The energy of perturbations for Vlasov plasmas*,[a)]

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The energy content of electrostatic perturbations about homogeneous equilibria is discussed. The calculation leading to the well-known dielectric (or as it is sometimes called, the wave) energy is revisited and interpreted in light of Vlasov theory. It is argued that this quantity is deficient because resonant particles are not correctly handled. A linear integral transform is presented that solves the linear Vlasov–Poisson equation. This solution, together with the Kruskal–Oberman energy [Phys. Fluids **1**, 275 (1958)], is used to obtain an energy expression in terms of the electric field [Phys. Fluids B **4**, 3038 (1992)]. It is described how the integral transform amounts to a change to normal coordinates in an infinite-dimensional Hamiltonian system.

I. INTRODUCTION

Since the advent of δW, energy principles have been an important mainstay of theoretical plasma physics research. Indefinite energy expressions imply either instability or the occurrence of negative energy modes, both of which are detrimental to confinement (see, for example, Ref. 1 and references therein). Therefore, precise expressions for the energy are desirable. Surprisingly, such energy expressions for the simplest of plasma models have not been obtained until recently.[2–6] In this paper I discuss the energy content of electrostatic perturbations within the confines of Vlasov–Poisson theory. In Sec. II the derivation of the energy of a dielectric is revisited, and comments about its inability to completely describe the energy of a Vlasov plasma are made. In Sec. III, the linear Vlasov equation is solved by an integral transform technique. This solution is then used to obtain an energy expression for Vlasov theory in terms of the electric field, which is Fourier transformed in space and time independently.[4] In Sec. IV it is briefly described how the energy obtained in Sec. III is the Hamiltonian for an infinite-dimensional Hamiltonian system and how the integral transform solution amounts to a canonical transformation to action-angle variables. Finally, in Sec. V, is the conclusion.

II. THE DIELECTRIC ENERGY

Consider a gedanken experiment in which a dielectric medium, i.e., a "plasma," spatially coexists with an artificial medium that carries an imposed current, J_e. It is assumed that the only interaction between the artificial medium and the plasma is by means of the electric field. The Maxwell equation that describes this situation is

$$\frac{\partial E}{\partial t}+4\pi(J+J_e)=0, \tag{1}$$

where E and J are the electric field and plasma current density, respectively. Assume

$$J_e \sim e^{-i\omega t+ikx}, \tag{2}$$

where $\omega=\omega_R+i\mu$ and, for now, $\mu>0$ and $-\infty<t\leqslant 0$. Note that E and J are generated solely by J_e; thus, their space and time dependencies are identical to those of J_e. According to the usual response theory, the plasma is assumed to be adequately described by a dielectric function $\epsilon(k,\omega)$,

$$E+i\frac{4\pi}{\omega}J=\epsilon(k,\omega)E, \tag{3}$$

and hence

$$J_e=\frac{i\omega}{4\pi}\epsilon(k,\omega)E. \tag{4}$$

Now the energy absorbed by the plasma, $\mathscr{E}_P$, due to J_e, is calculated from the power absorbed by the plasma. The latter quantity, which is equal to the power liberated by the artificial medium, is given by calculating the work done in maintaining J_e against the electric field that arises in the plasma:

$$\begin{aligned} P&=-\frac{1}{4}\int_V (J_e+J_e^*)(E+E^*)d^3x \\ &=-\frac{V}{4}(J_e^*E+J_eE^*) \\ &=i\frac{V}{4}\frac{|E|^2}{4\pi}[\omega^*\epsilon^*(k,\omega)-\omega\epsilon(k,\omega)]. \end{aligned} \tag{5}$$

Assuming $\epsilon(k,\omega_R)$ possesses real and imaginary parts; i.e.,

$$\epsilon(k,\omega_R)=\epsilon_R(k,\omega_R)+i\epsilon_I(k,\omega_R) \tag{6}$$

and

$$\epsilon(k,\omega_R+i\mu)\simeq\epsilon(k,\omega_R)+i\mu\frac{\partial\epsilon(k,\omega_R)}{\partial\omega_R}, \tag{7}$$

and noting that the power is related to the plasma energy by $P=2\mu\mathscr{E}_P$, yields

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[a)]Dedicated to the memory of Carl Oberman.
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$$\mathscr{E}_P = \frac{V}{16\pi}\,|E|^2\left(\frac{\partial}{\partial\omega_R}\,[\omega_R\epsilon_R(k,\omega_R)] + \frac{\omega_R}{\mu}\,\epsilon_I(k,\omega_R)\right). \tag{8}$$

So far no connection between k and ω_R has been assumed. This is put in by assuming $\epsilon(k,\omega_R+i\gamma)=0$. When $\epsilon_I\neq0$, this can be approximately solved, in the so-called small growth rate expansion, as follows:

$$\epsilon_R(k,\omega_R)=0, \quad \gamma=-\frac{\epsilon_I(k,\omega_R)}{\partial\epsilon_R/\partial\omega_R}. \tag{9}$$

Remember μ is a property of the current J_e, while in light of the above γ arises from the dispersion relation. Because of the expansions used, both quantities must be small. With (9), (8) becomes

$$\mathscr{E}_P = \frac{V}{16\pi}\,|E|^2\omega_R\,\frac{\partial}{\partial\omega_R}\,\epsilon_R(k,\omega_R)\left(1-\frac{\gamma}{\mu}\right). \tag{10}$$

The above expression is not quite that normally obtained in calculations of this sort, because of the factor $(1-\gamma/\mu)$. The dielectric energy, $\mathscr{E}_D$, is defined by $\mathscr{E}_P=\mathscr{E}_D(1-\gamma/\mu)$. The factor has some interesting consequences.

For unstable plasmas one can take $\mu=\gamma$ and obtain the result $\mathscr{E}_P=0$, a result that is, in fact, correct for a Vlasov plasma, as can be shown directly within Vlasov theory.[4] In this unstable case $E\neq0$ at $t=0$ is obtained with $J_e=0$; i.e., only the self-consistent E and J contribute. This case could be called *self-consistent "adiabatic" turn-on*. For a mode with $\gamma<0$, one can choose $\mu<0$, and in this case the time interval $0\leqslant t<\infty$ is considered. The energy at $t=0$ is given by the energy that has been transferred to the artificial medium during this time interval. If $\gamma=\mu$, then again $J_e=0$ and $\mathscr{E}_P=0$, which is again a valid result for a Vlasov plasma. This case could be called *self-consistent "adiabatic" turn-off.*

Some people find the above result unsettling, but it is really to be expected. Consider a simple one degree-of-freedom Hamiltonian system for a particle in an inverted potential well: $H=(p^2-q^2)/2$. This linear system has two eigenvalues $\pm\gamma$ and two corresponding eigenvectors. Evaluating H on the eigenvector corresponding to the growing mode (for example) yields

$$H=\tfrac{1}{2}\,e^{2\gamma t}(p_+^2-q_+^2)=0. \tag{11}$$

The last equality follows because the only way energy can be conserved is to have $(p_+^2-q_+^2)=0$. It is a simple matter to insert the eigenvector and check that this is so. As the particle falls down the well, its speed and displacement both increase as $e^{\gamma t}$, but sum to zero. This is analogous to what occurs for the dielectric medium. As the instability grows the energy in the field increases, but this is canceled by the energy of the medium, which is negative but grows in absolute magnitude. For Vlasov theory the medium energy is kinetic energy, which, in a sense, is being drawn from that contained in the unstable equilibrium state.

It is important to point out that the validity of the above results, for both the growing and damped modes, depends upon γ being the imaginary part of a root of the dielectric function. In the case of a Vlasov plasma such modes may exist, but these must be distinguished from solutions of the Landau problem, where the contour of integration is deformed. In the latter case, the above analysis is invalid. For a stable Vlasov plasma a dielectric function $\epsilon(k,\omega)$ strictly speaking *does not* exist. The expression with the deformed contour used for obtaining Landau damping is only asymptotically valid in the limit of large time, where the electric field decays exponentially, and one cannot self-consistently turn off, as in the above case of a stable mode, a perturbed electric field that is only asymptotically of the form $E\sim e^{-\gamma t}$.

In many places in the literature, attempts have been made to obtain energy expressions by solving the linearized Vlasov equation or other plasma models with the adiabatic turn-on assumption. Generally these expressions are deficient in two respects. First, they are not constants of motion so their use in energy arguments must be viewed with caution. Second, the presence of resonant particles leads to singularities. This is because a finite amount of energy is deposited in the plasma in each wave period over an infinite interval of time. This behavior is recovered from Eq. (10) by keeping γ fixed and taking the limit $\mu\to0$.

The limit where $\mu\gg|\gamma|$, but both still small, is also of interest, since in this case Eq. (10) reduces to $\mathscr{E}_D$. Although this limit can be appropriate for dielectric media, it is, in general, not valid for Vlasov plasma. In the case of weakly Landau damped modes, a self-consistent exponential adiabatic turn-on (or turn-off) is not possible and the dielectric or wave energy expression is inappropriate.

The physical reason that an expression like the dielectric energy is inadequate to describe the energy content of a Vlasov perturbation is that there can be an arbitrary amount of energy stored in the short time "transient" field. This energy can go into the particles in a complicated manner that is not representable by a formula like $\mathscr{E}_D$.

There is one instance in Vlasov theory where the dielectric energy is appropriate. This occurs for special neutral modes that exist for equilibrium distribution functions that possess stationary inflection points.[7] These have $\epsilon_I=0$ at the phase velocity of the wave and so there is no Landau damping. Also, in models without resonant particles, such as fluid models, the dielectric energy is exact and appropriate. Nevertheless, the question remains: What in general is the energy for stable Vlasov plasmas?

In closing this section I wish to point out that after the completion of Ref. 4 we become aware that the deficiency of the dielectric energy because of resonant particles was previously suggested by Best.[8] However, this work, which is based on a rather complicated second-order perturbation treatment, serves to underscore the use of Hamiltonian techniques that we have advocated, since this reference contains incorrect and incomplete results.

III. INTEGRAL TRANSFORM SOLUTION OF THE LINEAR VLASOV EQUATION AND THE NEW ENERGY EXPRESSION

Now we solve the linear Vlasov equation by an integral transform. This method is akin to the solution by Van

Kampen,[9] but has a distinct difference, as will be described below. The solution is used to obtain an energy expression.

Begin by expanding about an equilibrium as follows:

$$f(x,v,t)=f^0(v)+\sum_k f_k(v,t)e^{ikx}, \tag{12}$$

where it is assumed that f^0 is stable. The linear Vlasov–Poisson system, which is to be solved, is given by

$$ikE_k=4\pi e\int f_k\,dv,$$
$$\frac{\partial f_k}{\partial t}+ikf_k+\frac{e}{m}E_k\frac{\partial f^0}{\partial v}=0. \tag{13}$$

We define

$$\epsilon(k,v)\equiv\lim_{\nu\to0^+}1-\frac{\omega_p^2}{k^2}\int_{-\infty}^{\infty}\frac{\partial f_0/\partial v}{v'-v-i\nu}dv, \tag{14}$$

and observe that by stable we mean that $\epsilon(k,v)\neq0$ for $\mathrm{Im}(kv)>0$. Note that below we will often let $v=\omega/k$, and use $\epsilon(k,v)\equiv\epsilon(k,\omega)$. All integrations in this paper are along the real axis.

The solution of (13) is obtained easily by the following, specially designed, integral transform:

$$f_k(v,t)=\frac{ik}{4\pi e}\int_{-\infty}^{\infty}E_k(u,t)\mathscr{G}_k(u,v)du, \tag{15}$$

where the kernel of the transform is given by

$$\mathscr{G}_k(u,v)=\epsilon_I(k,v)\frac{P}{\pi}\frac{1}{u-v}+\epsilon_R(k,v)\delta(v-u). \tag{16}$$

This is a linear transformation that takes a function of u into one of v. It is important to note that no particular time dependence is assumed—time plays the role, at this stage, of a parameter. This is were we depart from Van Kampen, who assumed a particular time dependence. The transform of (15) can also be view as a coordinate change on an infinite dimensional space, a point of view that we take in Sec. IV below.

Integral transforms are not particularly useful unless they possess an inverse. The inverse of (15) is given by

$$E_k(u,t)=\frac{4\pi e}{ik}\int_{-\infty}^{\infty}f_k(v,t)\widetilde{\mathscr{G}}_k(u,v)dv, \tag{17}$$

where the kernel of the inverse transformation is given by

$$\widetilde{\mathscr{G}}_k(u,v)=\frac{\epsilon_I(k,v)}{|\epsilon|^2}\frac{P}{\pi}\frac{1}{u-v}+\frac{\epsilon_R(k,u)}{|\epsilon|^2}\delta(v-u). \tag{18}$$

That this is the inverse follows from the following completeness relations:

$$\int_{-\infty}^{\infty}\widetilde{\mathscr{G}}_k^{\epsilon}(u,v')\mathscr{G}_k^{\epsilon}(u,v)du=\delta(v-v') \tag{19}$$

and

$$\int_{-\infty}^{\infty}\widetilde{\mathscr{G}}_k^{\epsilon}(u,v)\mathscr{G}_k^{\epsilon}(u',v)dv=\delta(u-u'). \tag{20}$$

These relations are necessary to make the transform useful. They were, in principle, proved by Van Kampen[9] and are, in essence, equivalent to the orthogonality relations that appear in the works of Case.[10] Another property of the transform is obtained by integrating (15) over v. It is easily seen from the resulting charge density that E_k is the electric field associated with f_k.

In addition to the existence of an inverse, the above transform (15) (denoted by an overtilde) possess many properties as itemized:

(i) A generalized *convolution* theorem;
(ii) a generalized Parseval's theorem;
(iii) $[\widetilde{vf(v)}]=u\tilde{f}-(\epsilon_I/|\epsilon|^2)(1/\pi)\int_{-\infty}^{\infty}f\,dv$;
(iv) $(\widetilde{\epsilon_I})=\epsilon_I/|\epsilon|^2$;
(v) $\int_{-\infty}^{\infty}[\epsilon_I(k,u)/|\epsilon(k,u)|^2]\quad\mathscr{G}_k^{\epsilon}(u,v')\mathscr{G}_k^{\epsilon}(u,v)du=\epsilon_I(k,v)\delta(v-v')$; and
(vi) $\int_{-\infty}^{\infty}\epsilon_I(k,v)\widetilde{\mathscr{G}}_k^{\epsilon}(u,v)\quad\widetilde{\mathscr{G}}_k^{\epsilon}(u',v)dv=[\epsilon_I(k,u)/|\epsilon(k,u)|^2]\delta(u-u')$.

(Above we have omitted the factor $ik/4\pi e$ in the definition of the transform.) We will state and give the proofs of items (i) and (ii) elsewhere. The remaining items were proved in Ref. 4. Items (v) and (vi) will be alluded to in Sec. IV, where we discuss the Hamiltonian interpretation. They ensure that the integral transforms are, in fact, canonical transformations of a sort. They are new (and nontrivial) identities that do not appear in the works of Van Kampen, Case, and others.

Now let us proceed to transform (13). Using item (ii), we obtain

$$\frac{\partial E_k}{\partial t}+ikuE_k-ik\frac{\epsilon_I}{|\epsilon|^2}\frac{1}{\pi}\int_{-\infty}^{\infty}f_k\,dv+\frac{e}{m}E\widetilde{\frac{\partial f^0}{\partial v}}=0. \tag{21}$$

Solution of (21) is hampered by the presence of the last two terms, but using item (iii), these are seen to cancel, yielding

$$\frac{\partial E_k}{\partial t}+ikuE_k=0, \tag{22}$$

an equation that is trivially solved by

$$E_k=\hat{E}_k(u)e^{-ikut}, \tag{23}$$

where $\hat{E}$ is arbitrary and can be used to set the initial value. From the above, the solution $\delta f(x,v,t)$ is obtained by summing over k and performing the inverse transform. This, in the end, is equivalent to Van Kampen's solution; it amounts to solving a mathematics problem know as the Riemann–Hilbert problem.[11]

Let us now turn to the task of calculating the energy. We begin with an expression that was derived by Kruskal and Oberman,[12]

$$\mathscr{E}_{K0}=-\frac{m}{2}\int\int\frac{v\,\delta f^2}{\partial f^0/\partial v}dx\,dv+\frac{1}{8\pi}\int\delta E^2\,dx. \tag{24}$$

This expression was first obtained in a context more general than the present, but is easily obtained upon simplification. The derivation proceeds by expanding the energy subject to the general constraint

$$C[f]=\int \mathscr{C}(f)dx, \tag{25}$$

where $\mathscr{C}(f)$ is an arbitrary function that is mated to the equilibrium of interest. The expansion proceeds to second order and yields the constant of motion (24). The invariants C are an Eulerian manifestation of the conservation of phase space volume and are now often referred to as Casimir invariants. (Kruskal and Oberman proceeded to use their invariant to ascertain stability, a procedure that in the fluid mechanics literature is commonly and erroneously credited to Arnold.)

Although $\mathscr{E}_{\rm KO}$ is a conserved quantity, it is not at all obvious that it corresponds physically to the energy. In Refs. 2–4, $\mathscr{E}_{\rm KO}$ is generalized to a more general class of equilibria than that considered by Kruskal and Oberman, by imposing a condition referred to there as *dynamical accessibility* and, in addition, $\mathscr{E}_{\rm KO}$ is shown to be the energy by deriving it from time translation invariance in the context of an action principle. It is also shown how to derive it by an appropriate expansion of the full nonlinear energy for the Vlasov–Poisson system.

It is difficult to compare $\mathscr{E}_{\rm KO}$ with $\mathscr{E}_D$ or $\mathscr{E}_P$, since the former quantity depends upon both the perturbation of the distribution function and the perturbation of the electric field, rather than just the electric field. Thus, it is natural to expand δE in a Fourier series and to insert (15) into $\mathscr{E}_{\rm KO}$. This will yield a quantity quadratic in E_k; however, one that is ostensibly very messy looking. In our previous paper[4] we were able, with perseverance, to reduce this to a simple form. Now, with the aid of the identities listed above, notably the convolution and Parseval-type identities, this calculation has been significantly simplified (see Ref. 6). Here we will just state the result

$$\mathscr{E}=\frac{V}{32}\sum_k \int \omega \frac{|\epsilon(k,\omega)|^2}{\epsilon_I(k,\omega)} |E_k(\omega,t)|^2 \, d\omega, \tag{26}$$

where we have used $\omega=ku$. Observe in this formula that $E_k(\omega,t)$ depends explicitly upon time. If we insert the solution (23) then this time dependence disappears, a consequence of the fact that $\mathscr{E}$ is a constant of motion. The quantity $\hat{E}_k(\omega)$ is the Fourier expansion of the electric field, where space and time are treated independently.

The form of $\mathscr{E}$ seems familiar, but is really quite different in character from $\mathscr{E}_D$. The latter quantity requires a solution of the dispersion relation, and is evaluated on $\omega(k)$. It is natural to question: Is $\mathscr{E}$ near, in some sense, to $\mathscr{E}_D$? The answer to this question is, in general, no! To see this consider some electric field profiles. First, it is clear that the electric field at a point does not determine the state of a plasma. Suppose the electric field in the plasma is zero—positive charge neutralizing negative charge. In the future this may remain so. However, if a group of particles are given an initial velocity, then an instant later charge will be bared and an electric field will arise. Thus, specifying that the electric field is zero does not determine the state. However, it is true that specifying the electric field for all times, i.e., for $t\in(-\infty,\infty)$ does determine the state. In light of the above analysis this is clear, since upon Fourier transformation $\hat{E}_k(\omega)$ is obtained. Moreover, given any $E(x,t)$ for all space and time (with weak restrictions, i.e., within an appropriate Hoelder class of functions) one can find an initial condition $\delta f(x,v,0)$, where $\delta f(x,v,t)$ solves the linear Vlasov–Poisson system and gives rise to $E(x,t)$. We can use this fact to attempt to relate $\mathscr{E}$ to $\mathscr{E}_D$. If we specify an electric field that fits our intuition, i.e., has a form that looks like Landau damping, then perhaps they can be made to agree. An example that was treated in detail in Ref. 4 assumes $E(x,t)$ behaves as follows:

$$\lim_{t\to\pm\infty} E_k(t)=E_0(k,\omega_0)e^{-i\omega_0 t-\gamma|t|}, \tag{27}$$

where $\omega_0+i\gamma$ is a root of the Landau dispersion relation. Does this ensure that $\mathscr{E}\approx\mathscr{E}_D$? The answer to this question is no. The reason for this is that there can exist an arbitrary amount of energy in the decayed transient. This can go into the plasma kinetic energy in an arbitrary way, the details of which require the specification of the electric field during the short times as well. The formula for $\mathscr{E}_D$ does not contain this information.

IV. ACTION-ANGLE VARIABLES

In this last section, we briefly interpret the above results in terms of Hamiltonian theory. Recall in elementary mechanics classes we are taught how to transform a linear stable Hamiltonian system with a Hamiltonian of the form

$$H(Q,P)=\frac{1}{2}\sum_{i,j}(A_{ij}P_iP_j+B_{ij}Q_iQ_j+2C_{ij}P_iQ_j) \tag{28}$$

into *normal coordinates.* In terms of normal coordinates, the Hamiltonian takes the form

$$H(q,p)=\frac{1}{2}\sum_i \omega_i(p_i^2+q_i^2)\equiv\frac{1}{2}\sum_i \omega_i J_i, \tag{29}$$

where all the oscillator degrees of freedom are separated. In the definition above we have made a further transformation to action-angle variables, with the action $J\equiv(p^2+q^2)/2$. Now, the form of $\mathscr{E}$ suggests that the action variables for the linear Vlasov system should be $\mathscr{T}\sim|\epsilon(k,\omega)|^2/\epsilon_I(k,\omega)$. The sum in (29) is replaced in (26) by the sum over k and the integral over ω. Thus, the parallel here is one where the Vlasov system corresponds to a Hamiltonian system with an infinite number of degrees of freedom. I would like to emphasize that the action $\mathscr{T}$ is not an action for particles, but an action that describes the perturbation of the distribution function, $\delta f(x,v,t)$. It is an action for a field theory.

By now it is well known that the full nonlinear Vlasov–Poisson system is an infinite-dimensional Hamiltonian system.[13] However, when the dynamical variable is the distribution function the system is in noncanonical variables and the Poisson bracket takes a noncanonical form. In Ref. 4 we have begun from this complete nonlinear Hamiltonian theory and derived a Hamiltonian description for the linear theory discussed in this paper. The transformation from the variable f_k to the variable E_k amounts to a canonical transformation to normal coordinates for this infinite-dimensional system. The new identities that were given in Sec. III [items (v) and (vi)] ensure that the transformation is a canonical transformation. Recall in finite-dimensional theory there are two requirements of the transformation to normal coordinates. The first is that it diagonalizes the quadratic form of (28). This is clearly achievable by an orthogonal transformation. The second is that this be done by a canonical transformation. This, in essence, is the role of the new identities (v) and (vi). I refer the reader to Ref. 4 for details.

V. CONCLUSION

Since for finite systems the transformation to action-angle variables is general, i.e., can be done for all stable systems, the question is raised as to how general the construction given above is for infinite-dimensional fluid and plasma models. I believe it is quite general and can be done for essentially every nondissipative model, although, in general, it will not be as easy as the case presented here. Recent and ongoing efforts are along this line. For transverse electromagnetic perturbations about stable inhomogeneous equilibria, we have effected the transformations above and obtained the following form for the energy:

$$\mathscr{E}=\frac{V}{32}\sum_{k}\int v_{\parallel}\,\frac{|\epsilon_T-c^2/v_{\parallel}^2\,|^2}{\mathrm{Im}\,\epsilon_T}\,|E_{\perp}\,|^2\,dv_{\parallel}\,, \qquad (30)$$

with obvious definitions of symbols. The details of this calculation will be presented by Shadwick and the author elsewhere.

In the literature there exist very many calculations that produce or use formulas like $\mathscr{E}_D$. Generally, this is because assumptions similar to those of Sec. II have been made; i.e., that there exists an $\epsilon(k,\omega)$, and resonant particle effects are ignored or not treated in entirety. The dielectric is correct and complete in some models, for example, in fluid theories or in kinetic theories, where the distribution function vanishes identically for sufficiently large velocities. However, an arbitrarily small amount of resonant particles can contain an arbitrarily large amount of energy.

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