$. If we think of ϵ as a curve in the complex ϵ-plane, parameterized by ω, then the phase change in ϵ is given by the number of times $\epsilon(\omega)$ circles the origin as ω ranges from $-\infty$ to ∞

along the real axis. For stability N must be zero for all values of k^2. To encircle the origin, $\epsilon(\omega)$ must cross the real axis at least once to the left of the origin. Thus we are interested in the sign of $\mathrm{Re}(\epsilon)$ whenever $\epsilon(\omega)$ crosses the real axis. For $\mathrm{Im}(\omega)=0$,

$$\epsilon_r \equiv \mathrm{Re}(\epsilon) = 1 - \frac{\omega_p^2}{k^2} \int \mathrm{d}v \frac{f_0'}{v-\omega/k} \tag{13}$$

and

$$\epsilon_i \equiv \mathrm{Im}(\epsilon) = -\pi \frac{\omega_p^2}{k^2} f_0' \bigg|_{v=\omega/k} . \tag{14}$$

It is important to remember that, for this analysis, ω and k^2 must be viewed as independent variables.

For a neutral mode with phase velocity v_*, the dispersion relation $\epsilon(k, kv_*)=0$ gives two equations, $\epsilon_r=0$ and $\epsilon_i=0$, that relate k and ω. The second of these is trivially satisfied since $f_0'(v_*)=0$. To examine solutions of $\epsilon_r=0$, it is convenient to write ϵ_r as

$$\epsilon_r(k, kv_*) = 1 - \frac{\omega_p^2}{k^2} \Lambda , \tag{15}$$

where

$$\Lambda = \int \mathrm{d}v \frac{f_0'(v)}{v-v_*}$$

is a constant that depends only on the shape of f_0. The dispersion relation reduces to

$$\epsilon_r = 1 - \frac{\omega_p^2}{k^2} \Lambda = 0 , \tag{16}$$

which only has solutions when $\Lambda > 0$, since $k^2 > 0$. In this case the solutions are given by

$$k_* = \pm \omega_p \sqrt{\Lambda} . \tag{17}$$

The central question is: Does the presence of the neutral mode admit unstable solutions of $\epsilon=0$? This can be answered by observing that there are two generic possibilities: either f_0', and consequently ϵ_i, changes sign at v_* (i.e. v_* is a maximum or a minimum of f_0) or it does not. Let us consider the first case. Here ϵ_i changes sign as ω/k passes through v_*, that is, the curve $\epsilon(\omega)$ crosses the real axis as ω/k passes through v_*. Further, for $k=k_*$, $\epsilon(\omega)$ goes through the origin. Since $\epsilon_r(k_*, k_* v_*)=0$, the curve will cross to the left of the origin for $k<k_*$ and to the right for $k>k_*$. One of these values of k will result in the curve encircling the origin. Thus for a system with an unbounded spatial domain, it will always be possible to find a value of k^2 such that $\epsilon(\omega)$ encircles the origin. For an equilibrium distribution where v_* corresponds to a minimum, this is the Penrose criterion [11]. Hence the existence of this type of neutral mode results in the presence of unstable solutions of the dispersion relation.

The other possibility is that f_0' does not change sign at v_*. This requires that v_* be a zero of f_0' with even multiplicity. That is, for v near v_*,

$$f_0' \approx \mathrm{const} \times (v-v_*)^{2m} \tag{18}$$

for some m. Here, although $\epsilon_i(v_*)=0$, the curve $\epsilon(\omega)$ does not cross the axis, since $f_0'(v)$ does not change sign as v passes through v_* and the value of ϵ_r is unimportant. In essence the multiple root of f_0' at v_* prevents $\epsilon(\omega)$ from encircling the origin. Hence $\epsilon=0$ does not admit solutions with $\mathrm{Im}(\omega)>0$. Thus it is possible for neutral modes to exist at such points of f_0 without losing spectral stability. Further, we see that the only possible neutral modes for a stable equilibrium are modes of this type.

If the system is spatially bounded, the first possibility (i.e. where v_* corresponds to a minimum or maximum of f_0) is not necessarily excluded. Even if there exist exponentially growing solutions of $\epsilon(k, \omega)=0$ for $k<k_*$, it may happen that, the smallest value of k, $k_{\min}=k_*$. That is, the neutral mode has the longest wavelength allowed by the system. Thus no modes with $k<k_*$ exist and the neutral mode is not incompatible with linear stability. For such systems, as shown by Case [1], neutral modes with phase velocities corresponding to minima or maxima of the equilibrium distribution are possible. In what follows we will only consider unbounded systems.

3. Free energy of neutral modes

Free energy expressions can be obtained from either the Eulerian or Lagrangian variable descriptions. In the Eulerian case, the free energy comes from extremizing the Hamiltonian constrained by certain constants of motion known as Casimirs, which embody conservation of phase space volume. (Unconstrained stationary points of the Hamiltonian correspond to the "vacuum"; i.e. where all particles have zero kinetic energy and the perturbed fields are also zero and thus generally uninteresting.) This constraint is equivalent to demanding that the perturbation be dynamically accessible, since the Casimirs divide the phase space into constraint surfaces (symplectic leaves) which cannot be crossed by the phase space flow. Here $\delta^2 F$ is given by [4,5]

$$\delta^2 F=\tfrac{1}{2}\int dx\,dv\,[H_0,g]\,[g,f^0] + \frac{1}{8\pi}\int dx\,\delta E^2[g]\,. \tag{19}$$

It has the physical interpretation as the energy increase due to such perturbations. This expression is valid for any perturbation arising from a generating function g, according to $\delta f=[f^0,g]$, where f^0 is the equilibrium distribution function and [,] is the usual Poisson bracket. The argument of δE is shown explicitly in order to emphasize that δE is a (known) expression that depends on δf and thus g through Poisson's equation. Alternatively, the expression for $\delta^2 F$ can be obtained in the Lagrangian description starting from the Low Lagrangian and restricting to canonical perturbations; i.e., those perturbations derived from a generating function [5].

Using (3),

$$f_k\equiv ikg_k f^{0\prime}=-iE_k\frac{q}{m}\frac{f^{0\prime}}{\omega-kv}, \tag{20}$$

and the free energy for a single neutral mode with phase velocity v_p, corresponding to a stationary inflection point of the equilibrium distribution function, is given by

$$\begin{aligned}\delta^2 F&=\frac{1}{8\pi}|E_k|^2-\tfrac{1}{2}m\int dv\,v|g_k|^2k^2f^{0\prime}\\&=\frac{1}{8\pi}\left(|E_k|^2-\frac{\omega_p^2}{k^2}|E_k|^2\int dv\frac{vf_0'}{(v-v_p)^2}\right)\\&=\frac{1}{8\pi}|E_k|^2\left(\epsilon(k,\omega)-v_p\frac{\omega_p^2}{k^2}\int dv\frac{f_0'}{(v-v_p)^2}\right).\end{aligned} \tag{21}$$

Since $\epsilon(k,\omega)=0$,

$$\delta^2 F=-\frac{1}{8\pi}|E_k|^2 v_p\frac{\omega_p^2}{k^2}\int dv\frac{f_0'}{(v_p-v)^2}. \tag{22}$$

For the distributions that we are considering, $f_0''(v_p)=0$, thus the expression for ϵ can be safely differentiated with respect to ω, from which we find

$$\delta^2 F=\frac{1}{8\pi}|E_k|^2\omega\frac{\partial\epsilon}{\partial\omega}. \tag{23}$$

It is important to remember that in the above, v_p is the phase velocity in the frame where the energy is a minimum, i.e. in the center of mass frame. Note that the integrals are frame independent: in a frame moving with velocity V with respect to the original frame, the new velocity, $\tilde{v}$, is

$$\tilde{v}=v-V,\quad \tilde{f}_0(\tilde{v})=f_0(v-V), \tag{24}$$

giving

$$\int dv\frac{f_0'(v)}{(v_p-v)^2}\to\int d\tilde{v}\frac{f_0'(v-V)}{(\tilde{v}_p-\tilde{v})^2}, \tag{25}$$

where $\tilde{v}_p=v_p+V$ is the Doppler shifted phase velocity [12]. If the variable of integration is shifted, $\tilde{v}\to\tilde{v}-V$, we obtain the same expression as in the original frame. Thus the free energy depends on the frame only through the phase velocity, i.e. $\delta^2 F=-v_p\times\text{const}$ and $\delta^2\tilde{F}=\tilde{v}_p/v_p\delta^2 F$. The sign of $\delta^2 F$ is determined by

$$-v_p\int dv\frac{f_0'}{(v_p-v)^2} \tag{26}$$

and so depends on the precise shape of f_0.

Consider an equilibrium distribution with a single maximum which supports a neutral mode with phase velocity v_p. In some frame

$$f_0'=-v(v-v_p)^2P(v)\,, \tag{27}$$

where $P(v)\geqslant 0$ for all v. While this may not be the center of mass frame, the existence of the mode is frame independent. Let P_n denote the nth moment of P. Clearly

$$P_{2n}>0\,. \tag{28}$$

The wave number of the neutral mode is given by $\epsilon_r=0$, namely

$$\begin{aligned}k^2&=-\omega_p^2\int dv\frac{f_0'}{v-v_p}=\omega_p^2\int dv\,(v-v_p)P(v)\\&=\omega_p^2(v_pP_1-P_2)\,.\end{aligned} \tag{29}$$

For the mode to exist, $k^2>0$ which means $v_p P_1>0$ since $P_2>0$. In the chosen frame

$$\delta^2 F \propto v_p \int dv\, vP(v) = v_p P_1 > 0 .$$

The velocity of the center of mass of the plasma, v_{cm}, is given by

$$v_{cm} = \frac{1}{n_0} \int dv\, v f^0 = -\tfrac{1}{2} \int dv\, v^2 f'_0$$
$$= \tfrac{1}{2} v_p^2 P_3 - v_p P_4 + \tfrac{1}{2} P_5 .$$

The requirements that the distribution function be normalized and vanish at infinity constrains the first and third moments of P,

$$P_1 = \frac{1}{2v_p^3}\,(1+3v_p^3 P_2 - P_4) \tag{30}$$

and

$$P_3 = \frac{1}{2v_p}\,(v_p^2 P_2 + P_4 - 1) . \tag{31}$$

Using this in the expressions for k^2 and v_{cm} gives

$$k^2 = \frac{\omega_p^2}{2v_p^2}\,(1+v_p^2 P_2 - P_4) , \tag{32}$$

$$v_{cm} = \tfrac{1}{4} v_p (v_p^2 P_2 - 3P_4 - 1) + \tfrac{1}{2} P_5 . \tag{33}$$

Transforming to the center of mass frame, the phase velocity of the neutral mode becomes

$$\tilde{v}_p = \tfrac{1}{4} v_p (3 + v_p^2 P_2 - 3P_4) + \tfrac{1}{2} P_5 \tag{34}$$

and

$$\delta^2 \tilde{F} \propto \tfrac{1}{4} v_p P_1 (3 + v_p^2 P_2 - 3P_4 + 2P_5/v_p) . \tag{35}$$

Since these moments of P are arbitrary, the free energy in the center of mass frame can be either positive, negative or zero depending on f_0.

We now specialize to the case where the equilibrium distribution is symmetric in some frame. In that frame (also the center of mass frame) f'_0 has the form

$$f'_0 = -v(v^2 - v_p^2)^2 P(v) , \tag{36}$$

where $P(v)$ is a positive definite, symmetric function. Here

$$P_{2n} > 0, \quad P_{2n+1} = 0 , \tag{37}$$

giving

$$k^2 = \omega_p^2 (v_p^2 P_2 - P_4) \tag{38}$$

and

$$\delta^2 F \propto 2 v_p^2 P_2 > 0 . \tag{39}$$

For this class of distributions, all neutral modes have positive free energy.

We see that of the distributions that have a single maximum, those that are symmetric can only support neutral modes with positive free energy, while those that are nonsymmetric allow for neutral modes to have negative free energy. Further, for a symmetric distribution with more than one maximum, P_2 and P_4 are not necessarily positive (since P is no longer positive definite) thus admitting the possibility of the neutral mode having negative free energy [9].

4. Discussion

Previous work on neutral modes [1] ("class 1c") only considered modes that correspond to either minima or maxima of the equilibrium distribution. Such modes can only be supported by a linearly stable equilibrium if the spatial extent of the system is such that k_{min} is sufficiently large so that $\epsilon_r(\omega/k_{min} = v_*) \geqslant 0$. Stationary inflection point modes are the only linear undamped modes that can be supported by a spatially unbounded, stable equilibrium. Furthermore, we have shown that *only* for neutral modes is

$$\frac{1}{8\pi}\,|E_k|^2 \omega \frac{\partial \epsilon}{\partial \omega} \tag{40}$$

the correct expression for the free energy [9]. This energy expression is correct both for inflection point modes as well as the neutral modes of the type studied by Case. (The correct energy of perturbations about stable equilibria that do not support neutral modes is not the above but is given by an expression derived in ref. [6].)

The nonlinear stability of neutral modes has previously been examined [13]. Unfortunately, this analysis did not take into account that neutral modes of the type described here exist in linear theory and that the condition $k^2>0$ is necessary for their existence. The case of nonlinear undamped plasma waves has been recently explored both numerically [14] and analytically [15] for spatially nonhomogeneous equilibria. Inflection point neutral modes seem to be a likely candidate for the linear limit of these nonlinear undamped waves in the sense that a homogeneous equilibrium obtained by spatial averaging supports inflection point modes with the same phase velocity as the observed nonlinear oscillations.

In a model that includes trapped particles, BGK modes with phase velocities corresponding to the trapped particles become van Kampen modes in the limit that the trapped particle density goes to zero, provided that the equilibrium distribution function has a discontinuity [16]. If f^0 is smooth then instead of a van Kampen mode, one obtains a quasimode [17], which is subject to Landau damping. If f^0 has a stationary inflection point then in this limit one obtains a neutral mode. The neutral eigenmode limit seems to be a natural counterpart to the van Kampen (singular) eigenmode for the case of continuous f^0 in that both are true eigenmodes and persist for all time.

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