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Linear Stability of Stationary Solutions of the Vlasov-Poisson System in Three Dimensions

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Abstract

Rigorous results on the stability of stationary solutions of the Vlasov-Poisson system are obtained in the contexts of both plasma physics and stellar dynamics. It is proved that stationary solutions in the plasma physics (stellar dynamics) case are linearly stable if they are decreasing (increasing) functions of the local, i.e., particle, energy. The main tool in the analysis is the free energy, a conserved quantity of the linearized system. In addition, an appropriate global existence result is proved for the linearized Vlasov-Poisson system and the existence of stationary solutions which satisfy the above stability condition is established.

1. Introduction

The evolution considered in this paper is governed by the Vlasov-Poisson system

$$\partial_t f + v \cdot \partial_x f - \partial_x U \cdot \partial_v f = 0,$$

$$\Delta U = 4\pi(\rho^+ + \gamma\rho),$$

$$\rho(t,x) := \int f(t,x,v)\, dv,$$

where $t \geqq 0$ denotes time, $x \in \mathbb{R}^3$ position, and $v \in \mathbb{R}^3$ velocity. For $\gamma = -1$ this system describes a collisionless plasma of electrons, which move in the electrostatic field that arises self-consistently from the electron spatial charge density $\rho(t,x)$ and a fixed ion background with special charge density $\rho^+ = \rho^+(x) \geqq 0$. The case where $\gamma = 1$ and ρ^+ is set to zero describes a collisionless ensemble of self-gravitating point masses, e.g., stars in a galaxy or galaxies in a galactic cluster. In this case, $\rho(t,x)$ represents the spatial mass density. The function $f = f(t,x,v)$ denotes the phase-space density of either the electrons or stars, while $-U$ and U denote the electrostatic and gravitational potential, respectively.

The initial-value problem for this system, where the initial phase-space density $f(0,x,v) = \mathring{f}(x,v)$ is prescribed, is now well understood, and the existence of global, classical solutions for C^1 data with appropriate decay at infinity is established [10, 21, 27, 32, 35].

However, these rigorous results provide only limited information about the qualitative behavior of the solutions. The purpose of the present investigation is to clarify the question of stability of certain stationary solutions. Two main stability concepts have to be distinguished: A stationary solution is *nonlinearly stable* if solutions of the nonlinear Vlasov-Poisson system remain arbitrarily close to the stationary solution in some norm for all times, provided the Vlasov-Poisson solutions start sufficiently close to the stationary solution. The stationary solution is *linearly stable* if the solutions of the nonlinear system are replaced by the solutions of the corresponding linearized problem in the above "definition." Obviously, a global existence result — at least for initial data close to the steady state under consideration — is an integral part of both stability concepts.

If the solution is written as $f_0 + f(t)$, where f_0 is the distribution function of the steady state, and the term that is quadratic in $f(t)$ is neglected, one obtains the linearized Vlasov-Poisson system:

$$\partial_t f + v \cdot \partial_x f - \partial_x U_0 \cdot \partial_v f = \partial_x U_f \cdot \partial_v f_0,$$

$$\Delta U_f = 4\pi\gamma\rho_f,$$

$$\rho_f(t, x) := \int f(t, x, v)\, dv,$$

where the steady state (f_0, U_0) satisfies the stationary Vlasov-Poisson system:

$$v \cdot \partial_x f_0 - \partial_x U_0 \cdot \partial_v f_0 = 0,$$

$$\Delta U_0 = 4\pi(\rho^+ + \gamma\rho_0),$$

$$\rho_0(x) := \int f_0(x, v)\, dv.$$

Stability conditions are often expressed in terms of how f_0 depends on the local or particle energy $E(x, v) := \frac{1}{2}v^2 + U_0(x)$. Since E and, for spherical symmetry under which $U_0(x) = U_0(|x|)$, also $F := |x \times v|^2$ are constant along the characteristics of the stationary Vlasov equation, it is natural to represent $f_0(x, v) = \varphi(E)$ or $f_0(x, v) = \varphi(E, F)$ with some function φ. The present work is restricted to the first case.

There exists a large number of investigations of both linear and nonlinear stability: cf. [1, 2, 11, 12, 13, 16, 19, 20, 22, 36, 37, 38]. The general and long-standing opinion seems to be that — both in the plasma physics and in the stellar dynamics cases — a steady state is stable if φ is a decreasing function of the energy. Although these results are physically appealing and plausible, a distinction must be made between these results and rigorous mathematics. (Note, too, that certain conclusions drawn for anisotropic spherical systems are admittedly contradictory [13, p. 308].) Concerning nonlinear stability we mention the following rigorous results: In [9] it is shown, for the plasma physics case with spatial periodicity, that spatially homogeneous steady states are nonlinearly stable if φ is decreasing; the analogous result for the relativistic Vlasov-Maxwell system is shown in [25]. Both results are based on using the total energy of the system as a Lyapunov function; cf. also [28]. In [34] it is shown by REIN how to put the so-called energy-Casimir method as described in [20] into a rigorous mathematical setting for the plasma physics case, thereby obtaining a nonlinear stability result in this case. In [38] an attempt to prove nonlinear stability using the energy-Casimir method for the stellar dynamics case was made; however, shortcomings of this work were pointed out in [24]. (Bounds used in [38] require that solutions of the Vlasov-Poisson system have uniformly bounded gradients, a property that has not to date been established. Also, it is not clear whether steady states that are "regular" in the sense of Definition 1 exist.) As for linear stability, the only rigorous result we are aware of is [18] where the effect of Landau damping is established for a homogeneous steady state in the one-dimensional case. It is important to point out that in infinite-dimensional dynamical systems such as the Vlasov-Poisson system, the relationship between the two concepts of stability is in general not clear.

The present investigation is intended to fill part of the gap between what is done in more physically motivated papers and what is mathematically established. We proceed as follows: In the next section the basic assumptions on the steady states under consideration are collected. In Section 3 we prove a global existence and uniqueness result for the linearized Vlasov-Poisson system. Section 4 introduces a conserved quantity of the linearized system, the dynamical free energy, for solutions with initial conditions whose support lies inside the compact support of the steady state f_0. This is preparatory for Section 5 where we derive our stability results in terms of a weighted L^2-norm induced by the free energy: For the plasma physics case we obtain linear stability if $\varphi' < 0$ on the support of φ, and in the stellar dynamics case we obtain linear stability if $\varphi' > 0$ on its support. Since the steady state has to have finite total mass, this necessitates a jump discontinuity of φ. In Section 6 we show that there exist steady states which satisfy our assumptions. This is necessary since in the plasma physics case the existence of steady states in the above situation has not yet been demonstrated; we refer to [6, 14, 15, 33] for related results. In the stellar dynamics case the polytropes of the form $\varphi(E) = (E_0 - E)_+^\mu$ with $-1 < \mu < 0$, which are investigated in [7] and [8], do not satisfy our assumptions, but their approximations $\tilde{\varphi}(E) = \varphi(E)\chi_{]-\infty,E_1]}(E)$ with $E_1 < E_0$ do.

Since the restriction on the support of the initial distribution may seem unphysical, we show in an appendix that it is a natural assumption within the framework of a linearized theory. In the stellar dynamics case the result that f_0 is stable if φ is increasing on its support and has a jump discontinuity at its boundary may also seem unphysical. However, our analysis then at least shows that the results of a linearized theory have to be taken with care. This in turn should be a valuable insight for the plasma physics and stellar dynamics communities, since there stability questions are very often approached by linearization.

2. Assumptions on the stationary solutions

We consider stationary solutions (f_0, U_0) of the Vlasov-Poisson system such that

$$f_0(x, v) = \varphi(\tfrac{1}{2}v^2 + U_0(x)), \quad x, v \in \mathbb{R}^3,$$

where φ satisfies the assumptions

(φ1) $\varphi \in L^\infty_{\text{loc}}(\mathbb{R})$, $\varphi \geqq 0$,
(φ2) $E_0 := \inf\{E \in \mathbb{R} \mid \varphi(E') = 0 \text{ a.e. for } E' > E\} - \infty, \infty[$,
(φ3) $\varphi \in C^1(]-\infty, E_0[)$ with $\varphi' \in L^1_{\text{loc}}(]-\infty, E_0])$,

and U_0 satisfies the assumptions

(U1) $U_0 \in C^2(\mathbb{R}^3)$,
(U2) U_0 is bounded; $U_{\min} := \inf_{x \in \mathbb{R}^3} U_0(x) < E_0$,
(U3) the set $B := \{(x, v) \in \mathbb{R}^6 \mid \frac{1}{2}v^2 + U_0(x) \leqq E_0\}$ is bounded, and ∂B has measure zero.

Here $\varphi \in L^\infty_{\text{loc}}(\mathbb{R})$ means that $\varphi_{|K} \in L^\infty(K)$ for every compact interval $K \subset \mathbb{R}$, and $L^1_{\text{loc}}(]-\infty, E_0])$ is defined analogously.

Conditions (φ2) and (U2) imply that the energy levels of the distribution function f_0 vary between the values $U_{\min}$ and E_0. Together with (U3) this means that f_0 has phase-space support in the bounded set B. In particular, the steady state has finite radius, i.e., there exists a radius $R_0 > 0$ such that $f_0(x, v) = 0$ for $|x| > R_0$, and by (φ1) it has finite mass or change:

$$\iint_{\mathbb{R}^6} f_0(x, v)\, dv\, dx \leqq \operatorname{vol}(B) \sup_{E \in [U_{\min}, E_0]} \varphi(E) < \infty.$$

Condition (φ3) implies the estimates

$$\begin{aligned}\int_{\mathbb{R}^3} |\varphi'(E(x, v))|\, dv &= 4\pi \int_{U_0(x)}^{E_0} |\varphi'(E)| \sqrt{2(E - U_0(x))}\, dE \\ &\leqq 4\pi \sqrt{2(E_0 - U_{\min})} \int_{U_{\min}}^{E_0} |\varphi'(E)|\, dE \\ &< \infty, \quad x \in \mathbb{R}^3,\end{aligned}$$

and, with $z := (x, v) \in \mathbb{R}^6$,

$$\int_{\mathbb{R}^6} |\varphi'(E(z))|\, dz < (4\pi)^2 \sqrt{2(E_0 - U_{\min})} \int_{U_{\min}}^{E_0} |\varphi'(E)|\, dE \int_0^{R_0} r^2\, dr < \infty,$$

which are needed in the proof of global existence for the linearized Vlasov-Poisson system in Section 3.

We write $(f_0, U_0) \in \mathcal{S}$ if (f_0, U_0) is a stationary solution of the Vlasov-Poisson system satisfying the above assumptions. In Section 6 we show that stationary solutions of this type exist both in the plasma physics and in the stellar dynamics cases.

Throughout the paper constants which depend only on the steady state under consideration — such as the above integrals — and which may change from line to line are all denoted by C.

3. Global existence for the linearized Vlasov-Poisson system

Let $(f_0, U_0) \in \mathcal{S}$ and let $t \mapsto f_0 + f(t)$ be a solution of the Vlasov-Poisson system with initial condition $f_0 + \mathring{f}$. If the term which is quadratic in $f(t)$ in the

Vlasov equation is neglected, we arrive at the linearized Vlasov-Poisson system for the (small) perturbation $f(t)$

$$\partial_t f + v \cdot \partial_x f - \partial_x U_0 \cdot \partial_v f = \partial_x U_f \cdot \partial_v f_0, \tag{3.1}$$

$$\Delta U_f = 4\pi\gamma\rho_f, \tag{3.2}$$

$$\rho_f(t,x) := \int f(t,x,v)\, dv, \tag{3.3}$$

together with the initial condition $f(0) = \mathring{f}$. Assuming that U_f vanishes at infinity, we obtain

$$U_f(t,x) = -\gamma \int \frac{\rho_f(t,y)}{|x-y|}\, dy, \quad x \in \mathbb{R}^3. \tag{3.4}$$

Consider the system of characteristics corresponding to (3.1):

$$\dot{x} = v,\ \dot{v} = -\partial_x U_0(x). \tag{3.5}$$

Due to the regularity of U_0, for every $t \in \mathbb{R}$ and $z = (x,v) \in \mathbb{R}^6$ there exists a unique global solution $Z(\cdot, t, z) = (X, V)(\cdot, t, x, v)$ of (3.5) with $Z(t,t,z) = z$. The mapping Z is continuously differentiable in all variables and $Z(s,t,\cdot)\colon \mathbb{R}^6 \to \mathbb{R}^6$ is a measure-preserving diffeomorphism for all $s, t \in \mathbb{R}$. Using the flow Z, we can write (3.1) in the form

$$\frac{d}{ds} f(s, Z(s,t,z)) = (\partial_x U_f(s) \cdot \partial_v f_0)(Z(s,t,z)), \quad s,t \in \mathbb{R},\ z \in \mathbb{R}^6,$$

which upon integration yields

$$f(t,z) = \mathring{f}(Z(0,t,z)) + \int_0^t (\partial_x U_f(s) \cdot \partial_v f_0)(Z(s,t,z))\, ds, \quad t \geqq 0,\ z \in \mathbb{R}^6.$$

Since for steady states of class $\mathscr{S}$ we have $f_0(x,v) = \varphi(\frac{1}{2}v^2 + U_0(x))$ and since the energy $E(x,v) = \frac{1}{2}v^2 + U_0(x)$ is invariant under the characteristic flow, this relation becomes

$$f(t,z) = \mathring{f}(Z(0,t,z)) + \varphi'(E(z)) \int_0^t (\partial_x U_f(s) \cdot v)(Z(s,t,z))\, ds, \quad t \geqq 0,\ z \in \mathbb{R}^6. \tag{3.6}$$

This motivates the following definition:

Definition 3.1. Let $\mathring{f} \in L^1(\mathbb{R}^6)$. A function $f\colon [0,\infty[\times \mathbb{R}^6 \to \mathbb{R}$ is a solution of the linearized Vlasov-Poisson system with initial value $\mathring{f}$ if and only if

(i) $f \in C([0,\infty[, L^1(\mathbb{R}^6))$,
(ii) $\rho_f \in C([0,\infty[, L^\infty(\mathbb{R}^3))$,
(iii) $U_f \in C([0,\infty[, C_b^1(\mathbb{R}^3))$,
(iv) f satisfies (3.6) for $t \geqq 0$ and $z \notin \partial B$,
(v) $f(0) = \mathring{f}$.

Here ρ_f and U_f are defined by (3.3) and (3.4), respectively, and $C_b^1(\mathbb{R}^3)$ denotes the space of continuously differentiable functions which are bounded together with their first derivatives.

Note that a solution in this sense satisfies the linearized Vlasov-Poisson system classically if f and U_f are sufficiently regular.

Theorem 3.2. *Let $(f_0, U_0) \in \mathscr{S}$ and let $\mathring{f} \in L^1(\mathbb{R}^6)$ be a pointwise-defined, $\mathbb{R}$-valued function such that $\mathring{f} \circ Z(0,\cdot)$ satisfies conditions* (i) *and* (ii) *of Definition 3.1. Then there exists a unique solution of the linearized Vlasov-Poisson system with initial value $\mathring{f}$.*

Proof. We construct a converging sequence of iterates in the set

$$M := \{g : [0, \infty[\, \times \mathbb{R}^6 \to \mathbb{R} \,|\, g \text{ satisfies (i), (ii), (v) of Definition 3.1}\}.$$

Obviously, $f_1 := \mathring{f} \circ Z(0,\cdot) \in M$. Let $g \in M$. Then (i) implies that

$$\rho_g \in C([0, \infty[, L^1(\mathbb{R}^3)),$$

and the well-known estimates (cf. [4])

$$\begin{aligned} \| U_g(t) \|_\infty &\leqq 2(2\pi)^{1/3} \| \rho_g(t) \|_\infty^{1/3} \| \rho_g(t) \|_1^{2/3}, \\ \| \partial_x U_g(t) \|_\infty &\leqq 3(2\pi)^{2/3} \| \rho_g(t) \|_\infty^{2/3} \| \rho_g(t) \|_1^{1/3} \end{aligned} \tag{3.7}$$

yield (iii) for U_g. Define

$$(Tg)(t,z) := f_1(t,z) + \varphi'(E(z)) \int_0^t (\partial_x U_g(s) \cdot v)(Z(s,t,z))\, ds, \quad t \geqq 0,\ z \notin \partial B,$$

and zero otherwise. The estimate

$$\begin{aligned} &\int_{\mathbb{R}^6} \left| \varphi'(E(z)) \int_0^t (\partial_x U_g(s) \cdot v)(Z(s,t,z))\, ds \right| dz \\ &\qquad \leqq \int_B |\varphi'(E(z))|\, dz \sup_{z \in B} |v| \int_0^t \| \partial_x U_g(s) \|_\infty\, ds \\ &\qquad \leqq C \int_0^t \| \rho_g(s) \|_\infty^{2/3} \| \rho_g(s) \|_1^{1/3}\, ds \end{aligned}$$

shows that $(Tg)(t) \in L^1(\mathbb{R}^6)$ — recall that constants denoted by C may depend on the steady state under consideration. Let $0 \leqq \tau < t$; then

$$\begin{aligned} &\| (Tg)(t) - (Tg)(\tau) \|_1 \\ &\leqq \| f_1(t) - f_1(\tau) \|_1 + C \int_\tau^t \| \partial_x U_g(s) \|_\infty\, ds \\ &\quad + \int_0^\tau \int_B |\varphi'(E(z))| \, |(\partial_x U_g(s) \cdot v)(Z(s,t,z)) - (\partial_x U_g(s) \cdot v)(Z(s,\tau,z))| \, dz\, ds \\ &\to 0 \quad \text{for } \tau \to t, \end{aligned}$$

since f_1 satisfies condition (i) and $(\partial_x U_g(s)\cdot v)(Z(s,\tau,z))$ is uniformly continuous on $[0,t]^2\times B$. The case $\tau > t$ is analogous, and thus Tg satisfies condition (i). Next observe that

$$|\rho_g(t,x)| \leqq |\rho_{f_1}(t,x)| + C\int|\varphi'(E(z))|\,dv\int_0^t \|\partial_x U_g(s)\|_\infty\,ds$$

$$\leqq |\rho_{f_1}(t,x)| + C\int_0^t \|\partial_x U_g(s)\|_\infty\,ds,$$

which implies that $\rho_g(t)\in L^\infty(\mathbb{R}^3)$ for $t\geqq 0$. Furthermore, for $\tau<t$,

$$\begin{aligned}\|\rho_g(t)-\rho_g(\tau)\|_\infty &\leqq \|\rho_{f_1}(t)-\rho_{f_1}(\tau)\|_\infty + C\int_\tau^t \|\partial_x U_g(s)\|_\infty\,ds\\ &\quad + C\int_0^\tau \sup_{z\in B}|(\partial_x U_g(s)\cdot v)(Z(s,t,z))-(\partial_x U_g(s)\cdot v)(Z(s,\tau,z))|\,ds\\ &\to 0 \quad \text{for } \tau\to t,\end{aligned}$$

where the first term converges by the assumption on $\mathring{f}$, the second converges by (iii) and the last by the same argument as above. Since the case $\tau>t$ is analogous, we have (ii), and (v) being obvious we have shown that $Tg\in M$, i.e., T maps the set M into itself. Now let $g_1, g_2\in M$; then the above estimates show that

$$\|(Tg_1)(t)-(Tg_2)(t)\|_1 \leqq C\int_0^t \|\rho_{g_1}(s)-\rho_{g_2}(s)\|_\infty^{2/3}\,\|g_1(s)-g_2(s)\|_1^{1/3}\,ds,$$

$$\|\rho_{Tg_1}-\rho_{Tg_2}(t)\|_\infty \leqq C\int_0^t \|\rho_{g_1}(s)-\rho_{g_2}(s)\|_\infty^{2/3}\,\|g_1(s)-g_2(s)\|_1^{1/3}\,ds.$$

Hence, if we define $f_{n+1}:=Tf_n$, $n\geqq 1$, it follows that there exist functions $f\in C([0,\infty[, L^1(\mathbb{R}^6))$ and $\rho\in C([0,\infty[, L^\infty(\mathbb{R}^3))$ such that $f_n(t)\to f(t)$ in $L^1(\mathbb{R}^6)$, t-locally uniformly on $[0,\infty[$ and $\rho_n(t)\to\rho(t)$ in $L^\infty(\mathbb{R}^3)$, t-locally uniformly on $[0,\infty[$. Since $\rho_{f_n}(t)\to\rho_f(t)$ in $L^1(\mathbb{R}^3)$, we have $\rho=\rho_f$ and $\rho_{f_n}(t)\to\rho_f(t)$ in $L^1(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)$, t-locally uniformly on $[0,\infty[$. This implies that $U_{f_n}(t)\to U_f(t)$ in $C_b^1(\mathbb{R}^3)$, t-locally uniformly on $[0,\infty[$. Passing to the limit in the relation

$$f_{n+1}(t,z)=f_1(t,z)+\varphi'(E(z))\int_0^t(\partial_x U_{f_n}(s)\cdot v)(Z(s,t,z))\,ds,\ t\geqq 0,\quad z\notin\partial B,$$

we obtain

$$f(t,z)=\mathring{f}(Z(0,t,z))+\varphi'(E(z))\int_0^t(\partial_x U_f(s)\cdot v)(Z(s,t,z))\,ds,\ t\geqq 0,\quad z\notin\partial B,$$

after redefining f on a set of measure zero. Since condition (v) is clear, f is a solution in the sense of Definition 3.1. Uniqueness of the solution is obvious. □

Corollary 3.3. *The solution f obtained in Theorem 3.2 has the following properties for $t \geqq 0$:*

(a) $f(t,z) = \mathring{f}(Z(0,t,z))$ *for $z \notin B$, in particular, if $\mathring{f}$ vanishes outside B, then so does $f(t)$.*

(b) $\frac{d}{dt} f(t, Z(t,0,z)) = \varphi'(E(z))(\partial_x U_f(t) \cdot v)(Z(t,0,z)),\ z \notin \partial B.$

(c) $\int \mathring{f}(z)\,dz = \int f(t,z)\,dz.$

(d) *If $\mathring{f}$ has compact support or vanishes sufficiently rapidly at infinity, then*

$$\int U_f(t,x) f(t,z)\,dz = -\frac{\gamma}{4\pi}\int |\partial_x U_f(t,x)|^2\,dx.$$

Proof. (a) is obvious, (b) follows from replacing z in (3.6) by $Z(t,0,z)$ and differentiating the resulting equation, (c) follows by integrating (3.6) with respect to z and using the fact that the flow Z preserves measure and that the term $\partial_x U_f(s,x)\cdot v$ is odd in v. Note that the set B is invariant with respect to $(x,v) \mapsto (x,-v)$.

(d) If $\rho_f(t)$ is, in addition, Hölder-continuous, then

$$\begin{aligned}
&\int U_f(t,x) f(t,z)\,dz = \int U_f(t,x)\rho_f(t,x)\,dx \\
&= \lim_{r\to\infty} \int_{|x|\leqq r} U_f(t,x)\rho_f(t,x)\,dx = \frac{1}{4\pi\gamma}\lim_{r\to\infty}\int_{|x|\leqq r} U_f(t,x)\,\Delta U_f(t,x)\,dx \\
&= \frac{\gamma}{4\pi}\lim_{r\to\infty}\left[\int_{|x|=r} U_f(t,x)\,\partial_x U_f(t,x)\cdot n(x)\,d\omega(x) - \int_{|x|\leqq r} |\partial_x U_f(t,x)|^2\,dx\right] \\
&= -\frac{\gamma}{4\pi}\int |\partial_x U_f(t,x)|^2\,dx
\end{aligned}$$

if the decay of $U_f(t,x)\partial_x U_f(t,x)$ at spatial infinity is such that the boundary term vanishes; if $\mathring{f}$ has compact support, then $U_f(t,x)\partial_x U_f(t,x) = O(|x|^{-3})$. In case $\rho_f(t)$ is not Hölder-continuous, we can use a mollification of $\rho_f(t)$ to get the result. □

4. Conservation of free energy

Theorem 4.1. *Let $(f_0, U_0) \in \mathscr{S}$ and let $\mathring{f} \in L^1(\mathbb{R}^6)$ be as in Theorem 3.2 with $\mathring{f}(z) = 0$ for $z \notin B$ and*

$$\int_B \frac{\mathring{f}^2(z)}{|\varphi'(E(z))|}\,dz < \infty. \tag{4.1}$$

Then

$$\int_B \frac{f^2(t,z)}{|\varphi'(E(z))|}\,dz < \infty, \quad t \geqq 0,$$

$$\mathscr{F}(t) := -\int_B \frac{f^2(t,z)}{\varphi'(E(z))}\,dz + \int U_f(t,x) f(t,z)\,dz = \mathscr{F}(0), \quad t \geqq 0. \tag{4.2}$$

Here the quotient is to be understood as

$$\frac{g^2(z)}{\varphi'(E(z))} = \begin{cases} \infty & \text{for } \varphi'(E(z)) = 0 \text{ and } g(z) \neq 0, \\ 0 & \text{for } \varphi'(E(z)) \text{ arbitrary and } g(z) = 0. \end{cases}$$

Proof. Assume that

$$\int_B \frac{f^2(t,z)}{|\varphi'(E(z))|}\, dz < \infty$$

for some $t \geqq 0$. Then with Corollary 3.3,

$$\mathscr{F}(t) = -\int_B \frac{f^2(t, Z(t,0,z))}{\varphi'(E(z))}\, dz - \gamma \int\int \frac{f(t, Z(t,0,z)) f(t, Z(t,0,z'))}{|X(t,0,z) - X(t,0,z')|}\, dz\, dz'$$

$$= -\int_B \frac{\mathring{f}^2(z)}{\varphi'(E(z))}\, dz - \int_B \int_0^t \frac{\frac{d}{ds} f^2(s, Z(s,0,z))}{\varphi'(E(z))}\, ds\, dz - \gamma \int\int \frac{\mathring{f}(z)\mathring{f}(z')}{|x - x'|}\, dz\, dz'$$

$$- \gamma \int\int\int_0^t \frac{d}{ds} \frac{f(s, Z(s,0,z)) f(s, Z(s,0,z'))}{|X(s,0,z) - X(s,0,z')|}\, ds\, dz\, dz'$$

$$= \mathscr{F}(0) - 2 \int_B \int_0^t f(s, Z(s,0,z)) (\partial_x U_f(s) \cdot v)(Z(s,0,z))\, ds\, dz$$

$$+ \gamma \int\int\int_0^t \frac{X(s,0,z) - X(s,0,z')}{|X(s,0,z) - X(s,0,z')|^3} (V(s,0,z) - V(s,0,z'))$$

$$\times f(s, Z(s,0,z)) f(s, Z(s,0,z'))\, ds\, dz\, dz'$$

$$- 2\gamma \int_B \int_0^t \varphi'(E(z)) (\partial_x U_f(s) \cdot v)(Z(s,0,z))$$

$$\times \int \frac{f(s, Z(s,0,z'))}{|X(s,0,z) - X(s,0,z')|}\, dz'\, ds\, dz$$

$$= \mathscr{F}(0) - 2 \int_0^t \int_B f(s,z)\, \partial_x U_f(s,x) \cdot v\, dz\, ds$$

$$+ \gamma \int_0^t \int\int \frac{x - x'}{|x - x'|^3} (v - v') f(s,z) f(s,z')\, dz\, dz'\, ds$$

$$+ 2 \int_0^t \int_B \varphi'(E(z))\, \partial_x U_f(s,x) \cdot v\, U_f(s,x)\, dz\, ds =$$

$$= \mathscr{F}(0) - 2 \int_0^t \int_B f(s,z)\, \partial_x U_f(s,x) \cdot v \, dz \, ds$$

$$+ 2 \int_0^t \int_B f(s,z)\, \partial_x U_f(s,x) \cdot v \, dz \, ds$$

$$+ 2 \int_0^t \int_B \varphi'(E(z))\, \partial_x U_f(s,x) \cdot v \, U_f(s,x) \, dz \, ds$$

$$= \mathscr{F}(0);$$

the last integral vanishes because the integrand is odd in v and B-invariant under the mapping $v \mapsto -v$. Retracing all the steps of the argument, we observe that all the integrals exist by the boundedness of the term $\partial_x U_f(s,x) \cdot v$ on B and by the integrability of $\varphi' \circ E$, and Fubini's Theorem applies. □

Remark: The energy expression of (4.2), restricted to monotonically decreasing (nonvanishing) stationary phase-space densities, was first obtained in [26] in a plasma physics model more general than that of the Vlasov-Poisson system. Imposing condition (4.1) allows one to consider stationary solutions of compact support by restricting the class of initial conditions. In the Appendix we comment on the restriction that $\mathring{f}$ has to vanish outside the support of f_0.

5. Linear stability

Theorem 5.1. *Let $(f_0, U_0) \in \mathscr{S}$, assume that $\gamma\varphi'(E) > 0$ for $U_{\min} \leqq E < E_0$, and define the weighted L^2-norm*

$$\| g \|_{2,\varphi}^2 := \gamma \int_B \frac{g^2(z)}{\varphi'(E(z))} \, dz.$$

Then (f_0, U_0) is linearly stable in the following sense: For every $\mathring{f}$ as in Theorem 4.1 with $\| \mathring{f} \|_1^{2/3} \| \rho_{\mathring{f}} \|_\infty^{1/3} \leqq 1$ the corresponding solution f of the linearized Vlasov-Poisson system satisfies the estimate

$$\| f(t) \|_{2,\varphi}^2 \leqq c_0 \| \mathring{f} \|_{2,\varphi} + \| \mathring{f} \|_{2,\varphi}^2, \quad t \geqq 0,$$

where the constant c_0 depends only on the stationary solution (f_0, U_0).

Proof. $$\| f(t) \|_{2,\varphi}^2 = \gamma \int_B \frac{f^2(t,z)}{\varphi'(E(z))} \, dz$$

$$= -\gamma \mathscr{F}(t) - \frac{\gamma^2}{4\pi} \int | \partial_x U_f(t,x) |^2 \, dx$$

$$\leqq -\gamma\mathscr{F}(0) = \gamma \int_B \frac{\mathring{f}^2(z)}{\varphi'(E(z))}\,dz - \gamma \int_B U_{\mathring{f}}(x)\mathring{f}(z)\,dz$$

$$\leqq \|\mathring{f}\|_{2,\varphi}^2 - \gamma \int_B \frac{\mathring{f}(z)}{\sqrt{\gamma\varphi'(E(z))}}\sqrt{\gamma\varphi'(E(z))}\,U_{\mathring{f}}(x)\,dz$$

$$\leqq \|\mathring{f}\|_{2,\varphi}^2 + \left(\int_B |\varphi'(E(z))|\,U_{\mathring{f}}^2(x)\,dz\right)^{1/2} \|\mathring{f}\|_{2,\varphi}$$

$$\leqq \|\mathring{f}\|_{2,\varphi}^2 + 2(2\pi)^{1/3}\,\|\mathring{f}\|_1^{2/3}\,\|\rho_{\mathring{f}}\|_\infty^{1/3}\left(\int_B |\varphi'(E(z))|\,dz\right)^{1/2} \|\mathring{f}\|_{2,\varphi},$$

where the last estimate follows from (3.7). Thus, the proof is complete, with

$$c_0 := 2(2\pi)^{1/3}\left(\int_B |\varphi'(E(z))|\,dz\right)^{1/2}. \qquad \square$$

Remarks. 1. Using [38, Lemma 2] to estimate the potential energy corresponding to $\mathring{f}$ in the above proof we obtain the alternative stability estimate

$$\|f(t)\|_{2,\varphi}^2 \leqq c_1 \|\mathring{f}\|_2^2 + \|\mathring{f}\|_{2,\varphi}^2, \quad t \geqq 0,$$

for all initial data as in Theorem 4.1, where c_1 again depends on the stationary solution (f_0, U_0).

2. If $0 < c_- \leqq |\varphi'(E)| \leqq c_+ < \infty$ on $]-\infty, E_0[$, then the norm $\|\cdot\|_{2,\varphi}$ is equivalent to the usual L^2 norm, and we obtain the stability estimate

$$\|f(t)\|_2 \leqq c_3 \|\mathring{f}\|_2, \quad t \geqq 0$$

for all initial data as in Theorem 4.1.

3. The stellar dynamics case where $\varphi'(E) > 0$ is of particular interest. This result requires the jump discontinuity in φ and the restricted class of initial conditions $\mathring{f}$ described in Theorem 4.1. It is natural to question the physical relevance of and the sensitivity to these assumptions. One would expect collisions, i.e., the effect of short-range interactions, to smooth out the jump discontinuity in φ and produce a transition region where $\varphi'(E) > 0$ (and large). In this way collisions can provide a mechanism for the onset of instability.

6. Examples

In this section we establish the existence of a large class of stationary solutions $(f_0, U_0) \in \mathscr{S}$. Among these there are steady states satisfying the stability condition of Theorem 5.1, i.e., $\varphi'(E) < 0$ in the plasma physics case and $\varphi'(E) > 0$ in the stellar dynamics case.

Any f_0 of the form

$$f_0(x,v) = \varphi(E) = \varphi(\tfrac{1}{2}v^2 + U_0(x)) \tag{6.1}$$

automatically satisfies Vlasov's equation, since the energy E is constant along characteristics. Therefore, the stationary Vlasov-Poisson system is reduced to the semilinear Poisson equation

$$\Delta U_0(x) = 4\pi(\rho^+(x) + \gamma h_\varphi(U_0(x)), \quad x \in \mathbb{R}^3,$$

where

$$h_\varphi(u) := \int \varphi(\tfrac{1}{2}v^2 + u)\,dv.$$

Here we investigate spherically symmetric solutions of this problem, i.e., solutions of

$$\frac{1}{r^2}(r^2 U_0'(r))' = 4\pi(\rho^+(r) + \gamma h_\varphi(U_0(r)), \quad r > 0, \tag{6.2}$$

where

$$h_\varphi(u) := \frac{2\pi}{r^2} \int_0^\infty \int_0^\infty \varphi\left(\frac{1}{2}w^2 + \frac{F}{2r^2} + u\right) dF\,dw, \tag{6.3}$$

$r := |x|$, $w := x \cdot v/|x|$, $F := |x \times v|^2 = x^2v^2 - (x \cdot v)^2$. The distribution function f_0 is then a function of r, w, F, and $\rho_0(r) = h_\varphi(U_0(r))$ is a function of r, i.e., the whole steady state is spherically symmetric.

For the rest of this section let φ satisfy the conditions (φ1) and (φ2) from Section 2 and assume in addition that

(φ4) Case (S) (stellar dynamics case): $\varphi(E_0) := \lim_{E \nearrow E_0} \varphi(E)$ exists and $\varphi(E_0) > 0$,
Case (P) (plasma physics case): $\rho^+ \in C([0,\infty[)$, $\rho^+ \geqq 0$, $r^2\rho^+ \in L^1([0,\infty[)$, and there exist constants $r_0 > 0$ and $\rho_0^+ > 0$ such that $\rho^+(r) \geqq \rho_0^+$, $r \in [0, r_0]$.

Theorem 6.1. *Let the assumptions (φ1), (φ2), and (φ4) be satisfied. Then there exists a constant $\alpha_0 < E_0$ such that for $\alpha \in]\alpha_0, E_0[$ the problem (6.2) has a unique solution $U_0 \in C^2([0,\infty[)$ with $U_0(0) = \alpha$, where h_φ is defined by (6.3). U_0 is strictly increasing, $U_0'(0) = 0$, $E_0 < \lim_{r\to\infty} U_0(r) < \infty$, and there exists $R_0 > 0$ such that $U_0(R_0) = E_0$ and $U_0'(R_0) > 0$. Consequently, if φ in addition satisfies (φ3) and f_0 is defined by (6.1) and $\rho_0 := h_\varphi \circ U_0$, then $(f_0, U_0) \in \mathcal{S}$, $\rho_0 \in C^1([0,\infty[)$, and $\rho_0(r) = 0$ for $r \geqq R_0$, $\rho_0(r) > 0$ for $r < R_0$. Under the further assumption that $\gamma\varphi'(E) > 0$ for $E < E_0$, the steady state (f_0, U_0) is stable in the sense of Theorem 5.1.*

For the proof of this result the following lemma is useful:

Lemma 6.2. *Let φ satisfy the assumptions* (φ1) *and* (φ2). *Then* (6.3) *defines a function $h_\varphi \in C^1(\mathbb{R})$, $h_\varphi(u) = 0$ for $u \geqq E_0$, and*

$$h_\varphi(u) = 4\pi\sqrt{2} \int_u^\infty \varphi(E)\sqrt{E-u}\,dE,$$

$$h_\varphi'(u) = -\frac{4\pi}{\sqrt{2}} \int_u^\infty \varphi(E)\frac{dE}{\sqrt{E-u}}, \quad u \in \mathbb{R}.$$

The proof of this lemma is an easy application of Lebesgue's theorem on dominated convergence and therefore omitted.

Proof of Theorem 6.1. Local existence and uniqueness of the solution for arbitrary $\alpha \in \mathbb{R}$ follow by the contraction mapping principle, applied to the following reformulation of the problem:

$$U_0'(r) = \frac{4\pi}{r^2} \int_0^r s^2(\rho^+(s) + \gamma h_\varphi(\alpha + \int_0^s U_0'(\tau)\,d\tau))\,ds.$$

Let $U_0 \in C^1([0,R[)$ be the solution, extended to its maximal interval of existence $[0,R[$, and $\rho_0(r) := h_\varphi(U_0(r))$. Then $U_0 \in C^2(]0,R[)$, $U_0''(r)$ has a limit for $r \to 0$, $U_0'(0) = 0$, and this implies the regularity assertions for U_0 and ρ_0. For the rest of the proof, we have to treat the two cases (P) and (S) separately.

Case (P): Take $\alpha_0 < E_0$ such that

(i) $h_\varphi(u) < \rho_0^+/2$ for $u \in]\alpha_0, E_0[$,

(ii) $E_0 - \alpha_0 < \frac{\pi}{3}\rho_0^+ r_0^2$,

and let $\alpha \in]\alpha_0, E_0[$. Then by (i) $U_0'(r) > 0$, and U_0 is strictly increasing on $[0,r_0] \cap [0,R]$; in fact, because h_φ is decreasing,

$$U_0'(r) = \frac{4\pi}{r^2} \int_0^r s^2(\rho^+(s) - h_\varphi(U_0(s))\,ds$$

$$> \frac{4\pi}{r^2} \int_0^r s^2\left(\rho_0^+ - \frac{\rho_0^+}{2}\right) ds = \frac{2\pi}{3}\rho_0^+ r.$$

Since $h_\varphi(u) = 0$ for $u \geqq E_0$, this implies that $R > r_0$, and by condition (ii),

$$U_0(r_0) > \alpha + \tfrac{\pi}{3}\rho_0^+ r_0^2 > E_0.$$

Thus there exists $R_0 \in]0,r_0[$ with $U_0(R_0) = E_0$ and $U_0(r) > E_0 (< E_0)$ for $r > R_0 (< R_0)$ which implies the assertions on ρ_0.

Case (S): First of all we note that for any $\alpha < E_0$ the potential U_0 is strictly increasing on $[0,R[$. Either $U_0(r) < E_0$ for $r \in [0,R[$ in which case U_0 is bounded and thus exists globally, or $U_0(r) \geqq E_0$ for $r \geqq R_0$ and some R_0 in which case ρ_0 vanishes for $r \geqq R_0$, and U_0 again exists globally. To prove that actually the latter case holds, we rely on the analysis in [5]. The existence of R_0 follows if we show that the (possibly infinite) limit

$$L := \lim_{r \to \infty} U_0(r) > E_0.$$

Assume $L < E_0$. Then the monotonicity of U_0 and h_φ implies that $h_\varphi(U_0(r)) \geqq h_\varphi(L)$ for $r \geqq 0$, and thus,

$$U_0'(r) \geqq \frac{4\pi}{r^2} \int_0^r s^2 h_\varphi(L)\, ds = \frac{4\pi}{3} h_\varphi(L) r, \quad r \geqq 0.$$

But this means that $U_0(r) \to \infty$ for $r \to \infty$, a contradiction. Thus it remains to show that the assumption $L = E_0$ leads to a contradiction as well. To this end, define $y(r) := E_0 - U_0(r)$, $r \geqq 0$; then

(i) $y(r) > 0$ and $y'(r) < 0$ for $r \geqq 0$,

(ii) $\lim_{r \to \infty} y(r) = 0$, $\lim_{r \to \infty} r y(r) > 0$, $\lim_{r \to 0} \frac{y'(r)}{r} < 0$,

(iii) $(r^2 y'(r))' = -H(r, y(r))$, $r > 0$, where $H(r, y) := 4\pi r^2 h_\varphi(E_0 - y)$.

Here the assertions in (i) follow from the strict monotonicity of U_0, and the first assertion in (ii) is our assumption $L = E_0$. The second assertion in (ii) follows by L'Hospital's rule:

$$\begin{aligned} \lim_{r \to \infty} r(L - U_0(r)) &= \lim_{r \to \infty} \frac{L - U_0(r)}{1/r} = \lim_{r \to \infty} \frac{U_0'(r)}{1/r^2} \\ &= \lim_{r \to \infty} 4\pi \int_0^r s^2 \rho_0(s)\, ds = 4\pi \int_0^\infty s^2 \rho_0(s)\, ds > 0. \end{aligned}$$

Finally,

$$\frac{y'(r)}{r} = -\frac{U_0'(r)}{r} = -\frac{4\pi}{r^3} \int_0^r s^2 \rho_0(s)\, ds \to \frac{4\pi}{3} \rho_0(0) = -\frac{4\pi}{3} h_\varphi(\alpha) < 0$$

for $r \to 0$. Condition (iii) is Equation (6.2), rewritten for y. Obviously, H satisfies the relation

(iv) $r \partial_r H(r, y) = 2H(r, y)$, $r \geqq 0$, $y \in \mathbb{R}$.

The assumption (φ4) in case (S) implies the existence of constants $\alpha_0 < E_0$ and $0 < c_1 < c_2$ such that

$$c_1 \leqq \varphi(E) \leqq c_2 \quad \text{for } E \in [\alpha_0, E_0[.$$

Take $\alpha \in]\alpha_0, E_0[$; then $0 < y(r) \leqq E_0 - \alpha < E_0 - \alpha_0$ or $\alpha_0 < E_0 - y(r) < E_0$ for $r \geqq 0$. Thus,

$$\begin{aligned}\partial_y H(r, y(r)) &= -4\pi r^2 h'_\varphi(E_0 - y(r)) \\ &= \frac{1}{\sqrt{2}}(4\pi)^2 r^2 \int_{E_0 - y(r)}^{E_0} \varphi(E) \frac{dE}{\sqrt{E - E_0 + y(r)}} \\ &\leqq \frac{(4\pi)^2}{\sqrt{2}} r^2 c_2 2\sqrt{y(r)} = (4\pi)^2 \sqrt{2} c_2 r^2 \sqrt{y(r)},\end{aligned}$$

$$H(r, y(r)) \geqq (4\pi)^2 \sqrt{2} c_1 r^2 \int_{E_0 - y(r)}^{E_0} \sqrt{E - E_0 + y(r)}\, dE = (4\pi)^2 \sqrt{2}\, \tfrac{2}{3} c_1 r^2 y(r)^{3/2}.$$

For α_0 sufficiently close to E_0 we can assume that

$$\frac{c_2}{c_1} < \frac{10}{3}.$$

Then there exists a constant $m \in]1, 5[$ such that $c_2/c_1 \leqq 2m/3$ i.e., $c_2 \leqq \frac{2}{3} c_1 m$, which in view of the above estimates for $H(r, y(r))$ and $\partial_y H(r, y(r))$ implies that

(v) $y(r)\partial_y H(r, y(r)) \leqq mH(r, y(r))$, $r \geqq 0$.

Conditions (i) to (v) now lead to the desired contradiction in the following way; cf. also [5]: Define

$$\begin{aligned}q(r) &:= \frac{r y^{(m+1)/2}(r)}{y'(r)}, \quad r \geqq 0, \\ Q(r) &:= r^2 y(r)^{(1-m)/2} y'(r)^2 q'(r) \\ &= 3r^2 y(r) y'(r) + \frac{m+1}{2} r^3 y'(r)^2 + r y(r) H(r, y(r)), \quad r \geqq 0;\end{aligned}$$

then

$$\begin{aligned}Q'(r) &= \frac{5-m}{2} r^2 y'(r)^2 + (r\partial_r H - 2H) y(r) + (y\partial_y H - mH) r y'(r) \\ &\geqq \frac{5-m}{2} r^2 y'(r)^2, \quad r \geqq 0\end{aligned}$$

by (iv) and (v), and

$$\lim_{r \to 0} Q(r) = 0.$$

Thus

$$r^2 y(r)^{(1-m)/2} y'(r)^2 q'(r) = Q(r) = \int_0^r Q'(s)\, ds > 0,$$

which implies that $q'(r) > 0$ for $r > 0$, and we have shown that q is strictly increasing. The third assertion in (ii) yields

$$\lim_{r \to 0} q(r) = -A, \quad A > 0,$$

and we conclude that

$$-A < q(r) = \frac{r y^{(m+1)/2}(r)}{y'(r)}, \quad r > 0.$$

Therefore,

$$\begin{aligned}(-A) y(r)^{-(m-1)/2} &< (-A)(y(r)^{-(m-1)/2} - y(0)^{-(m-1)/2}) \\ &= (-A) \int_0^r \left(-\frac{m-1}{2}\right) y(s)^{-(m+1)/2} y'(s)\, ds \\ &< \frac{1-m}{2} \int_0^r s\, ds = \frac{1-m}{4} r^2, \quad r > 0,\end{aligned}$$

which implies that

$$r y(r) < C r^{(m-5)/(m-1)}, \quad r > 0$$

for some constant $C > 0$. But this contradicts the second assertion in (ii). Thus, the only remaining possibility is $L > E_0$ which implies that $U_0(R_0) = E_0$ for some $R_0 > 0$ also in the stellar dynamics case. In both cases U_0 is strictly increasing and $U_0'(r) \sim r^{-2}$ for $r > R_0$ so that $E_0 < \lim_{r \to \infty} U_0(r) < \infty$. Furthermore, $U_0'(R_0) > 0$ so that $\partial_z E(z) = (\frac{x}{r} U_0'(r), v) \neq 0$ for $z = (x, v) \in \partial B$. This shows that ∂B is a C^1-submanifold of $\mathbb{R}^6$ and is of measure zero. Since $U_0 \in C^2([0, \infty[)$ and $U_0'(0) = 0$, U_0 is in C^2 when interpreted as a function on $\mathbb{R}^3$, and the proof is complete. □

Appendix. Some comments on the class of admissible perturbations

The purpose of this section is to show that the restriction that $\mathring{f}$ vanish outside the support of f_0 is a natural one within the present, linearized setting.

A class of perturbations that one certainly wishes to include are perturbations caused by external forces. For example, one could think of f_0 as representing some stationary galaxy which is perturbed by some other, distant galaxy. The particle orbits are then given by a perturbed characteristic system

$$\dot{x} = v, \quad \dot{v} = -\partial_x P,$$

where the (time-dependent) potential P of the perturbed force field can be thought of as the sum of the self-consistent potential caused by the (perturbed) f_0, and the potential of the outside force. Let $Z_P = (X_P, V_P)$ be the flow of the perturbed characteristic system. The perturbed distribution function is then given by

$$f_0 \circ Z_P(0, \tau, \cdot),$$

and a natural class of perturbations would be the set

$$M := \{f_0 \circ Z_P(0, \tau, \cdot) - f_0 \mid P \in C(]-\delta, \delta[; C_b^2(\mathbb{R}^3)),\ \tau \in]-\delta, \delta[,\ \delta > 0\}.$$

It is quite obvious that this set contains perturbations whose support is not contained in the support of f_0. However, in a linearized theory it is consistent to approximate the above perturbations to first order in τ, that is, to take as perturbations the tangent vectors to all curves

$$r \mapsto f_0 \circ Z_P(0, \tau, \cdot)$$

at $\tau = 0$ where $P \in C(]-\delta, \delta[; C_b^2(\mathbb{R}^3))$. In more geometric language: We replace the "manifold" M by its "tangent space" at f_0. Now

$$\begin{aligned}\frac{d}{d\tau}\Big|_{\tau=0} f_0(Z_P(0, \tau, z)) &= \partial_x f_0(z) \cdot \frac{d}{d\tau}\Big|_{\tau=0} X_P(0, \tau, z) + \partial_v f_0(z) \cdot \frac{d}{d\tau}\Big|_{\tau=0} V_P(0, \tau, z) \\ &= -\partial_x f_0(z) \cdot v + \partial_v f_0(z) \cdot \partial_x P(0, x) \\ &= [g, f_0](z),\end{aligned}$$

where $[\cdot, \cdot]$ denotes the usual Poisson bracket, and

$$g(z) := \tfrac{1}{2} v^2 + P(0, x);$$

note that $Z_P(0, \cdot, z)$ satisfies

$$\begin{aligned}&\frac{d}{d\tau} Z_P(0, \tau, z) \\ &= (-v, \partial_x P(\tau, x)) + \int_\tau^0 \left(\frac{d}{d\tau} V_P(\sigma, \tau, z), \partial_x^2 P(\sigma, X_P(\sigma, \tau, z)) \cdot \frac{d}{d\tau} X_P(\sigma, \tau, z)\right) d\sigma.\end{aligned}$$

Therefore, the set

$$\{[g, f_0] \mid g(z) = \tfrac{1}{2} v^2 + P(x), P \in C_b^2(\mathbb{R}^3)\}$$

is natural as the set of admissible perturbations in a linearized setting, and obviously these functions have support in the support of f_0.

If we take $f_0 \in C_c^1(\mathbb{R}^6)$, then the curve

$$\tau \mapsto f_0 \circ Z_P(0, \tau, \cdot)$$

is differentiable as a curve with values in any $L^p(\mathbb{R}^6)$, $1 \leqq p \leqq \infty$. In general, we do not require this regularity of f_0, and the corresponding curve need not be differentiable. However, neither do we require the form $[g, f_0]$ for the initial perturbation, but keep only the restriction on the support which makes sense without any differentiability assumption on f_0.

Perturbations of the form $[g, f_0]$ were called *dynamically accessible* perturbations in [29, 30, 31]. (See also [3, 17, 23].) For such perturbations, condition (4.1) turns into the following condition on g:

$$\int [g, E]^2(z)\,|\varphi'(E(z))|\,dz < \infty,$$

and the singularity in (4.1) due to the vanishing of φ' outside B disappears.

A second argument justifying our assumption on the support of the perturbation goes as follows. Let us for the moment return to the original nonlinear system and write the solution as $F = f_0 + f$. If we insert this into the nonlinear Vlasov equation and use the fact that f_0 is a steady state, we obtain the Vlasov equation

$$\partial_t f + v \cdot \partial_x f - \partial_x U_0 \cdot \partial_v f - \partial_x U_f \cdot \partial_v f_0 - \partial_x U_f \cdot \partial_v f = 0, \quad (x, v) \in \mathbb{R}^6,$$

with obvious meaning of U_f. If one linearizes this equation, one drops certain terms (quadratic in f) with the justification in mind that for small f they are small compared to other terms (linear in f) which are not dropped. On the support of f_0 the term $\partial_x U_f \cdot \partial_v f$ is small compared to the term $\partial_x U_f \cdot \partial_v f_0$ and thus can be dropped. However, outside the support of f_0 the latter term is zero so that the quadratic term in f can be dropped only if it is small compared to zero, i.e., if it is zero itself, which is the case if and only if $\mathring{f}$ vanishes outside the support of f_0. It would be questionable to drop the term $\partial_x U_f \cdot \partial_v f$ outside $\operatorname{supp} f_0$ by comparing it to the term $\partial_x U_0 \cdot \partial_v f$ since $\partial_x U_0$ can be zero outside $\operatorname{supp} f_0$, for example, if f_0 represents a spherically symmetric plasma with zero net charge.

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