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Fluid element relabeling symmetry

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Abstract

Lagrangian symmetries are found for hydrodynamics and magnetohydrodynamics, which result in conservation of potential vorticity and of cross helicity, respectively. These symmetries, which persist in the reduction from Lagrangian to Eulerian variables, directly give rise to Casimir invariants of the Hamiltonian formalism. The mechanism of spontaneous symmetry breaking in a fluid is also presented.

1. Introduction

We find, in Section 2, a symmetry of the ideal compressible fluid Lagrangian, which is an *infinite* continuous group parametrized by an *arbitrary function*. The corresponding generalized Bianchi identity, which must exist by Noether's second theorem, gives rise to Ertel's theorem of conservation of potential vorticity [1]. The symmetry we find is a velocity-dependent transformation, unlike the point symmetries of fluid field theories, and their consequences due to Noether's (first) theorem that have been explored earlier [2-5]. The symmetries here involve only a transformation of the fluid element labels, hence we follow Ref. [5] in naming them "relabeling symmetries". (See Ref. [6] for more details, where a related but less general symmetry is discussed.)

In Section 3 we show that the potential energy functional obtained by expanding about a stationary equilibrium possesses a Bianchi identity. Also, in this section a relationship to spontaneous symmetry breaking and the existence of null eigenfunctions are shown.

Section 4 is concerned with the Hamiltonian framework. We show that the map from Lagrangian variables (or material variables) to Eulerian variables for a fluid has the same relabeling symmetry. This symmetry is then used to directly construct the Casimir invariants for the noncanonical Poisson bracket [7,8] for the fluid in Eulerian form. This rounds out the usual picture of reduction from Lagrangian to Eulerian variables (see e.g. Refs. [8,9]).

We deal with magnetohydrodynamics (MHD) in Section 5. Relabeling symmetry of MHD results in the cross helicity invariant for barotropic flows. This invariant has previously been linked to Lagrangian symmetries [10], but not to fluid element relabeling. Other symmetries of the reduction from material to Eulerian variables give rise to Casimir invariants too, including a little explored family of invariants that incorporates magnetic helicity as a special case.

2. Relabeling symmetry in hydrodynamics

The variables $q(a,t)$ keep track of the position of the fluid element labeled a. At any time the mapping between q and a is an invertible mapping of a domain,

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D, and to simplify matters, D is assumed time independent although the fluid is compressible. The fluid Lagrangian density, $\mathcal{L}$, may be written as [6,11–13]

$$\mathcal{L}(a,q,\dot{q},\partial q) = \rho_0[\tfrac{1}{2}\dot{q}^2 - U(\rho,s) - \Phi(q)], \qquad (1)$$

where $\rho_0 = \rho_0(a)$ is the initial density distribution and $\dot{q}$ denotes the time derivative of q keeping the label fixed. The internal or potential energy per unit mass is denoted by U and is assumed to be a function of two thermodynamic quantities, viz. the density, ρ and the entropy, s. Additional forces on the fluid can be accounted for by including a potential, $\Phi(q)$. We also assume adiabaticity, that is $s = s_0(a)$ only. Conservation of mass implies $\rho\,\mathrm{d}^3q = \rho_0\,\mathrm{d}^3a$, hence we have $\rho(a,t) = \rho_0(a)/\mathcal{J}$, where $\mathcal{J}$ denotes the Jacobian of the transformation, $\partial(q)/\partial(a)$, and is restricted to be positive. Dependence of $\mathcal{L}$ on derivatives of q is indicated by ∂q.

Upon seeking an infinitesimal *relabeling* transformation $\hat{a} = a + \delta a$ and $\Delta q := \hat{q}(\hat{a},t) - q(a,t) \equiv 0$, the action, $S = \int\int_D \mathcal{L}\,\mathrm{d}^3a\mathrm{d}t$, is invariant when

$$\delta a(a,\partial q,\partial\dot{q},\partial^2 q,\partial^2\dot{q}) = \frac{\nabla s_0 \times \nabla\epsilon}{\rho_0}, \qquad (2)$$

where $\epsilon = \epsilon(Q_s,a)$ is an infinitesimal, arbitrary function of its arguments and Q_s is an abbreviation for

$$Q_s := \frac{1}{\rho_0}\nabla\dot{q}_i \times \nabla q^i \cdot \nabla s_0. \qquad (3)$$

The relabeling transformation defined by Eq. (2) satisfies

$$\nabla\cdot(\rho_0\,\delta a) = 0, \quad \delta a\cdot\nabla s_0 = 0, \qquad (4)$$

which assure that the relabeling does not alter the mass and lies within isentropic surfaces, however it *does* alter the velocity field, $\Delta\dot{q} = -\nabla s_0 \times \nabla(\partial\epsilon/\partial t)\cdot\nabla q/\rho_0$. Later it will be clear that Q_s, and hence $\epsilon(Q_s,a)$, has no time dependence on the orbits, so the velocity field is unchanged *only on the orbits*. (There does exist a simpler point symmetry [6] for which $\delta\dot{a}$ vanishes identically and there is no change in the velocity field. For this symmetry, ϵ depends only on a and the Lagrangian *density* itself is invariant. The symmetry given by Eq. (2) depends on space and time derivatives of q and is more general than the point symmetry.) Evidently, relabeling means that each component of q transforms as a scalar.

This symmetry being of an infinite continuous group, there exists a generalized Bianchi identity,

$$\rho_0\frac{\partial Q_s}{\partial t} + \nabla\left(\frac{S_i}{\rho_0}\right)\cdot\nabla q^i \times \nabla s_0 = 0, \qquad (5)$$

where S_i denotes the functional derivative of the action, $\delta S/\delta q^i$. Generalized Bianchi identities are valid for *any* $q(a,t)$ and indicate that not all equations of motion are independent. (For more on this see Refs. [14–16]; for a summary see e.g. Ref. [6].) When the equations of motion are satisfied $S_i \equiv 0$ and Eq. (5) reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\rho}\tilde{\nabla}s\cdot\tilde{\nabla}\times v\right) = 0, \qquad (6)$$

where the chain rule has been used to convert a derivatives to q derivatives to yield the Eulerian expression above. The gradient operator in q space is denoted by $\tilde{\nabla}$, the velocity $v(q,t) := \dot{q}(a,t)$, the entropy $s(q,t) := s_0(a)$, and the density $\rho(q,t)$ are obtained from the inverse map, $a(q,t)$. The Lagrangian or material derivative is $\mathrm{d}/\mathrm{d}t := \partial/\partial t|_a = \partial/\partial t|_q + v\cdot\tilde{\nabla}$.

In Eq. (6) the quantity, $\tilde{Q}_s(q,t) := \tilde{\nabla}s\cdot\tilde{\nabla}\times v/\rho$, which is the Eulerian expression of $Q_s(a,t)$ on the orbits, is called the potential vorticity associated with the advected quantity, s, and Eq. (6), which expresses the advection of $\tilde{Q}_s$, is called Ertel's theorem of conservation of potential vorticity. The conservation of potential vorticity was derived from a (different) Lagrangian point symmetry in Ref. [4] for incompressible stratified flows. In Ref. [5] conservation of potential vorticity is derived from a constrained variational principle; the relabeling transformation is not stated explicitly. The treatment in Ref. [3] expresses the symmetry in terms of δq rather than δa, however the arbitrary dependence of ϵ on Q_s is not stated. For the symmetry presented here, we can express δq Eulerianly as: $\delta q = \Delta q - \delta a\cdot\nabla q = (\tilde{\nabla}\epsilon\times\tilde{\nabla}s)/\rho$, where ϵ is understood to be the Eulerian equivalent to $\epsilon(Q_s,a)$. The use of relabeling symmetry seems to have been made first in Ref. [2] where an "exchange symmetry" is found for an incompressible, ideal fluid without internal energy, U. In Refs. [2] and [3] relabeling symmetry is related to Kelvin's circulation theorem. One can easily proceed to show that Ertel's theorem gives rise to Kelvin's circulation theorem on material surfaces of constant entropy [6]. (In the barotropic case the circulation theorem holds on *any* material surface since the entropy in Ertel's theorem may be replaced by any advected quantity.)

3. Spontaneous symmetry breaking

In the stability analysis of stationary fluid equilibria (in particular MHD) one often considers the second variation of potential energy functionals. Consider the fluid potential energy functional,

$$W[q] := \int_D \rho_0[U(\rho, s) + \Phi(q)]\, \mathrm{d}^3a\,, \tag{7}$$

for which,

$$\delta_* W = \int_D \frac{\delta W}{\delta q^i} \delta_* q^i\, \mathrm{d}^3a \equiv 0\,, \tag{8}$$

where $\delta_* q = -\delta a \cdot \nabla q$ and δa is given by Eq. (2), but without any restriction on the time dependence of ϵ, i.e. $\epsilon = \epsilon(a, t)$. (The symbol δ_* specifies the symmetry transformation for the sake of clarity in what follows.) Eq. (8) leads to a generalized Bianchi identity,

$$\nabla\left(\frac{1}{\rho_0}\frac{\delta W}{\delta q^i}\right) \times \nabla q^i \cdot \nabla s_0 = 0\,. \tag{9}$$

The functional derivatives of W, which are set to zero to obtain the extremal point, are thus not entirely independent of each other.

Taking a second variation of (8) and evaluating on an equilibrium point, q_e, yields

$$\delta_*^2 W_e = \int_D \delta q^j \frac{\delta^2 W[q_e]}{\delta q^i \delta q^j} \cdot \delta_* q_e^i\, \mathrm{d}^3a \equiv 0\,, \tag{10}$$

where the dot indicates that the operator on the left acts on the quantity to the right and since the second δq is arbitrary, it follows that

$$\frac{\delta^2 W[q_e]}{\delta q^i \delta q^j} \cdot \delta_* q_e^i \equiv 0\,. \tag{11}$$

The case $\delta_* q_e^i = 0$ is trivial (since it implies $q_e = q_e(s_0)$ alone), therefore $\delta^2 W[q_e]/\delta q^i \delta q^j$ has $\delta_* q_e^i$ as a null eigenvector, and symmetry is "spontaneously broken". (See e.g. Ref. [17]; in the context of noncanonical Hamiltonian theory see Ref. [18].) Observe that $\delta_* q_e$ is a zero frequency eigenfunction of the linearized equations of motion written in Lagrangian variables.

Since relabeling is a symmetry group, it is clear that one can make a finite displacement from the equilibrium point and remain on the same level set of H. For example, the next variation evaluated on q_e gives

$$\delta_*^3 W_e = \int_D \delta_* q^i \left(\frac{\delta^3 W[q_e]}{\delta q^k \delta q^j \delta q^i} \cdot \delta q^j\right) \cdot \delta q^k\, \mathrm{d}^3a \equiv 0\,. \tag{12}$$

This procedure is analogous to Taylor expanding a potential energy function about an equilibrium of a finite system that lies in a trough. This was worked out explicitly to all orders for the special case of toroidal geometry in Ref. [19]. Although in terms of Lagrangian variables the equilibria that are connected by the relabeling transformation are distinct, it is evident by the definition of relabeling that in the Eulerian description these equilibria are identical.

The argument outlined above also holds for the potential energy functional for barotropic MHD. However, the symmetry is not an infinite parameter group; thus there does not exist a generalized Bianchi identity for MHD.

4. Symmetry of the Eulerian variables

We now consider the Hamiltonian formulation of hydrodynamics (see e.g. Ref. [8]). Expressed in Lagrangian variables the Hamiltonian has the form

$$H[\pi, q; a] := \int_D \mathcal{H}(\pi, q, \partial q, a)\, \mathrm{d}^3a$$
$$:= \int_D \rho_0\left[\frac{1}{2}\left(\frac{\pi}{\rho_0}\right)^2 + U\left(\frac{\rho_0}{\mathcal{J}}, s_0\right) + \Phi(q)\right] \mathrm{d}^3a\,, \tag{13}$$

which together with the canonical Poisson bracket,

$$[F, G] = \int_D \left(\frac{\delta F}{\delta q} \cdot \frac{\delta G}{\delta \pi} - \frac{\delta G}{\delta q} \cdot \frac{\delta F}{\delta \pi}\right) \mathrm{d}^3a\,, \tag{14}$$

produces the ideal fluid equations of motion. On making the transformation $\hat{a} = a + \delta a(a, t)$, $\Delta q := \hat{q}(\hat{a}, t) - q(a, t) \equiv 0$ and $\Delta \pi := \hat{\pi}(\hat{a}, t) - \pi(a, t) =$

$(\delta a \cdot \nabla \rho_0)(\pi/\rho_0)$ (so that $\Delta(\pi/\rho_0) \equiv 0$) it is seen that the action and the Hamiltonian density are invariant for the relabeling symmetry given by Eq. (2), but with $\dot{q}$ replaced by π/ρ_0. The existence of this symmetry of the Hamiltonian density indicates that one may be able to obtain an alternative formulation of the dynamics in terms of variables which inherently possess this symmetry. This is indeed the case for the reduction (see e.g. Refs. [8,9] and references therein) to Eulerian variables,

$$\rho(q,t) := \frac{\rho_0(a)}{\mathcal{J}}, \quad \sigma(q,t) := \frac{\rho_0(a)\, s_0(a)}{\mathcal{J}}, \\ M(q,t) := \frac{\pi(a,t)}{\mathcal{J}}, \tag{15}$$

where $a = a(q,t)$. The variations of the Eulerian variables ρ, σ, and M, that are *induced by relabeling*, $\mathcal{J}\delta\rho = \nabla \cdot (\rho_0 \delta a)$, $\mathcal{J}\delta\sigma = s_0 \nabla \cdot (\rho_0 \delta a) + \rho_0 \delta a \cdot \nabla s_0$ and $\rho_0 \mathcal{J} \delta M = \pi \nabla \cdot (\rho_0 \delta a)$, vanish for the relabeling transformation under consideration.

In the framework resulting from the reduction to Eulerian variables, we are naturally interested in functionals expressible in terms of the Eulerian variables, $F[q,\pi] = \bar{F}[\rho,\sigma,M]$. For a functional, $F[q,\pi]$, to be thus expressible, it is necessary (although not sufficient) that it possess the symmetry of the Euler–Lagrange map. This enables us to construct Casimir invariants, special invariants that arise in the Eulerian framework, from knowledge of the symmetry. The variation of F must vanish when the variations δq and $\delta \pi$ arise from the relabeling symmetry, δa, hence

$$\delta F = \int_D \left(\frac{\delta F}{\delta q} \cdot \delta q + \frac{\delta F}{\delta \pi} \cdot \delta \pi \right) \mathrm{d}^3 a = 0. \tag{16}$$

(Note that δ is to be distinguished from Δ; Δq is the first order change, $\hat{q}(\hat{a},t) - q(a,t)$, at the *transformed* point, while δq represents the first order change, $\hat{q}(a,t) - q(a,t)$, at the same point.) It is clear that if there exists a functional, $C[q,\pi]$, such that $\delta q = -\delta C/\delta \pi$ and $\delta \pi = \delta C/\delta q$, then its Poisson bracket with any $F[q,\pi]$ belonging to the class of functionals satisfying Eq. (16), vanishes. This will be the case when the Poisson bracket is expressed in terms of Eulerian, noncanonical variables [7] and therefore, by definition, C is a Casimir invariant. Evidently, Casimir invariants are constants of motion for any dynamics with a Hamiltonian that can be expressed in terms of Eulerian variables. It is easily checked that the functional defined by

$$C[q,\pi] := \int_D \rho_0(a)\, \bar{\epsilon}(Q_s, a)\, \mathrm{d}^3 a, \tag{17}$$

is the generator of the symmetry, i.e. it satisfies $[C, q^i] = -\delta C/\delta \pi_i = -\delta a \cdot \nabla q^i =: \delta q^i$ and $[C, \pi_i] = \delta C/\delta q^i = \Delta \pi_i - \delta a \cdot \nabla \pi_i =: \delta \pi_i$. We note that Q_s is written in terms of π/ρ_0 rather than $\dot{q}$ and the ϵ that appeared earlier in the expression for the symmetry is related to $\bar{\epsilon}$ by $\epsilon(Q_s, a) = \partial \bar{\epsilon}(Q_s, a)/\partial Q_s$. The Eulerian expression for the Casimir invariants yields

$$C[\rho, s, v] := \int_D \rho \mathcal{C}(s, \tilde{Q}_s)\, \mathrm{d}^3 q, \tag{18}$$

where $\mathcal{C}$ is an arbitrary function of both arguments and $s(q,t) := \sigma(q,t)/\rho(q,t) = s_0(a(q,t))$.

In the noncanonical Hamiltonian formulation of the fluid, a Casimir has to satisfy the conditions

$$\tilde{\nabla} \cdot \left(\rho \frac{\delta C}{\delta M} \right) = 0, \quad \frac{\delta C}{\delta M} \cdot \tilde{\nabla} \left(\frac{\sigma}{\rho} \right) = 0, \tag{19}$$

$$M_j \tilde{\nabla} \frac{\delta C}{\delta M_j} + \frac{\delta C}{\delta M} \cdot \tilde{\nabla} \left(\frac{M}{\rho} \right) + \rho \tilde{\nabla} \frac{\delta C}{\delta \rho} + \sigma \tilde{\nabla} \frac{\delta C}{\delta \sigma} \\ = 0. \tag{20}$$

The equivalence of these conditions to the symmetry conditions, Eqs. (4), is seen when one notes that if C can be expressed as a functional of ρ, σ, and M, then

$$\frac{\delta C}{\delta \pi} = \frac{\delta C}{\delta M}, \\ \frac{\delta C}{\delta q} = \mathcal{J} \left(\rho \tilde{\nabla} \frac{\delta C}{\delta \rho} + \sigma \tilde{\nabla} \frac{\delta C}{\delta \sigma} + M_i \tilde{\nabla} \frac{\delta C}{\delta M_i} \right). \tag{21}$$

Relating the above expressions to the symmetry generated by C, i.e. setting $\delta C/\delta \pi = \delta a \cdot \nabla q$ and $\delta C/\delta q = -\rho_0\, \delta a \cdot \nabla(\pi/\rho_0)$, leads to Eqs. (19) and (20) when δa satisfies Eqs. (4) and vice versa.

For barotropic flow, s may be replaced by any advected τ. Therefore one can use Q_τ to generate yet another advected quantity, Q_{Q_τ} and so on; from one advected quantity we can generate an infinite family of advected quantities. Thus the Casimir has the form

$$C[\rho, \tau, v] = \int_D \rho f(\tau, Q_\tau, Q_{Q_\tau}, \ldots)\, \mathrm{d}^3 q, \tag{22}$$

where $f(\tau, Q_\tau, Q_{Q_\tau}, \ldots)$ is an arbitrary function of the arguments.

5. Relabeling symmetry in MHD

The Lagrangian density for MHD [13] is given by

$$\mathcal{L}_{\mathrm{MHD}} = \mathcal{L} - \frac{1}{2\mathcal{J}} \partial_j q^i \partial_k q_i B_0^j B_0^k, \tag{23}$$

where $\mathcal{L}$ is the fluid Lagrangian density given by Eq. (1) and $B_0^i(a)$ are components of the magnetic field as a function of the labels, e.g. the initial magnetic field. The introduction of the magnetic field term leads to overspecification of conditions on δa, and consequently there is no relabeling symmetry. (It is for this reason that the potential energy functional for MHD does not exhibit spontaneous symmetry breaking, unlike the fluid case discussed in Section 3, and is thus reminiscent of the Higgs mechanism in quantum field theory.)

Any integral of a function of the labels alone, if it has a representation in terms of ρ, s, and B, is expected to be a Casimir of the noncanonical Hamiltonian structure in Eulerian variables. This is because $\delta C/\delta q = 0 = \delta C/\delta \pi$ for such integrals, C, hence the Poisson bracket of C with any functional of q and π vanishes trivially. It is easily verified that $B \cdot \tilde{\nabla}\tau/\rho = B_0 \cdot \nabla\tau_0/\rho_0$, where $\tau(q,t) := \tau_0(a)$ is an arbitrary advected quantity. Similarly, the Lagrange–Euler map for the magnetic field, $\mathcal{J}B^i(q,t) := B_0^j(a)\, \partial q^i/\partial a^j$, and its corresponding vector potential representation, $A_i(q,t) = A_{0j}(a)\, \partial a^j/\partial q^i$, lead to the conclusion that $A \cdot B/\rho = A_0 \cdot B_0/\rho_0$, within a gauge restriction. We note that one may add to $A_0(a)$, the gradient of a gauge, $\phi_0(a,t)$, which leads to a corresponding gauge choice, $\phi(r,t) := \phi_0(q^{-1}(r,t),t)$, for $A(r,t)$. But for the validity of $A \cdot B/\rho = A_0 \cdot B_0/\rho_0$, we must restrict the gauge to be advected, $\phi(r,t) := \phi_0(q^{-1}(r,t))$, which is equivalent to demanding that all explicit time dependence be removed from A_0. With this choice it can be seen that the vector potential in Eulerian coordinates satisfies the equation

$$\frac{\partial A}{\partial t} = v \times B - \tilde{\nabla}(A \cdot v). \tag{24}$$

(This gauge choice and the corresponding invariant is discussed by Gordin and Petviashvili [20].) Thus, more generally, the Casimir invariants are expressed by

$$C[\rho, s, A] := \int_D \rho g\left(s, \frac{A \cdot B}{\rho}, \frac{B \cdot \tilde{\nabla} s}{\rho}, \frac{B \cdot \tilde{\nabla}}{\rho}\left(\frac{B \cdot \tilde{\nabla} s}{\rho}\right), \frac{B \cdot \tilde{\nabla}}{\rho}\left(\frac{A \cdot B}{\rho}\right), \ldots\right) \mathrm{d}^3 q, \tag{25}$$

where B is understood to be an abbreviation for $\tilde{\nabla} \times A$. Operating within the restricted choice of gauges mentioned earlier, we note that the addition of a gauge, $A \to A + \tilde{\nabla}\phi$, changes $A \cdot B/\rho$ by the term $B \cdot \tilde{\nabla}\phi/\rho$, which is also advected. The numerical value of $C[\rho, s, A]$ thus depends on the gauge, but after the initial choice of the gauge has been made, it nevertheless is a constant of the motion. It is clear that magnetic helicity, $\int A \cdot B\, \mathrm{d}^3 a$, is a special case of this family of invariants.

A nontrivial symmetry can, however, be found if one eliminates the second of Eqs. (4) by considering a barotropic flow, i.e. U and hence p depend only on the density, ρ. (A solution can also be found without imposing the restriction of barotropicity in the case where the entropy, s_0, is a flux label, i.e. $B_0 \cdot \nabla s_0 = 0$.) Then one has the symmetry

$$\delta a = \epsilon(x_0, y_0) \frac{B_0}{\rho_0}, \tag{26}$$

where $x_0(a)$ and $y_0(a)$ are flux labels. In other words, the initial magnetic field is expressible as $\nabla x_0 \times \nabla y_0$. However the existence of flux labels $x_0(a)$ and $y_0(a)$ is not crucial; if they do not exist one simply thinks of ϵ as an infinitesimal constant parameter.

For this symmetry, Noether's (first) theorem gives

$$\frac{\partial}{\partial t}\left(\dot{q}_j B_0 \cdot \nabla q^j\right) + \nabla \cdot \left[B_0\left(\frac{\dot{q}^2}{2} - U - \rho\frac{\mathrm{d}U}{\mathrm{d}\rho} - \Phi\right)\right] = 0. \tag{27}$$

Integrating over the domain and passing over to the Eulerian form using the relation $B_0^k \partial_k = \mathcal{J} B^i \tilde{\partial}_i$, we get the conservation law

$$\frac{\mathrm{d}}{\mathrm{d}t} C[v, B] := \frac{\mathrm{d}}{\mathrm{d}t} \int_D v \cdot B\, \mathrm{d}^3 q = 0, \tag{28}$$

where $C[v, B]$ is commonly referred to as cross helicity. The existence of this symmetry also leads to spontaneous symmetry breaking for a stationary barotropic plasma, the argument for which closely follows that of Section 3. Prior to this work conservation of cross helicity was derived from a Lagrangian symmetry involving Clebsch potentials and the polarization in Ref. [10]. (See also Ref. [21].)

The Casimir for the barotropic case is written most generally as

$$C[\rho,v,A] = \int_D \rho\left[\frac{v\cdot B}{\rho} + f\left(\frac{A\cdot B}{\rho}, \frac{B\cdot\tilde{\nabla}}{\rho}\left(\frac{A\cdot B}{\rho}\right), \ldots\right)\right] \mathrm{d}^3q, \tag{29}$$

where f is an arbitrary function of its argument. In the case where flux labels exist globally, the Casimir is given by

$$C[v,x,y] = \int_D f(x,y)\, v\cdot\nabla x\times\nabla y\, \mathrm{d}^3q, \tag{30}$$

where f is an arbitrary function of the flux labels, $x(q,t) := x_0(a(q,t))$ and $y(q,t) := y_0(a(q,t))$.

6. Conclusions

We have described the consequences of Noether's theorems associated with the relabeling transformation for the ideal fluid and MHD. The action and Hamiltonian were seen to be invariant under the contact transformation we presented for hydrodynamics and the point transformation for barotropic MHD. These transformations were also seen to be symmetries of the Lagrange–Euler map, giving rise directly to Casimir's of the reduced Hamiltonian description of the fluids in terms of Eulerian variables. In addition Ertel's theorem, the Kelvin circulation theorem, cross and magnetic helicity, and other Casimir invariants, including a little known family of invariants in MHD, were discussed.

The formalism described is quite general and applies to a large class of ideal fluid models. More exotic fluids such as the Chew–Goldberger–Low model and gyroviscous fluids [2] possess a similar development.

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