Elliptical vortices in shear: Hamiltonian moment formulation and Melnikov analysis

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The equations of motion for interacting elliptical vortices in a background shear flow are derived from a Hamiltonian moment formulation. The equations reduce to the sixth order system of Melander *et al.* [J. Fluid Mech. **167**, 95 (1986)] when a pair of vortices is considered and shear is neglected. The equations for a pair of identical vortices are analyzed using a number of methods, with particular emphasis on the implications for vortex merger. The splitting distance between the stable and unstable manifolds connecting the hyperbolic fixed points of the intercentroidal motion—the separatrix splitting—is estimated with a Melnikov analysis. This analysis differs from the standard time-periodic Melnikov analysis on two counts: (a) the ''periodic'' perturbation arises from a second degree of freedom in the system which is not wholly independent of the first degree of freedom, the intercentroidal motion; (b) this perturbation has a faster time scale than the intercentroidal motion. The resulting Melnikov integral appears to be exponentially small in the perturbation as the latter goes to zero. Numerical simulations, notably Poincaré sections, provide a global view of the dynamics and indicate that, as observed in previous studies, there are essentially two modes of merger. The effect of the shear on chaotic motion is also discussed. © *1996 American Institute of Physics.* [S1070-6631(96)00104-1]

I. INTRODUCTION

In this paper, we first present a simple approximate model of the dynamics of N elliptical vortices in a two-dimensional shear flow. We subsequently use this model to examine the conditions under which a pair of vortices can merge in the presence of a shear flow. This model has many similarities with other discrete vortex models, most notably with the uniform elliptical vortex in shear of Kida (1981)[1] and the interacting uniform elliptical vortices of Melander *et al.* (1986)[2] (hereafter referred to as MZS). The essential difference lies in the simultaneous presence of vortex-vortex and vortex-shear interactions.[3,4] These processes both complement and compete with one another. Vortex-vortex and vortex-shear interactions are especially common in geophysics,[5–7] and they may also be found in other settings. The emergence of quasi-uniform vortices and their complex interaction with one another is recognized as an important feature of geostrophic (two-dimensional) turbulence.[8–10]

As in MZS, the vortices in our model are approximated as elliptical patches of uniform vorticity and an expansion based on spatial moments of the vorticity distribution is employed. However, the derivation of the resulting Hamiltonian system is considerably different (Sec. II). Instead of deriving the equations of motion by manipulating the moments, the procedure of Flierl *et al.* (1995)[11] (hereafter referred to as FMM) is followed; this is a powerful approach which provides a simple and generalizable route to the equations of motion. Beginning with a Hamiltonian description of the full (infinite-dimensional) system, approximations are made within this framework so as to obtain a reduced Hamiltonian description. In this way, the Hamiltonian structure of the problem is preserved in a natural way. (See Appendix A for background information on noncanonical Hamiltonian dynamics and the method of reduction.) The Hamiltonian nature of the equations of motion is more explicit, and the derivation is somewhat simpler than in MZS because of reduction and the noncanonical formalism.[12]

The analysis of the model is motivated by the phenomenon of vortex merger (Sec. III). In the absence of any external flow, like-signed vortices will merge when close together.[13–15] But when background shear is present, it is possible for even well separated vortices to merge if they are oriented appropriately.[3] Conversely, these two basic interactions can interfere: the shear flow may sweep the vortices past one another before they can merge, or each vortex may advect the other in a direction normal to the shear, making it easier for the shear to separate them.

An inherent limitation of our analysis is the approximate nature of the model. In the derivation, it is assumed that the vortices are small and well separated and that they remain elliptical for all time; during a vortex merger event, however, the model loses its asymptotic consistency since real vortices deviate increasingly from ellipticity as a merger event proceeds. It has been shown that the ''elliptical model'' of Legras and Dritschel (1991),[16] which is a Galerkin-like approximation to the contour dynamics equations, gives a better approximation than the model of MZS to some of the deformations seen during merger.[15] Nevertheless, in the absence of background shear, both the elliptical model and the model of MZS give similar predictions for the onset of merger.

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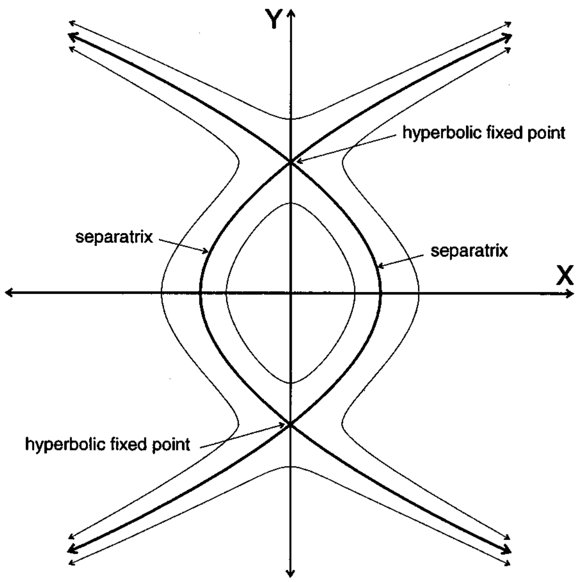

FIG. 1. Phase space geometry for a point vortex pair in shear with $\omega - e < 0$.

Thus, there is reason to believe that the model considered in this paper should provide some insight into the interaction of uniform vortices in shear.

For simplicity, the analysis is restricted to a pair of identical vortices in shear. Its centerpiece is a Melnikov analysis of the separatrix splitting between the stable and unstable manifolds connecting the hyperbolic fixed points of the intercentroidal motion (Sec. IV). The equations of motion for a pair of identical vortices reduce to those for a point vortex pair in shear (see Appendix B and Ref. 3), when the vortices' internal degrees of freedom are eliminated; for sufficiently strong strain, the phase plane of this system exhibits a pair of hyperbolic fixed points which are connected by a separatrix (see Fig. 1). Thus, a question which immediately comes to mind is what happens to the separatrix under the perturbation due to the elliptical vortices' internal degrees of freedom: in dynamical systems theory, the separatrix splitting, besides leading to chaotic motion, controls transport across the (unperturbed) separatrix.[17–19] This is interesting because it suggests a route by which widely separated vortices could interact at close range and possibly merge.

For the unperturbed point vortex pair in shear, the closed orbits in the interior are divided from the open orbits in the exterior by the separatrix (see Fig. 1); with a perturbation, a trajectory could pass from the exterior to the interior (and vice versa). However, because of the unusual form of the perturbation and the presence of multiple time scales, a standard Melnikov analysis cannot be applied. The approach we have developed differs from some previous applications of Melnikov's method to fluid dynamics[20–23] because the perturbation is not imposed externally, but rather arises naturally from the internal structure of the problem. A consequence of the separation of time scales between the perturbed and the unperturbed motion is that the Melnikov integral should be exponentially small as the perturbation amplitude goes to zero,[24,25] and this is what we find.

Although it is convenient (and natural) to present Melnikov's method in the Hamiltonian context, it is the model's underlying phase space geometry that is fundamental. Our numerical study of vortex merger and chaotic motion is based upon this general geometric approach to the dynamics (Sec. V). Two-dimensional Poincaré sections of the intercentroidal motion are the primary tool. They provide a global view of the dynamics, and as they are constructed at constant energy, they complement previous interpretations of vortex merger which are based on energy arguments (e.g., Ref. 4). In particular, they enable one to make a useful geometrical distinction between a predominantly vortex-vortex mode of merger and a predominantly vortex-shear one. In addition, it is observed that the chaotic dynamics in the vicinity of the separatrix are weaker than in the interior, in agreement with the asymptotic Melnikov analysis.

II. HAMILTONIAN MOMENT FORMULATION

In this section, the equations of motion for N elliptical vortices in a background shear flow are derived. Using a Hamiltonian moment formulation in which the quadratic vorticity moments are the dynamical variables, FMM were able to derive the equations of motion for the Kida vortex. Our work generalizes that of FMM by extending the analysis to N interacting vortices. Briefly, our derivation proceeds by (i) expressing the Poisson bracket for the two-dimensional (2-D) Euler equations in terms of the first and second order vorticity moments; (ii) determining the cosymplectic matrix J^{jk} from the bracket; (iii) computing the Hamiltonian in terms of the moments; and (iv) obtaining the equations of motion from H and J^{jk}. Background information on noncanonical Hamiltonian dynamics and on the notation adopted here may be found in Appendix A.

A. Poisson bracket

First, consider a 2-D Euler flow with a spatially and temporally varying vorticity distribution, $q(\mathbf{x},t)$. We make the assumption that q approaches a uniform, constant value, say ω, sufficiently rapidly as $|\mathbf{x}|\rightarrow\infty$, and we set $q(\mathbf{x},t)=\omega+q'(\mathbf{x},t)$.

The Poisson bracket for 2-D Euler flow is[26,27]

$$\{F,G\}=\int q'\left[\frac{\delta F}{\delta q'},\frac{\delta G}{\delta q'}\right]dx\,dy, \tag{1}$$

where F and G are functionals of vorticity, $\delta/\delta q'$ denotes a functional derivative, $[a,b]=a_x b_y - b_x a_y$ is the two-dimensional Jacobian, and the constant background vorticity does not appear. (The background vorticity will enter in the Hamiltonian by virtue of the dynamical role associated with the background flow.) We introduce a finite set of functionals of the perturbation vorticity which we call *moments*, $\{a^j[q']: j=0,\ldots,K-1\}$. For a subset of functionals of q' depending only on q' as *functions* of the moments, e.g., $F[q']=f(a^0[q'],\ldots,a^{K-1}[q'])$,

$$\frac{\delta F}{\delta q'}=\frac{\partial f}{\partial a^j}\frac{\delta a^j}{\delta q'},$$

and (1) takes the form

$$\{F,G\}=\frac{\partial f}{\partial a^j}J^{jk}\frac{\partial g}{\partial a^k},\tag{2a}$$

where

$$J^{jk}=\int q'\left[\frac{\delta a^j}{\delta q'},\frac{\delta a^k}{\delta q'}\right]dxdy.\tag{2b}$$

(Repeated sum notation is used here and henceforth.) The success of this approach depends on whether we are able to approximate the Hamiltonian of the system as a function of the moments, $\{a^j\}$. This will in turn depend both on the nature of the scalar field $q'(\mathbf{x})$ and on our particular choice of moments.

Let us move in the direction of *spatial* moments by introducing a finite (but as yet, arbitrary) set of time-dependent disjoint areas $\{S_j(t):j=0,\ldots,N-1\}$ and defining the set of moments, $\{\tilde{a}^j\}$, in terms of them by

$$\tilde{a}^j:=\int q'\chi_k(\mathbf{x})\chi_l(\mathbf{x})x^r x^s dxdy,\tag{3}$$

where r and s are non-negative integers and $r+s\leqslant 2$ for $0\leqslant k,l\leqslant N-1$. Here $\chi_j(\mathbf{x})=1$ if $\mathbf{x}\in S_j$ and $\chi_j(\mathbf{x})=0$ otherwise. The monomials x^r and x^s may be associated with $\chi_k(\mathbf{x})$ and $\chi_l(\mathbf{x})$, respectively. Provided that q' is such that we can choose the $\{S_j\}$ so that $q'=0$ on their boundaries, the Jacobians in the integrands of (2b) are polynomials of at most second degree. This ensures that J^{jk} takes the form

$$J^{jk}=c^{jk}_{\ell}\tilde{a}^{\ell}.\tag{4}$$

The c^{jk}_{ℓ} are the structure constants of a Lie algebra and the bracket, (2a), is a Lie-Poisson bracket with cosymplectic matrix J.

If the perturbation vorticity field has the form of "clumps," so that q' is nonzero only on a set of compact, disjoint regions, $\{D_i:i=0,\ldots,N-1\}$, then we may obtain a further simplification. Choosing the S_i so that each S_i completely contains the corresponding D_i but does not intersect any of the remaining D's, the moments in (3) are zero unless $k=l$.

With these simplifications, we can think of our model as approximating the vorticity distribution by a collection of elliptical patches of uniform vorticity, one patch being assigned to each of the disjoint clumps in the original q' distribution. More formally, there is a simple correspondence between the instantaneous state of the moments and the configuration of a collection of uniform elliptical patches. First order moments determine the positions of the centroids, and second order moments define the aspect ratio and orientation of the equivalent ellipses. Each vortex embodies an infinite number of degrees of freedom corresponding to the shape of each region D_i and the distribution of the vorticity within it. The moment reduction, as we shall see, restricts this number to only two degrees of freedom per vortex, one associated with the vortex centroid and one with its ellipticity and orientation.

For the specific problem considered in this paper, we adopt the perspective of N elliptical patches of uniform vorticity, $q_i'(i=0,\ldots,N-1)$, each with area, A_i, and circulation, Γ_i. The steady uniform background vorticity is associated with a flow that combines both background rotation and strain:

$$\Psi=\frac{1}{4}\omega(x^2+y^2)+\frac{1}{4}e(x^2-y^2),\tag{5}$$

ω and e being constants. This is the same background flow used in the Kida problem.

We now label the moments with a single suffix that combines information about both the polynomial used to generate the moment and the vortex with which it is associated. (To avoid unnecessary confusion, we lower the indices on the a's.) After the reduction sketched above, we find that there are six moments associated with each vortex. One is the circulation of the vortex,

$$\Gamma_i:=\tilde{a}_{1+6i}:=\int_{D_i}q_i'dxdy;\tag{6a}$$

two are the first moments of the vorticity,

$$\tilde{a}_{2+6i}=\int_{D_i}q_i'xdxdy,\quad \tilde{a}_{3+6i}=\int_{D_i}q_i'ydxdy;\tag{6b}$$

and the remaining three are second order moments,

$$\begin{aligned}&\tilde{a}_{4+6i}=\int_{D_i}q_i'x^2dxdy,\quad \tilde{a}_{5+6i}=\int_{D_i}q_i'xydxdy,\\ &\tilde{a}_{6+6i}=\int_{D_i}q_i'y^2dxdy.\end{aligned}\tag{6c}$$

The i's identify the vortices.

It is convenient to define functions $\tilde{m}_j$ associated with the integrands of the $\tilde{a}_j$:

$$\begin{aligned}&\tilde{m}_{4+6i}=x^2,\quad \tilde{m}_{5+6i}=xy,\quad \tilde{m}_{6+6i}=y^2,\\ &\tilde{m}_{2+6i}=x,\quad \tilde{m}_{3+6i}=y,\quad \tilde{m}_{1+6i}=1.\end{aligned}\tag{7a}$$

The structure constants c^{jk}_{ℓ} can then be evaluated from the relations

$$\begin{aligned}&[\tilde{m}_{4+6i},\tilde{m}_{6+6i}]=4\tilde{m}_{5+6i},\quad [\tilde{m}_{4+6i},\tilde{m}_{5+6i}]=2\tilde{m}_{4+6i},\\ &[\tilde{m}_{5+6i},\tilde{m}_{6+6i}]=2\tilde{m}_{6+6i},\quad [\tilde{m}_{2+6i},\tilde{m}_{3+6i}]=1,\\ &[\tilde{m}_{2+6i},\tilde{m}_{4+6i}]=0,\quad [\tilde{m}_{3+6i},\tilde{m}_{6+6i}]=0,\end{aligned}\tag{7b}$$

and

$$\begin{aligned}&[\tilde{m}_{2+6i},\tilde{m}_{5+6i}]=\tilde{m}_{2+6i},\quad [\tilde{m}_{2+6i},\tilde{m}_{6+6i}]=2\tilde{m}_{3+6i},\\ &[\tilde{m}_{3+6i},\tilde{m}_{4+6i}]=-2\tilde{m}_{2+6i},\quad [\tilde{m}_{3+6i},\tilde{m}_{5+6i}]=-\tilde{m}_{3+6i}.\end{aligned}\tag{7c}$$

(Any Jacobian with $\tilde{m}_{1+6i}$ as one of its arguments is clearly zero.)

B. Cosymplectic matrix

The cosymplectic matrix $\tilde{J}^{jk}$ is defined by

$$\{F,G\}=\frac{\partial F}{\partial \tilde{a}_j}\tilde{J}^{jk}\frac{\partial G}{\partial \tilde{a}_k}. \tag{8a}$$

From (2b), we see that

$$\tilde{J}^{jk}=\int q_j'[\tilde{m}_j,\tilde{m}_k]\chi_j\chi_k dx dy. \tag{8b}$$

(Note that $\delta\tilde{a}_j/\delta q_j'=\tilde{m}_j\chi_j$.)

Because of the factor of $\chi_j\chi_k$ in the preceding expression, moments of different vortices do not couple together. It follows from the products (7b)–(7c) and the definitions of $\tilde{a}_j$ that $\tilde{J}$ is a direct sum over $\tilde{J}_i$, the single-vortex cosymplectic matrices:

$$\tilde{J}=\bigoplus_{i=0}^{N-1}\tilde{J}_i. \tag{9}$$

For $N=2$ vortices, $\tilde{J}$ takes the block-diagonal form

$$\tilde{J}=\begin{pmatrix}\tilde{J}_0 & 0\\ 0 & \tilde{J}_1\end{pmatrix}.$$

Here $\tilde{J}_i$, a 6×6 matrix, has the following structure:

$$\tilde{J}_i=\begin{pmatrix}0 & 0 & 0\\ 0 & \tilde{\mathscr{A}}_i & \tilde{\mathscr{C}}_i\\ 0 & -\tilde{\mathscr{C}}_i^T & \tilde{\mathscr{B}}_i\end{pmatrix}, \tag{10}$$

with

$$\tilde{\mathscr{A}}_i=\begin{pmatrix}0 & \tilde{a}_{1+6i}\\ -\tilde{a}_{1+6i} & 0\end{pmatrix}, \tag{11a}$$

$$\tilde{\mathscr{B}}_i=\begin{pmatrix}0 & 2\tilde{a}_{4+6i} & 4\tilde{a}_{5+6i}\\ -2\tilde{a}_{4+6i} & 0 & 2\tilde{a}_{6+6i}\\ -4\tilde{a}_{5+6i} & -2\tilde{a}_{6+6i} & 0\end{pmatrix}, \tag{11b}$$

and

$$\tilde{\mathscr{C}}_i=\begin{pmatrix}0 & \tilde{a}_{2+6i} & 2\tilde{a}_{3+6i}\\ -2\tilde{a}_{2+6i} & -\tilde{a}_{3+6i} & 0\end{pmatrix}. \tag{11c}$$

Before turning to the Hamiltonian, we first note that the system has some symmetries that are independent of the form of the Hamiltonian. These symmetries are manifested in Casimir invariants C, which are solutions of

$$0=\tilde{J}^{jk}\frac{\partial C}{\partial \tilde{a}^k}. \tag{12}$$

The Casimirs arise when the cosymplectic matrix is singular and they correspond to constants of the motion. There are infinitely many Casimirs for the 2-D Euler equations, the materially conserved functionals of vorticity, but only $2N$ Casimirs for a system of N elliptical vortices in shear. Given the zeroes in the first column and row of $\tilde{J}$, the net circulation of each vortex, $\tilde{a}_{1+6i}$, is clearly a Casimir.

Since one of the coordinates in (10) is a Casimir, we can treat it as a constant parameter and reduce the dimension of the submatrices $\tilde{J}_i$ by one. This leaves five remaining variables per vortex, $\tilde{a}_{2+6i}$, $\tilde{a}_{3+6i}$, $\tilde{a}_{4+6i}$, $\tilde{a}_{5+6i}$, and $\tilde{a}_{6+6i}$. The submatrices can themselves be rendered block diagonal by using a transformation that replaces the second order moments with second order moments *about the vortex centroid*. We set

$$a_{1+6i}=\tilde{a}_{1+6i},\quad a_{2+6i}=\tilde{a}_{2+6i},\quad a_{3+6i}=\tilde{a}_{3+6i},$$
$$x_i^*=a_{2+6i}/a_{1+6i},\quad y_i^*=a_{3+6i}/a_{1+6i}, \tag{13a}$$

$$a_{4+6i}=\tilde{a}_{4+6i}-x_i^{*2}a_{1+6i},\quad a_{5+6i}=\tilde{a}_{5+6i}-x_i^*y_i^*a_{1+6i},$$
$$a_{6+6i}=\tilde{a}_{6+6i}-y_i^{*2}a_{1+6i}. \tag{13b}$$

The variables (x_i^*,y_i^*) are just the coordinates of the centroid of the ith vortex. In the new coordinates $\{a_j\}$, we denote the cosymplectic matrix by J^{jk}. After defining new functions m_j, the mixed products corresponding to (7c) vanish when integrated over D_i because

$$\int_{D_i} q_i'(x-x_i^*)dx dy=\int_{D_i} q_i'(y-y_i^*)dx dy=0.$$

The elements of the submatrix $\mathscr{C}_i$ are thus identically zero and the new cosymplectic matrix takes the form

$$J=\bigoplus_{i=1}^{N} J_i;\quad J_i=\begin{pmatrix}\mathscr{A}_i & 0\\ 0 & \mathscr{B}_i\end{pmatrix}. \tag{14}$$

The block diagonal submatrices $\mathscr{A}_i$, $\mathscr{B}_i$ are given by (11a) and (11b) after replacing the $\tilde{a}$'s by a's. Here $\mathscr{A}_i$ is, to within a normalization factor, the canonical cosymplectic matrix for point vortex motion; $\mathscr{B}_i$ is the cosymplectic matrix for a Kida vortex (cf. FMM). The block diagonal form of J shows that, in this coordinate system, the vortices are not coupled through the cosymplectic matrix. Coupling between the vortices arises through the Hamiltonian.

The existence of a second Casimir for each vortex now becomes apparent since $J^{jk}(\partial C/\partial a^k) = 0$ also has the solution

$$C^i=a_{4+6i}a_{6+6i}-a_{5+6i}^2. \tag{15}$$

For the particular case of uniform elliptical vortices, this is again related to the circulation of an individual vortex: $C^i=\Gamma_i^2A_i^2/16\pi^2$. This is not true in the general case, however (cf. FMM).

C. Hamiltonian

We now seek an approximation to the Hamiltonian written wholly in terms of the a_j. For the 2-D Euler equations, the excess energy is an invariant quantity.[28,29] For point vortex motion, the excess energy is the Hamiltonian. With the ansatz that the excess energy is the Hamiltonian for our system, we obtain

$$H=-\frac{1}{2}\sum_{i=0}^{N-1}\left\{\int_{D_i}2\Psi q_i' dxdy+\int_{D_i}\psi_i' q_i' dxdy+\sum_{j=0}^{N-1}{}'\int_{D_i}\psi_i' q_i' dxdy\right\}, \tag{16}$$

where

$$\psi_i'(\mathbf{x})=\frac{1}{2\pi}\int_{D_i}q_i'\ln|\mathbf{x}-\mathbf{x}'|dx'dy' \tag{17}$$

is the streamfunction induced by vortex i. The first term in (16) corresponds to interactions of the background flow with the vortices, the second to interactions of the vortices with themselves, and the third to interactions of the vortices with one another. (The notation $\Sigma_{j=0}^{N-1}{}'$ stands for $\Sigma_{j=0,j\neq i}^{N-1}$.)

Letting $H=H_1+H_2+H_3$, the first term in (16) may be written as

$$H_1=-\sum_{i=0}^{N-1}\left\{\frac{1}{4}(\omega+e)\left[a_{4+6i}+\frac{a_{2+6i}^2}{a_{1+6i}}\right]+\frac{1}{4}(\omega-e)\times\left[a_{6+6i}+\frac{a_{3+6i}^2}{a_{1+6i}}\right]\right\}. \tag{18}$$

For H_2 and H_3, two approximations are required. (a) The vortices are close to elliptical in shape with close to uniform vorticity. (The existence of the circulation Casimir then implies conservation of individual vortex area.) A constant area ellipse can be characterized by four parameters, for example, its aspect ratio, λ (the ratio of the semi-major and semi-minor axes), its orientation, ϕ (the angle between the fixed coordinate axes and the rotating body frame), and the x and y centroids. These four parameters are uniquely determined by the first and second order spatial moments of the ellipse.

The second order moments are related to the aspect ratio and orientation by

$$a_{4+6i}=(\lambda_i^{-1}\cos^2\phi_i+\lambda_i\sin^2\phi_i)\frac{\Gamma_i A_i}{4\pi},$$

$$a_{5+6i}=(\lambda_i^{-1}-\lambda_i)\sin\phi_i\cos\phi_i\frac{\Gamma_i A_i}{4\pi}, \tag{19}$$

$$a_{6+6i}=(\lambda_i^{-1}\sin^2\phi_i+\lambda_i\cos^2\phi_i)\frac{\Gamma_i A_i}{4\pi}.$$

(b) The vortices remain well separated in the sense that the vortex separations R_{ij} and the length scales of the vortices, characterized by the length of their semi-major axes, b_i, satisfy $b_i\ll R_{ij}$.

To evaluate H_2, only the first approximation is needed. Using (17),

$$H_2=-\frac{1}{8\pi}\sum_{i=0}^{N-1}\Gamma_i^2\ln\left[(a_{4+6i}+a_{6+6i})\frac{4\pi}{\Gamma_i A_i}+2\right]. \tag{20}$$

Invoking the second approximation to expand the Green's function to second order in b_i/R_{ij}, the final term is

$$\begin{aligned}H_3=-\frac{1}{8\pi}\sum_{i=0}^{N-1}\sum_{j=0}^{N-1}{}'\Bigg\{&\Gamma_i\Gamma_j\ln R_{ij}^2+\frac{2}{R_{ij}^2}\Bigg[\frac{1}{2}\Gamma_i(a_{4+6j}\\&+a_{6+6j})+\frac{1}{2}\Gamma_j(a_{4+6i}+a_{6+6i})\\&-\cos^2\theta_{ij}(\Gamma_i a_{4+6j}+\Gamma_j a_{4+6i})\\&-\sin^2\theta_{ij}(\Gamma_i a_{6+6j}+\Gamma_j a_{6+6i})\\&-\sin 2\theta_{ij}(\Gamma_i a_{5+6j}+\Gamma_j a_{5+6i})\Bigg]\Bigg\},\end{aligned} \tag{21a}$$

where

$$(R_{ij}\cos\theta_{ij},R_{ij}\sin\theta_{ij})=(x_i^*-x_j^*,y_i^*-y_j^*). \tag{21b}$$

Combining terms,

$$\begin{aligned}H=-\frac{1}{4}\sum_{i=0}^{N-1}\Bigg\{&\Bigg[(\omega+e)a_{4+6i}+(\omega-e)a_{6+6i}+(\omega+e)\frac{a_{2+6i}^2}{\Gamma_i}+(\omega-e)\frac{a_{3+6i}^2}{\Gamma_i}\Bigg]\\&+\frac{\Gamma_i^2}{2\pi}\ln\left\{\left[(a_{4+6i}+a_{6+6i})\frac{4\pi}{\Gamma_i A_i}+2\right]\right\}+\frac{1}{2\pi}\sum_{j=0}^{N-1}{}'\left[\Gamma_i\Gamma_j\ln R_{ij}^2+\frac{2}{R_{ij}^2}(\alpha_{ij}\cos 2\theta_{ij}-2\beta_{ij}\sin 2\theta_{ij})\right]\Bigg\},\end{aligned} \tag{22}$$

where

$$\alpha_{ij}=\Gamma_i(a_{6+6j}-a_{4+6j}),\quad \beta_{ij}=\Gamma_i a_{5+6j}.$$

Like the cosymplectic matrix, the Hamiltonian possesses several symmetries: the first and second order moments are uncoupled, and the Hamiltonian is invariant under a change of vortex labels.

D. Equations of motion for a_i

We now compute the equations of motion from (A5) and (14). The equations of motion for the quadratic moments are

$$\dot{a}_{4+6i}=\sum_{j=0}^{N-1}{}' \frac{a_{4+6i}\Gamma_j \sin 2\theta_{ij}}{\pi R_{ij}^2}+a_{5+6i}\left[-(\omega-e) - \frac{2q_i'}{[(a_{4+6i}+a_{6+6i})(4\pi/\Gamma_i A_i)+2]} - \sum_{j=0}^{N-1}{}' \frac{\Gamma_j \cos 2\theta_{ij}}{\pi R_{ij}^2}\right],$$

$$\dot{a}_{5+6i}=\frac{1}{2}\omega(a_{4+6i}-a_{6+6i})+\frac{1}{2}e(a_{4+6i}+a_{6+6i}) + q_i' \frac{(a_{4+6i}-a_{6+6i})}{[(a_{4+6i}+a_{6+6i})(4\pi/\Gamma_i A_i)+2]} - \sum_{j=0}^{N-1}{}' \frac{\Gamma_j \cos 2\theta_{ij}}{2\pi R_{ij}^2}(a_{4+6i}+a_{6+6i}),$$

$$\dot{a}_{6+6i}=\sum_{j=0}^{N-1}{}' -\frac{a_{6+6i}\Gamma_j \sin 2\theta_{ij}}{\pi R_{ij}^2}+a_{5+6i} \times\left[(\omega+e)+\frac{2q_i'}{[(a_{4+6i}+a_{6+6i})(4\pi/\Gamma_i A_i)+2]} - \sum_{j=0}^{N-1}{}' \frac{\Gamma_j \cos 2\theta_{ij}}{\pi R_{ij}^2}\right]. \tag{23a}$$

After some simplification, the first order equations are

$$\dot{x}_i^*=-\frac{1}{2}(\omega-e)y_i^*-\sum_{j=0}^{N-1}{}' \frac{\Gamma_j}{2\pi R_{ij}}\sin\theta_{ij} + \frac{1}{\Gamma_i}\sum_{j=0}^{N-1}{}' \frac{1}{2\pi R_{ij}^3}\{(\alpha_{ij}+\alpha_{ji})\sin 3\theta_{ij} + 2(\beta_{ij}+\beta_{ji})\cos 3\theta_{ij}\},$$

$$\dot{y}_i^*=\frac{1}{2}(\omega+e)x_i^*+\sum_{j=0}^{N-1}{}' \frac{\Gamma_j}{2\pi R_{ij}}\cos\theta_{ij} + \frac{1}{\Gamma_i}\sum_{j=0}^{N-1}{}' \frac{1}{2\pi R_{ij}^3}\{-(\alpha_{ij}+\alpha_{ji})\cos 3\theta_{ij} + 2(\beta_{ij}+\beta_{ji})\sin 3\theta_{ij}\}. \tag{23b}$$

The equations (23a) and (23b) constitute a set of $5N$ coupled ODEs. They are a closed set even though they do not contain explicit evolution equations for R_{ij} and θ_{ij} because R_{ij} and θ_{ij} may be determined from (21b).

The equations can be simplified in the following way. Since the N quantities $C^i=a_{4+6i}a_{6+6i}-a_{5+6i}^2$ are Casimirs, a_{4+6i}, a_{5+6i}, and a_{6+6i} are not all independent of one another. This can be made explicit by employing a transformation of variables wherein the Casimirs act as dependent variables, thereby leaving a set of only $4N$ independent equations of motion (N equations reduce to $dC^i/dt=0$). A further simplification may be had by noting that the equations of motion do not depend on the global centroid position; one is left with a system of $4N-2$ equations after appropriate linear combinations are taken.

Note added in proof. Analogous equations have recently been obtained by Riccardi *et al.*[30] using the method of MZS.

E. Equations of motion in physical variables

By transforming to the more intuitive variables, $(x_i^*, y_i^*, \lambda_i, \phi_i)$, a set of equations analogous to those of MZS is obtained.

The equations for the evolution of (x_i^*, y_i^*) are (23b). Using (19), we find that λ_i and ϕ_i evolve according to

$$\dot{\lambda}_i=-\lambda_i\sum_{j=0}^{N-1}{}'\left[\frac{\Gamma_j}{\pi R_{ij}^2}\sin 2(\theta_{ij}-\phi_i)+e\sin 2\phi_i\right],$$
$$\dot{\phi}_i=\frac{q_i'\lambda_i}{(1+\lambda_i)^2}-\frac{1}{2}\frac{1+\lambda_i^2}{1-\lambda_i^2} \times\left\{\frac{\Gamma_j}{\pi R_{ij}^2}\cos 2(\theta_{ij}-\phi_i)-e\cos 2\phi_i\right\}+\frac{\omega}{2}. \tag{24}$$

In the $\dot{\phi}_i$ equation, there is an apparent singularity when $\lambda_i=1$. This is not of dynamical significance. As noted by MZS, this singularity arises from the fact that the orientation of a circular vortex is not well-defined. MZS point out that one way to "desingularize" these equations is to introduce new variables,

$$(\delta_i, \gamma_i)=\left(\frac{A_i}{8\pi\lambda_i}\right)^{1/2}(\lambda_i-1)(\cos 2\phi_i, \sin 2\phi_i).$$

MZS further note that $((\lambda_i-1)^2/\lambda_i, 2\phi_i)$ is one set of canonical variables for this problem. This set was later used by Ide and Wiggins[31] in a study of the motion of a single elliptical vortex in a time-dependent linear background flow; an alternative set is introduced in FMM. However, there is a singularity at $R_{ij}=0$ which cannot be removed by a coordinate transformation. Following MZS we take $R_{ij}\rightarrow 0$ as being indicative of vortex merger, but it should be noted that the model ceases to be consistent in this limit because the assumption of well-separated vortices breaks down.

For reference, the equations of motion for $N=2$ vortices may be found in Appendix C.

III. PRELIMINARY ANALYSIS

In this section, we begin the analysis of our model. Aspect ratio–orientation variables are used because they clearly display physical processes. We will restrict the analysis to a system of two identical vortices. While more complicated configurations can exhibit behavior that a symmetric vortex pair cannot, the vortex-vortex and vortex-shear interactions analyzed below are still present. The Hamiltonian for N vortices is not fundamentally different from that for two vortices: there are no multipole interactions at the order of our truncation.

A. Nondimensionalization

Our starting point is the system (C1) for $N=2$ vortices from Appendix C. Letting D denote a characteristic separation scale, we nondimensionalize as follows: $R^2=uD^2$, $e=\tilde{e}q_0$, $\omega=\alpha\tilde{e}q_0$, $\delta=A_1/A_0$, and time is scaled by q_0^{-1}. We define a nondimensional perturbation parameter $\epsilon=A_0/(\pi D^2)$, which is assumed to be small (i.e., the vortices are assumed to be well separated). Specializing to the symmetric case of identical vortices, with $\lambda_0=\lambda_1=\lambda$, $\phi_0=\phi_1=\phi$, $A_1=A_0$, and $q_1=q_0$:

$$\begin{aligned}
\dot{u}&=\tilde{e}u\sin 2\theta-\epsilon^2u^{-1}\frac{1-\lambda^2}{\lambda}\sin 2(\theta-\phi),\\
\dot{\theta}&=\frac{\tilde{e}}{2}(\alpha+\cos 2\theta)+\epsilon u^{-1}+\epsilon^2\frac{1}{2}u^{-2}\frac{1-\lambda^2}{\lambda}\cos 2(\theta-\phi),\\
\dot{\lambda}&=-\lambda\{\tilde{e}\sin 2\phi+\epsilon u^{-1}\sin 2(\theta-\phi)\},\\
\dot{\phi}&=\frac{\lambda}{(1+\lambda)^2}+\frac{1}{2}\frac{1+\lambda^2}{1-\lambda^2}\{\tilde{e}\cos 2\phi-\epsilon u^{-1}\cos 2(\theta-\phi)\}\\
&\quad+\alpha\frac{\tilde{e}}{2}.
\end{aligned}\tag{25}$$

The terms in (25) have simple physical interpretations. The terms at $O(1)$ represent (a) the self-rotation of the vortices (the first term in the ϕ equation), and (b) the effects of the background flow on the vortices (the terms involving $\tilde{e}$ in each equation). At $O(\epsilon)$, interactions between the vortices modify the evolution of the aspect ratio and orientation but have little effect on the separation. As with point vortices, these interactions produce a constant change in the rotation rate of the separation vector, but no change in its length. At $O(\epsilon^2)$, the shape and separation of the vortices are tightly coupled.

The first two equations in (25), the pair that govern the separation of the vortices, have terms up to $O(\epsilon^2)$, while the remaining four have terms to $O(\epsilon)$. The truncation implicit in (25) arises from the truncation of the Hamiltonian: an infinite moment hierarchy is closed at second order by approximating the vortices as ellipses.

B. Integrable basic states: Perturbed point vortex pair in shear

The system (25) is a fourth order (two degree-of-freedom), nonlinear, Hamiltonian system. In the absence of shear, the 2-D Euler equations conserve angular impulse and the x and y centroids; this leads to two independent integrals of motion when the total circulation is nonvanishing.[32] These integrals of motion are destroyed by background shear and the Hamiltonian is the only one which remains. From our derivation, we know that this is a Hamiltonian system represented in noncanonical coordinates (see FMM for a discussion of canonical coordinates); we anticipate that there is a possibility of chaotic motion because the number of degrees of freedom exceeds the number of integrals of motion.[33]

This system can be regarded as a perturbation to any of three integrable basic states: (i) the MZS model; (ii) a pair of isolated Kida vortices; and (iii) a point vortex pair in shear. The equations of motion for the MZS model are obtained by setting $\tilde{e}=0$ in (25); those for a Kida vortex are obtained from the λ and ϕ equations when $\epsilon=0$; and the equations describing a point vortex pair in shear correspond to the u and θ equations when terms in ϵ^2 are neglected. The perturbations to these basic states then represent the addition of (i) vortex-shear interactions; (ii) vortex-vortex interactions; and (iii) internal degrees of freedom (aspect ratio and orientation).

We choose to regard (25) as a perturbed point vortex pair in shear. There are two reasons for this choice. The phase space geometry of a point vortex pair in shear provides a convenient framework for studying vortex merger. For sufficiently strong strain, and in the absence of a perturbation, a separatrix divides closed orbits from unbounded ones (see Appendix B for details). When they are perturbed, we will find that the vortices can merge and that some of the closed orbits will disappear; furthermore, the separatrix splits apart into distinct stable and unstable manifolds. (The connection between these phenomena is discussed in Sec. V.) The second reason is a practical one: there exists an analytical tool, namely the Melnikov function, which may be applied to systems with heteroclinic orbits.

In preparation for the Melnikov analysis, we apply one more scaling. We are particularly interested in what happens in the vicinity of the separatrix of the point vortex pair. It will prove convenient if the distance from the separatrix to the origin, determined by balancing the first two terms in the θ equation of (25), is scaled to be $O(1)$. This is equivalent to choosing $\tilde{e}=\epsilon\gamma$ where $\gamma=O(1)$. The equations (25) then become

$$\begin{aligned}
\dot{u}&=\epsilon\gamma u\sin 2\theta-\epsilon^2u^{-1}\frac{1-\lambda^2}{\lambda}\sin 2(\theta-\phi)+O(\epsilon^4),\\
\dot{\theta}&=\epsilon\left\{\frac{\gamma}{2}(\alpha+\cos 2\theta)+u^{-1}\right\}+\epsilon^2\frac{1}{2}u^{-2}\frac{1-\lambda^2}{\lambda}\\
&\quad\times\cos 2(\theta-\phi)+O(\epsilon^4),\\
\dot{\lambda}&=-\epsilon\lambda\{\gamma\sin 2\phi+u^{-1}\sin 2(\theta-\phi)\}+O(\epsilon^3),\\
\dot{\phi}&=\frac{\lambda}{(1+\lambda)^2}+\epsilon\frac{1}{2}\frac{1+\lambda^2}{1-\lambda^2}\{\gamma\cos 2\phi-u^{-1}\\
&\quad\times\cos 2(\theta-\phi)\}+\epsilon\alpha\frac{\gamma}{2}+O(\epsilon^3).
\end{aligned}\tag{26}$$

These equations express the following sequence of interactions. On the $O(1)$ time scale, the elliptical vortices rotate at a rate $\lambda/(1+\lambda)^2$. On a longer time scale, $O(\epsilon^{-1})$, the aspect ratio and rotation rate of the vortices vary slowly, and the vortices move in the background shear flow under their mutual interaction. On an even longer time scale, $O(\epsilon^{-2})$, the finite sizes of the vortices induce small perturbations to the rate at which the intercentroidal separation vector changes.

IV. ASYMPTOTIC ANALYSIS

In this section, we investigate the system (26) by means of an asymptotic analysis in the small parameter ϵ. As noted above, we use the Melnikov method[34–36] and appeal to analogies between this system and a rapidly forced pendulum. We begin with a brief discussion of the Melnikov method.

The Melnikov integral measures the distance between the stable and unstable manifolds formed by the splitting of a homoclinic or heteroclinic orbit. Its most commonly encountered variant is for a periodically perturbed, one degree-of-freedom (canonical) Hamiltonian system:

$$\dot{q}=\frac{\partial H}{\partial p}(q,p), \quad \dot{p}=-\frac{\partial H}{\partial q}(q,p), \tag{27}$$

where $H=H_0(q,p)+\epsilon H_1(q,p,t)$, and H_1 is an explicit periodic function of the time variable, t. Let the period of H_1 be τ and let this be $O(1)$. Setting $z(t)=(q(t),p(t))$, we consider the case in which the unperturbed system has two hyperbolic fixed points that are joined by a heteroclinic orbit $z(t)=z_0(t)$. (The homoclinic case is analogous.) If one defines a function M of $t_0 \in (-\infty,\infty)$ by

$$M(t_0)=\epsilon\int_{-\infty}^{\infty}\{H_0(z_0(t)),H_1(z_0(t),t+t_0)\}dt, \tag{28}$$

where $\{f,g\}:=f_q g_p - f_p g_q$, then the signed separation between the stable and unstable manifolds along the normal to the unperturbed separatrix is given by

$$d(t_0)=\frac{M(t_0)}{|\nabla H_0(z_0(t_0))|}+O(\epsilon^2). \tag{29}$$

The existence of one zero of $M(t_0)$ implies the existence of infinitely many zeroes, since $M(t_0)$ is periodic with period τ. If $\partial M(t_0)/\partial t_0 \neq 0$ at the zeroes, then there are infinitely many transverse manifold crossings and, by the Smale-Birkhoff homoclinic theorem, there exists chaotic motion in the vicinity of the separatrix. The quantity M is referred to as the Melnikov function or Melnikov integral.

Although we shall be working in terms of the periodically perturbed one degree-of-freedom Melnikov function, the Melnikov approach is applicable to non-Hamiltonian systems. (See Refs. 34 and 36 for more discussion.) We expect the methods and results described in this section to generalize.

The system (26) is not of the form required for the standard Hamiltonian Melnikov analysis. To begin with, the variables are not canonical; but this is just a question of coordinates. Next, the basic state that we would like to perturb around, that of a pair of point vortices in shear, is a one degree-of-freedom system, while the full system (26) has two degrees of freedom. Most importantly, there are two distinct time scales: the natural time scale for the intercentroidal motion, which is $O(\epsilon^{-1})$, and the time scale for the rotation of the vortices, which is $O(1)$. The terms containing ϕ are thus $O(1)$. The slow intercentroidal motion is coupled with fast variations in the vortices' orientation and shape, complicating the analysis.

These difficulties can be resolved by multiple time-scale perturbation theory. An explicit time-periodic perturbation to the Hamiltonian is not given, but in some parameter ranges the second degree of freedom, (λ,ϕ), behaves like an oscillator and the variable ϕ increases monotonically with time. One can consider the first degree of freedom, (u,θ), to be "perturbed" by this second degree of freedom. By transforming the autonomous two degree-of-freedom system (26) into a nonautonomous one degree-of-freedom system, where the slow intercentroidal motion is perturbed by the fast oscillatory terms of the $\dot{u}$, $\dot{\theta}$, and $\dot{\lambda}$ equations, a multiple time-scale analysis can then be performed.

There is a growing body of research on one degree-of-freedom systems with rapidly oscillating perturbations. The implications of this research for our system are briefly examined in the next section.

A. The rapidly forced pendulum

Since there are two time scales in the system of (26), and the fast oscillatory terms occur at higher order in ϵ, we expect behavior similar to that of a nonlinear pendulum forced by a weak but rapid oscillation, viz.

$$\frac{d^2x}{dt^2}+\sin x=\delta \sin\left(\frac{t}{\epsilon}\right), \tag{30}$$

where $\delta\epsilon^{-n}\to 0$ for an appropriate positive power n.[24,37,25] This is a Hamiltonian system of the form of (27) with

$$q=x, \quad p=\dot{x}, \quad H_0=\frac{1}{2}p^2-\cos q, \quad H_1=-q\,\delta\sin\left(\frac{t}{\epsilon}\right).$$

When $\delta=0$, the unperturbed system is a one degree-of-freedom Hamiltonian system with a hyperbolic fixed point at $(x=0,\dot{x}=0)$ and an emanating homoclinic trajectory (identifying $x=2\pi$ with $x=0$). As is standard for nonautonomous systems, one can define a Poincaré section by strobing the system at the period of the forcing, $2\pi\epsilon$. For sufficiently weak forcing, the associated Poincaré map has a hyperbolic fixed point that lies close to the unperturbed one. In certain cases, it can be proven that the stable and unstable manifolds persist, lie close to the unperturbed homoclinic orbit, and intersect transversally.[38]

In the periodically forced Hamiltonian system described at the beginning of Sec. IV, the splitting distance between the stable and unstable manifolds is $O(\epsilon)$ and is given in terms of the Melnikov integral by (29). For the rapidly forced system (30), the Melnikov integral is exponentially small; but the meaning of this is uncertain because formally the Melnikov theory is only accurate to $O(\epsilon^2)$. It is, however, now thought that under properly specified conditions the Melnikov analysis generally does provide a good estimate of the splitting distance, even in rapidly forced problems. Delshams and Seara[25] were able to establish that the leading order term in the splitting distance for the forced pendulum equation is indeed given by the Melnikov integral when $n\geqslant 0$. They also show[39] that the same is true for the class of second order equations, $\ddot{x}+f(x)=\mu\epsilon^p g(t/\epsilon)$, where g is 2π-periodic with zero mean, when the unforced equation has a homoclinic orbit which satisfies certain analyticity requirements. Fontich[40] proved, under less restrictive conditions, that the Melnikov function in these systems is indeed exponentially small and obtained upper bounds on the splitting distance.

These proofs exploit the explicit closed-form representation of the unperturbed system's homoclinic trajectory. (This allows the contours of integration of certain integrals to be moved in the complex plane.) In the problem at hand, we cannot provide such proofs, largely because we have an implicit representation only. Furthermore, our system is slightly different from those considered previously: it belongs to the general class of rapidly forced second order equations, $\ddot{x}+f(x,\dot{x})=\mu\epsilon^p g(t/\epsilon)$. Nevertheless, we conjecture that its structural similarity to the rapidly forced pendulum suggests that its manifolds should cross transversally and that the separatrix splitting should be exponentially small. We provide support for this by performing a Melnikov analysis and examining the leading order contribution to the splitting distance. This does not constitute a proof because in the asymptotic analysis an infinite series of integrals contributes to the splitting distance and they may not all be exponentially small. We will later turn to numerical simulations to see if they are in accord with this picture.

B. Calculating a Melnikov function

We now present a Melnikov analysis that freely exploits a number of assumptions. First, we exploit the autonomous nature of the system and restrict attention to situations in which the vortex orientation, ϕ, increases monotonically with time. This is reasonable when λ can be bounded away from unity, as suggested by the form of (26) for small ϵ. Thus we replace the independent variable t with ϕ:

$$\frac{du}{d\phi}=Q\left[\epsilon\gamma u\ \sin 2\theta-\epsilon^2u^{-1}\frac{1-\lambda^2}{\lambda}\sin 2(\theta-\phi)\right],$$

$$\frac{d\theta}{d\phi}=Q\left[\epsilon\left\{\frac{\gamma}{2}(\alpha+\cos 2\theta)+u^{-1}\right\}\right.$$
$$\left.+\epsilon^2\frac{1}{2}u^{-2}\frac{1-\lambda^2}{\lambda}\cos 2(\theta-\phi)\right], \tag{31a}$$

$$\frac{d\lambda}{d\phi}=-Q[\epsilon\lambda\{\gamma\sin 2\phi+u^{-1}\ \sin 2(\theta-\phi)\}],$$

where

$$Q=\left\{\frac{\lambda}{(1+\lambda)^2}+\epsilon\frac{1}{2}\frac{1+\lambda^2}{1-\lambda^2}\{\gamma\cos 2\phi-u^{-1}\right.$$
$$\left.\times\cos 2(\theta-\phi)\}+\epsilon\alpha\frac{\gamma}{2}\right\}^{-1}. \tag{31b}$$

Next, the conservation of the Hamiltonian is invoked[41] to express $\lambda=\lambda(u,\theta,\phi;h)$ and eliminate λ from the first two equations (31a). The resulting system may be written in the form

$$\frac{du}{d\phi}=F_1(u,\theta,\lambda(u,\theta,\phi))+f_1(u,\theta,\lambda(u,\theta,\phi),\phi),$$
$$\frac{d\theta}{d\phi}=F_2(u,\theta,\lambda(u,\theta,\phi))+f_2(u,\theta,\lambda(u,\theta,\phi),\phi), \tag{32a}$$

where

$$F_1=\epsilon\Omega\gamma u\ \sin 2\theta,\quad F_2=\epsilon\Omega\left\{\frac{\gamma}{2}(\alpha+\cos 2\theta)+u^{-1}\right\},$$

$$\Omega=\Omega(\lambda)=\left\{\frac{\lambda}{(1+\lambda)^2}+\epsilon\alpha\frac{\gamma}{2}\right\}^{-1},$$

$$f_1=Q\left[\epsilon\gamma u\ \sin 2\theta-\epsilon^2u^{-1}\frac{1-\lambda^2}{\lambda}\sin 2(\theta-\phi)\right]$$
$$-\epsilon\Omega\gamma u\ \sin 2\theta, \tag{32b}$$

$$f_2=Q\left[\epsilon\left\{\frac{\gamma}{2}(\alpha+\cos 2\theta)+u^{-1}\right\}\right.$$
$$\left.+\epsilon^2\frac{1}{2}u^{-2}\frac{1-\lambda^2}{\lambda}\cos 2(\theta-\phi)\right]$$
$$-\epsilon\Omega\left\{\frac{\gamma}{2}(\alpha+\cos 2\theta)+u^{-1}\right\}.$$

Our goal is to obtain a one degree-of-freedom system with a basic state that describes the motion of a point vortex pair in shear. The functions F_1 and F_2 have the correct form, but they do not constitute a proper basic state, as they are a coupled to the perturbed motion through $\Omega(\lambda)$. Therefore, we expand λ as $\lambda=\lambda_0+\epsilon\lambda_1(\phi)+\ldots$, where λ_0 is a constant and can be treated as a parameter. (This is permissible for trajectories with λ bounded away from 0 and 1.) The function $\Omega(\lambda)$ can then be expanded in ϵ, and its leading order component,

$$\Omega_0=\left\{\frac{\lambda_0}{(1+\lambda_0)^2}+\epsilon\alpha\frac{\gamma}{2}\right\}^{-1}, \tag{33}$$

used to define an appropriate basic state. Hence we write

$$\frac{du}{d\phi}=G_1(u,\theta;\Omega_0)+g_1(u,\theta,\phi)=\mathscr{G}_1,$$
$$\frac{d\theta}{d\phi}=G_2(u,\theta;\Omega_0)+g_2(u,\theta,\phi)=\mathscr{G}_2, \tag{34a}$$

with

$$G_1=\tilde{G}_1+\tilde{g}_1=\epsilon\Omega_0\gamma u\sin 2\theta+\tilde{g}_1,$$
$$G_2=\tilde{G}_2+\tilde{g}_2=\epsilon\Omega_0\left\{\frac{\gamma}{2}(\alpha+\cos 2\theta)+u^{-1}\right\}+\tilde{g}_2. \tag{34b}$$

By construction, G_1 and G_2—and thus the tilde quantities—have no explicit ϕ dependence, while g_1 and g_2 have an explicit π-periodic dependence on ϕ. The functions $\tilde{g}_1$ and g_1 may be obtained from (34b) and

$$\mathscr{G}_1=Q\left[\epsilon\gamma u\ \sin 2\theta-\epsilon^2u^{-1}\frac{1-\lambda^2}{\lambda}\sin 2(\theta-\phi)\right];$$

g_2 and $\tilde{g}_2$ may be obtained from (34b) and

$$\mathscr{G}_2=Q\left[\epsilon\left\{\frac{\gamma}{2}(\alpha+\cos 2\theta)+u^{-1}\right\}\right.$$
$$\left.+\epsilon^2\frac{1}{2}u^{-2}\frac{1-\lambda^2}{\lambda}\cos 2(\theta-\phi)\right].$$

[Note that while F_1, F_2, G_1 and G_2 are $O(\epsilon)$, f_1, f_2, g_1 and g_2 are $O(\epsilon^2)$.] The basic state is given by $(du/d\phi=\tilde{G}_1, d\theta/d\phi=\tilde{G}_2)$ and corresponds to the equations of motion for a pair of point vortices in shear when $\Phi=\epsilon\phi$ is identified as a slow time. The quantity Ω_0 can be interpreted as a parameter set by the initial conditions. Without loss of generality, we assume that $\gamma>0$ ($\gamma<0$ just rotates the phase space through 90°), and that $\alpha<1$ (a sufficient condition for the existence of hyperbolic fixed points; see Appendix B).

We note that the procedure described above, whereby a one degree-of-freedom nonautonomous Hamiltonian system (often called a one- and one-half degree-of-freedom system) is obtained from a two degree-of-freedom autonomous system, is quite old.[42] Except for the complication due to the second time scale, this procedure is analogous to the one described in Holmes and Marsden.[43]

We can now proceed along the lines of the standard Melnikov analysis. By analogy with other one- and one-half degree-of-freedom systems, a Poincaré section is defined on the plane $\phi=\phi_p(\mathrm{mod}\,\pi)$, $\phi_p \in (-\infty,\infty)$ being an arbitrary constant. [Formally, we transform to an autonomous third-order system and plot intersections of its trajectories with $\phi=\phi_p(\mathrm{mod}\,\pi)$.] In the unperturbed limit, the associated Poincaré map has hyperbolic fixed points at $(u,\theta)=(2/[\gamma(1-\alpha)],\pm\pi/2)$ that are joined by a pair of invariant heteroclinic manifolds. The smoothness of the system means that for any closed, compact range of λ that does not include $\lambda=0$ or 1, the hyperbolic points persist for sufficiently small ϵ. It is therefore expected that a perturbation will split the heteroclinic manifolds into distinct stable and unstable manifolds.

Let $q_0(\phi)=(u_0(\epsilon\phi),\theta_0(\epsilon\phi))$ be a heteroclinic trajectory of the unperturbed system. Since the unperturbed system is autonomous, $q_0(\phi)$ passes through all the points on the unperturbed heteroclinic manifold as ϕ increases from $-\infty$ to ∞ (i.e., from fixed point to fixed point). In the usual way, a set of coordinates on the heteroclinic manifold is then defined by a choice of the point $q_0(0)$. Trajectories on the perturbed stable and unstable manifolds can be obtained by expanding around this heteroclinic trajectory:

$$q^j(\phi;\phi_p,\epsilon)=q_0(\phi-\phi_p)+q_1^j(\phi;\phi_p),$$

where $j=s$ or u. The deviation q_1^u is asymptotically small (in ϵ) compared to q_0 as $\phi\to-\infty$; q_1^s is asymptotically small compared to q_0 as $\phi\to+\infty$. Actually, $u_1\sim O(\epsilon^2)$ and $\theta_1\sim O(\epsilon^2)$. The autonomous nature of the unperturbed system is once more used to shift the time origin $\phi=0$: for each particular choice of ϕ_p, the heteroclinic trajectory is now given by $q_0(\phi-\phi_p)$.

We seek an estimate of the distance between these manifolds at a point $q_0(0)$ and in a direction normal to the unperturbed heteroclinic orbit. Letting $G=(G_1(q),G_2(q))$, (34a) can be rewritten as

$$\frac{dq}{d\phi}=G(q)+g(q,\phi), \tag{35}$$

and the normal to the unperturbed invariant manifold is

$$\hat{\mathbf{n}}=(-G_2(q_0),G_1(q_0))/(G_1(q_0)^2+G_2(q_0)^2)^{1/2} =\mathbf{n}/(G_1(q_0)^2+G_2(q_0)^2)^{1/2}.$$

The transverse splitting distance at $q_0(\phi=0)$ is then given by

$$d(\phi_p)=M(\phi_p)/[G_1^2(q_0(0))+G_2^2(q_0(0))]^{1/2}, \tag{36a}$$

where

$$M(\phi_p)=\Delta^s(\phi_p;\phi_p)-\Delta^u(\phi_p;\phi_p) \tag{36b}$$

and

$$\Delta^j(\phi;\phi_p)=\mathbf{n}\cdot q_1^j(\phi;\phi_p) =G_1(q_0(\phi-\phi_p))\theta_1^j(\phi;\phi_p)-G_2(q_0(\phi-\phi_p))u_1^j(\phi;\phi_p). \tag{36c}$$

In order to calculate $M(\phi_p)$, and thus the splitting distance, the system (34a) must be expanded. Since $G(q_0)=F(q_0,\lambda_0)$ and similarly for g and f, the quantities Δ^j and q_1^j may be expressed in terms of f and F, simplifying the calculations. Expanding $d\Delta_j/d\phi$ and $dq_1^j/d\phi$, and collecting terms,

$$\begin{aligned}\frac{d\Delta^j}{d\phi}&=F_1(q_0(\phi-\phi_p),\lambda_0)f_2(q_0(\phi-\phi_p),\lambda_0,\phi)\\&\quad-F_2(q_0(\phi-\phi_p),\lambda_0)f_1(q_0(\phi-\phi_p),\lambda_0,\phi)\\&\quad+\lambda_1\left(F_1\frac{\partial f_2}{\partial\lambda}-F_2\frac{\partial f_1}{\partial\lambda}\right)+u_1\left(F_1\frac{\partial f_2}{\partial u}-F_2\frac{\partial f_1}{\partial u}\right)\\&\quad+\theta_1\left(F_1\frac{\partial f_2}{\partial\theta}-F_2\frac{\partial f_1}{\partial\theta}\right)+\cdots,\end{aligned}$$

which is formally $O(\epsilon^3)$ at leading order. For brevity this is written as

$$\frac{d\Delta^j}{d\phi}=\epsilon^3T_1+\epsilon^4T_2+\epsilon^5T_3+\cdots\ . \tag{37}$$

While the forms of the f's are rather complicated, they can be simplified by expanding in ϵ, e.g.,

$$f_2(q_0,\lambda_0,\phi)=f_{20}(q_0,\lambda_0,\phi)+\epsilon f_{21}(q_0,\lambda_0,\phi)+\cdots\ .$$

This induces corresponding expansions in the T's, $T_j=T_{j0}+\epsilon T_{j1}+\ldots$, and (37) thus becomes

$$\frac{d\Delta^j}{d\phi}=\epsilon^3T_{10}+\epsilon^4(T_{20}+T_{11})+\cdots\ .$$

Integrating the preceding equation over ϕ yields $M(\phi_p)$:

$$M(\phi_p)=\epsilon^3M_{10}(\phi_p)+\epsilon^4(M_{20}(\phi_p)+M_{11}(\phi_p))+\cdots\ . \tag{38}$$

The leading order contribution is fairly easy to calculate. We find that

$$\begin{aligned}M_{10}=-\frac{1}{2}\,\frac{1-\lambda_0^2}{\lambda_0}\Omega_0^2\sin 2\phi_p\int_{-\infty}^{\infty}u_0^{-1}\{\gamma\cos[4\theta_0(\epsilon\phi)\\-2\phi]+(\gamma\alpha+2u_0^{-1}(\epsilon\phi))\cos[2\theta_0(\epsilon\phi)-2\phi]\}d\phi.\end{aligned} \tag{39}$$

We have chosen the origin of the ϕ coordinate so that $\theta_0(0)=0$, i.e., the midpoint of the heteroclinic trajectory. From the leading $\sin 2\phi_p$ factor, we see that M_{10} crosses zero infinitely many times, suggesting the existence of chaotic motion.

The width of the chaotic region is related to the magnitude of the integral. By a generalized Riemann-Lebesgue lemma,[44] the integral is $\ll O(\epsilon^n)$ for any n; i.e., it is exponentially small in ϵ. A necessary (but not sufficient) condition for this to be the dominant part of $M(\phi_p)$ is that the higher-order corrections, M_{11}, M_{20}, etc. must also be, at most, exponentially small. This is clearly possible for those T_{mn} that are rapidly oscillating functions of ϕ. However, the contribution to M of any parts of T_{mn} that have no fast time dependence must be considered more carefully. Below we find the leading order term considering only slow ϕ dependence and show that it makes zero contribution to M.

We need to compute $T_{20}+T_{11}$. This requires a knowledge of the first asymptotic correction to the path of the perturbed stable and unstable manifolds. Upon substitution for λ_1, we find that the sum $T_{20}+T_{11}$ consists of two parts: a rapidly oscillating component, T_{2f}, which like T_{10} yields an exponentially small contribution upon integration, and another component, T_{2s}, which does not oscillate rapidly. In principle, this should result in a contribution to the Melnikov integral that is not exponentially small and so would dominate M_{10}. However, if we calculate T_{2s} explicitly, we find that

$$T_{2s}=-\left\{\frac{\Omega_0^4(1-\lambda_0)^2}{(1+\lambda_0)^2}+2\Omega_0^3\frac{1+\lambda_0^2}{\lambda_0}\right\} \times\frac{1}{8}\gamma u_0^{-1}\{\gamma\sin(4\theta_0)+(\gamma\alpha+u_0^{-1})\sin(2\theta_0)\}. \tag{40}$$

This is symmetric about $\phi=\phi_p$, so the integral $\int_{-\infty}^{\infty}d\phi$ is identically zero, and the slow component of $M_{20}+M_{11}$ thus vanishes. As for slow perturbations to the positions of the stable and unstable manifolds, there is a part of u_1 and θ_1 that does not oscillate rapidly, but it affects both manifolds in the same way and so does not contribute to the splitting distance at this order. We note that the correction $\epsilon^4(M_{20}+M_{11})$ formally occurs at the same order as would the corrections associated with deviations of the shape of the vortices from ellipticity [i.e., $O(\epsilon^3)$].

V. NUMERICAL RESULTS

The Melnikov analysis suggests that a chaotic band should form around the unperturbed separatrix, though the (exponentially small) scaling is tentative. Here we use numerical simulations to both support this and provide a picture of the global dynamics. For simplicity, only the ϵ dependence will be considered at length. The equations of motion (25) are used for the integrations; however, it is convenient to scale time by q_0. The singularity at $\lambda=1$ has not presented any difficulty.

A. Poincaré sections

We construct Poincaré sections for the two degree-of-freedom Hamiltonian system (26) by first choosing a value of H_0. A Poincaré section is then built up by computing trajectories from a set of initial conditions that satisfy $H(\lambda;u,\theta,\phi=0)=H_0$. Whenever $\phi(t)$ crosses zero (modπ) in a specified sense (increasing for $\Gamma_i>0$),[45] a point appears on the three-dimensional Poincaré section spanned by u, θ, and λ; it is then projected onto the $(u,\ \theta)$ plane. The results are shown in (x,y) coordinates by applying the transformation $(x,y)=u^{1/2}(\cos\theta,\ \sin\theta)$.

For each Poincaré section, H_0 is fixed by evaluating H for a given $(x=x_0,y=y_0,\lambda=\lambda_0;\phi=0)$ using (C2). A set of initial conditions satisfying $H=H_0$ is obtained by specifying x,y,ϕ and solving the nonlinear equation

$$H(\lambda;u,\theta,\phi=0)=H_0,$$

for λ. For $\epsilon=0.01$ to $\epsilon=0.03$, we have observed a maximum of 3 roots, the number depending on the value of H_0 and the limits, $\lambda_{\min},\lambda_{\max}$, between which roots are found. The limits span the range of λ over which the model is expected to be (initially) valid. In the cases discussed below, H_0 corresponds to a point on the unperturbed separatrix, i.e., $(x_0,y_0,\lambda_0)=(0,1,1.5)$. With a linear shear flow $(e,\omega)=(1/\pi,-1/\pi)$ and $\Gamma=1$, the fixed points are located at $(x_0,y_0)=(0,\pm 1/(\pi e))=(0,\pm 1)$. We thus take $D=1$ in our definition of ϵ.

Poincaré sections are shown in Fig. 2. For the MZS model, closed orbits encircle the origin, but at some distance [Fig. 2(a)]. There is a large gap in the interior, where there are no (closed) orbits at all; initial conditions that are too close to the origin merge before appearing on a Poincaré section. Moving away from the origin, one sees that closed orbits do not extend beyond $u^2=1$. For $\epsilon=0.01$, there are, as with the point vortex pair in shear, closed orbits inside the separatrix, and unbounded orbits outside [Fig. 2(b)]. There is also a gap in the interior. When ϵ is increased, bounded and unbounded orbits remain, but there are far fewer of them: the fraction of merging initial conditions increases rapidly with ϵ. For $\epsilon=0.03$, every point inside the unperturbed separatrix merges [Fig. 2(c)].

In the absence of shear, the innermost orbit of the Poincaré section divides initial conditions that merge from those that do not: it defines a critical merger criterion. This criterion is *not*, however, the same as the classical criterion for identical circular vortices; i.e., that the critical separation, $r_c=3.3r_v$, where r_v is the vortex radius (see MZS). The Poincaré section is defined at constant H, not constant λ. In general, it is expected that the critical merger threshold defined on a constant Hamiltonian surface, $r_{c,H}$, will be greater than r_c, with $\min_H(r_{c,H})=r_c$. Even for a Poincaré section associated with near-circular vortices at the separatrix, $r_{c,H}$ is quite different from r_c because the vortices become strongly elongated away from the separatrix. For example, $r_{c,H}\approx 0.61=0.61r_v$ and $\lambda(r_{c,H})\approx 10$, when the Poincaré section is defined by $\lambda_0=1.001$ and $\epsilon=0.01$. In Fig. 3, we show a merging trajectory corresponding to Fig. 2(a)—note that r is initially less than $r_{c,H}$.

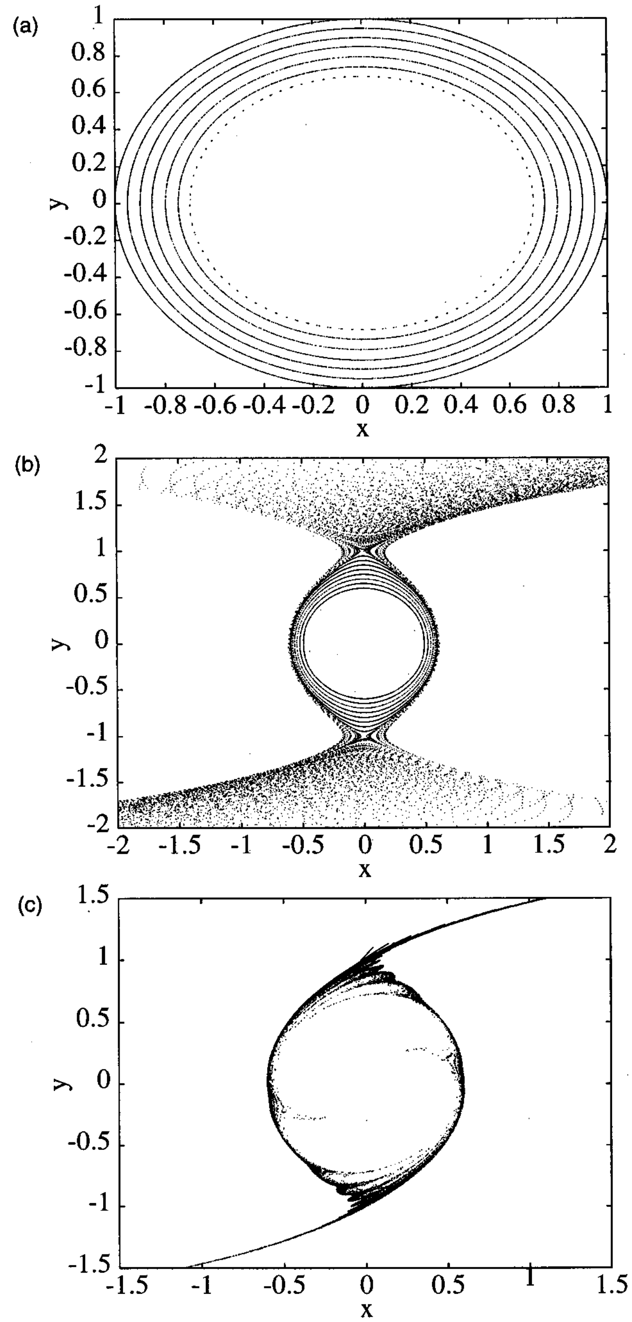

FIG. 2. Poincaré sections corresponding to $\Gamma=1$ and $(x_0,y_0,\lambda_0)=(0,1,1.5)$. (a) MZS model ($\epsilon=0.01$, no shear). (b) $\epsilon=0.01$ and $(\omega,e)=(-1/\pi,1/\pi)$. (c) $\epsilon=0.03$ and $(\omega,e)=(-1/\pi,1/\pi)$.

An innermost orbit can also be distinguished in Fig. 2(b) (i.e., $\epsilon=0.01$ with shear), but it cannot be associated with a critical merger threshold in the same way as for the MZS model. Because of the presence of the background shear, it is now possible for points outside the separatrix to merge; the regions of the Poincaré section inside and outside the unperturbed separatrix are no longer perfectly separated. Most of the exterior trajectories are analogous to the unbounded orbits for a point vortex pair in shear—physically, the vortices approach one another, reach a minimum separation, and are carried away by the shear. The combined effect of vortex-vortex and vortex-shear interactions now makes merger possible, but only for a small fraction of the orbits (Fig. 4). In cases where ϵ is small, the innermost orbit defines a critical radius that separates *most* of the merging orbits from *most* of the non-merging orbits [e.g., Fig. 2(b)], with the exchange between the interior and the exterior being rather limited.

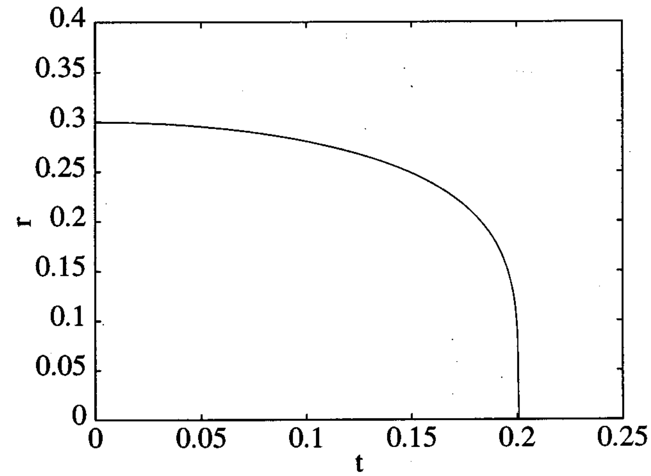

FIG. 3. Typical merging trajectory for Fig. 2(a), i.e., $\epsilon=0.01$. The initial condition is $(r_i,\theta_i,\lambda_i)=(0.400,1.57,0.111)$. This is an example of vortex-vortex merger.

The kind of merger represented by Fig. 4 is very different from that in Figs. 2(a) and 3. Whereas vortex-vortex interactions dominated the previous mode of merger, vortex-shear interactions are crucial to this one. Vortex-shear interactions stretch the vortices out and bring them together; vortex-vortex interactions then initiate the actual merger. For a typical merging trajectory that is initially outside the separatrix (Fig. 5), $\lambda \ll \lambda_{min}$ by the time the trajectory crosses the separatrix. (These trajectories do not appear on the Poincaré section because they merge before ϕ goes through π.) During the course of merger, the approximation that the vortices are small and well separated breaks down; however, this kind

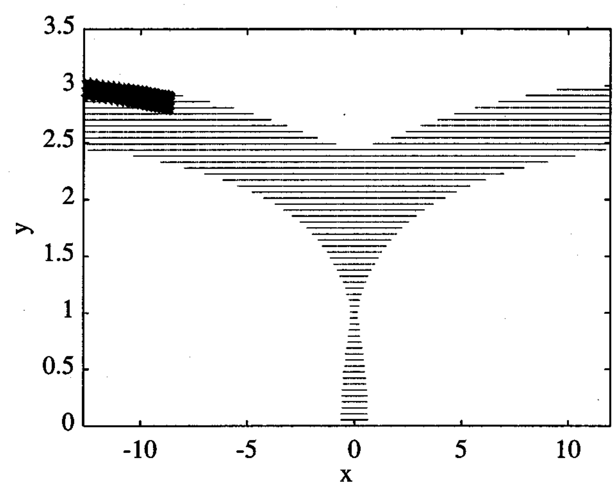

FIG. 4. Merging initial conditions well outside the separatrix plotted on top of a 2-D projection of the Hamiltonian surface corresponding to Fig. 2(a).

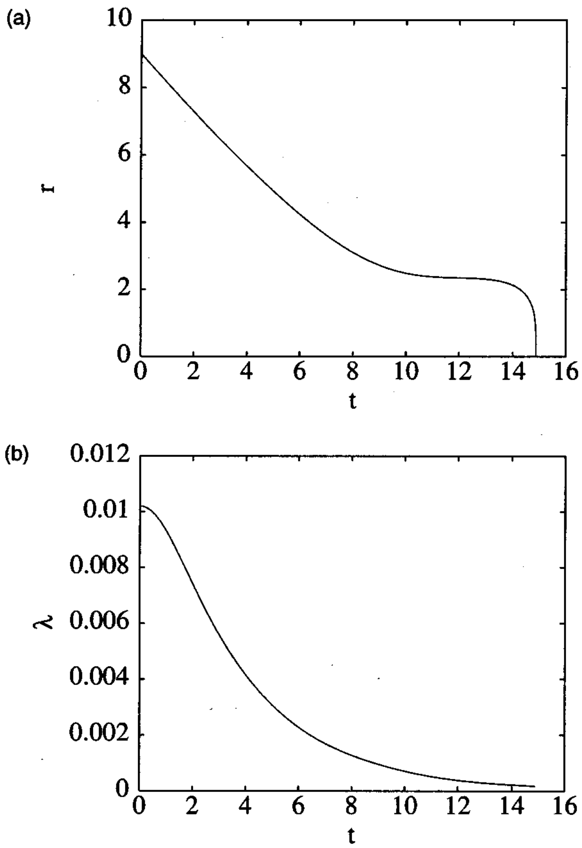

FIG. 5. Typical merging trajectory corresponding to Fig. 4. (a) $r:t$; and (b) $\lambda:t$. The initial conditions are $(r_i, \theta_i, \lambda_i) = (9.00, 2.81, 0.0102)$. This is an example of vortex-shear merger.

of merger is certainly physically plausible. Such a mechanism has also been observed in contour dynamics simulations of circular vortices in shear.[3] This suggests two distinct modes of merger: (i) a predominantly vortex-vortex one; and (ii) a predominantly vortex-shear one. Such a division was noted by Waugh[3] and it is implicit in the work of Marcus.[4] The latter study, however, differs in that it emphasizes energetics (exchanges between the vortices and between the vortices and the shear flow) rather than the geometrical configuration of the vortices. Because we are working on a constant-energy Poincaré surface and because our flow has a strain component (it is not simply an azimuthal shear flow), it is useful to think in terms of the vortices' geometrical configuration.

Melander *et al.*[14] derived a merger criterion for the MZS model that agrees fairly well with contour dynamics simulations. When ambient shear is present, there does not appear to be an analytical expression, based on the initial configuration of the vortex pair, that is necessary and sufficient for merger. A merger criterion for initial conditions inside the separatrix could be determined by estimating the position of the innermost orbit on the Poincaré section; but the situation is much more complicated for initial conditions outside the separatrix. Outside the separatrix, vortex merger is the result of a complex interplay between vortex-vortex and vortex-shear interactions. Many initial conditions yield trajectories that approach the separatrix, but most of these do not merge; a small displacement in (u, θ) can make the difference between merger and separation by the shear. Another complicating factor is that for larger values of ϵ, there is vortex-vortex merger for points lying just outside the separatrix and initial $\lambda \gg 1$.

B. Separatrix splitting and chaos

Because the Melnikov result is both asymptotic and approximate [the perturbation in (34a) is truncated], and because the Melnikov function is exponentially small, numerical verification of the formation of a heteroclinic tangle is needed.

Verification of the exponentially small nature of the numerator in the expression for the separatrix splitting distance is difficult because the position of the folds is a function of ϵ (through the slow time $\Phi = \epsilon\phi$), and the folds narrow and squeeze together as ϵ is reduced. Moreover, the denominator in (36a) becomes exponentially small as one approaches the hyperbolic points, where the splitting is greatest. The important result here is not the precise scaling, but the existence of the heteroclinic tangle and the fact that the splitting is generally small. It could be possible for the stable and unstable manifolds to split apart but not intersect transversally.[40]

Figure 6(a) shows a blow-up of the $\epsilon = 0.01$ Poincaré section near the separatrix, while Fig. 6(b) shows a typical trajectory in this region. Though we anticipate that a heteroclinic tangle is present, its width is too narrow to be resolved by these pictures; one must look very closely at the separatrix for any evidence of the separatrix splitting. For larger values of ϵ, the separatrix splitting is clearly evident, there are distinct fold-like structures for $\epsilon = 0.03$ [Fig. 2(c)]. This is suggestive of a heteroclinic tangle. Most trajectories, however, do not follow these structures indefinitely: they usually merge after a short while.

The separatrix splitting is relevant for vortex merger, insofar as there are only a limited number of initial conditions outside the separatrix that lead to merger; however, the relationship is not simple. The separatrix splitting applies to that portion of the Hamiltonian surface where $\lambda = O(1)$, not to the regions where $\lambda \ll 1$, as in vortex-shear merger beyond the separatrix. (Recall that the Melnikov analysis only deals with small perturbations about a basic state.)

The vicinity of the separatrix is not the only chaotic region. Figure 7(a) is a blow-up of the Poincaré section between $r = 0.50$ and $r = 0.40$; Fig. 7(b) is a typical trajectory in this region. The trajectories hop around chaotically until they fall into the origin (i.e., the vortices merge). This inner chaotic region provides an example of chaos associated with a higher-order resonance[46] in which the ratio of the vortices' rotational frequency to their librational frequency around the origin is 9:1. Somewhat unusually, chaos around the secondary resonances is stronger than chaos around the separatrix. This is because the separatrix splitting does not correspond to a primary resonance. For systems in which the time scale of the perturbation is the same as that of the basic state motion, the chaotic motion around the separatrix usually corresponds to the dominant 1:1 resonance; in our system there is a separation of time scales between the fast rotation of the

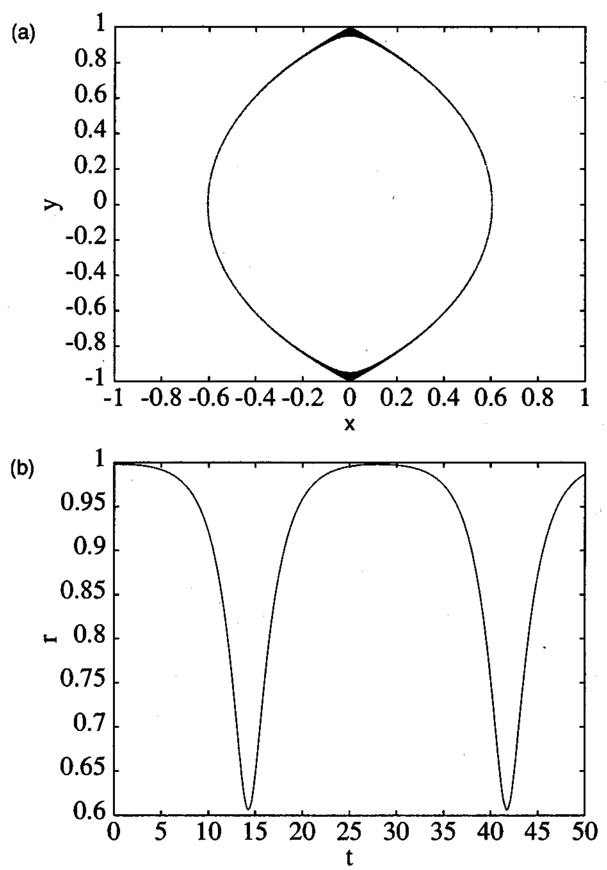

FIG. 6. Blow-up of the region around the separatrix for $\epsilon=0.01$. (a) Poincaré section for initial conditions $(r_i,\theta_i)=(0.99-1.01,\pi/2)$. (b) Typical trajectory just inside the separatrix: $(r_i,\theta_i,\lambda_i)=(0.998,1.57,0.7255)$. Other parameters are as in Fig. 2.

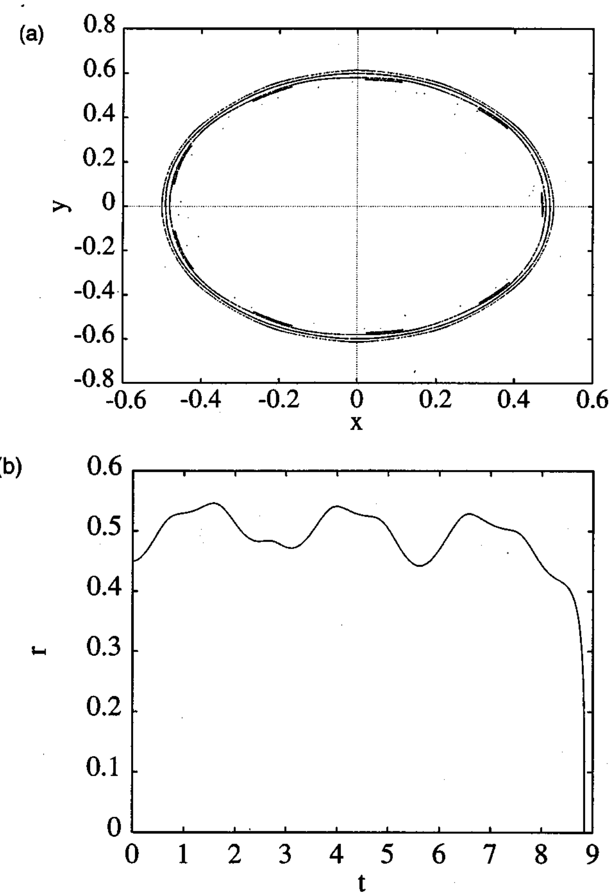

FIG. 7. Chaos in the interior for $\epsilon=0.01$. (a) Poincaré section for $r=0.40-0.50$. (b) Typical "chaotic" trajectory. The initial condition is $(r_i,\theta_i,\lambda_i)=(0.45,0.00,0.152)$. Other parameters are as in Fig. 2.

vortices and the slow centroidal motion. The order of the resonances increases as one approaches the separatrix and chaotic motion around the separatrix is thus severely curtailed. The existence of a separation of time scales for finite ϵ is consistent with an exponentially small Melnikov integral in the asymptotic limit.

VI. DISCUSSION

We began this paper by deriving the equations of motion for N elliptical vortices in a background shear flow using a Hamiltonian moment formulation. This model generalizes that of Melander *et al.*[2] to include a background shear that combines rotation and strain. Motivated by the phenomenon of vortex merger, we considered the case of two identical vortices, both analytically and numerically. A Melnikov analysis of the separatrix splitting between the stable and unstable manifolds connecting the hyperbolic fixed points of the intercentroidal motion was performed, with the expectation that it would provide insight into the interaction of initially widely-separated vortices. The numerical results have, by means of two-dimensional Poincaré sections, addressed the relative importance of vortex-vortex and vortex-shear interactions in vortex merger and the relationship between the separatrix splitting and chaotic motion (around the separatrix and in the interior).

The derivation of the model in Sec. II exposes the model's Hamiltonian structure. It makes explicit the origin of the approximations (they are made in the Hamiltonian), and the model's relation to point vortex, Kida, and MZS models. It would be interesting to see if a related procedure could be applied to the higher-order non-Hamiltonian model of Dritschel and Legras[15] and the Hamiltonian elliptical model of Legras and Dritschel.[16] The Hamiltonian structure of the latter has not been elucidated, and a Hamiltonian analog of the former should, at least in principle, be possible.

The Melnikov analysis of Sec. IV was motivated by vortex merger, but it is interesting in its own right. In dynamical systems theory, one usually deals with nonintegrable perturbations to integrable basic states; in this work, our starting point is a chaotic system. This system is then re-expressed as an integrable basic state plus a rapidly varying perturbation (which may be constructed order by order in the perturbation parameter). The resulting system is an approximate one, though it does avoid the dynamical consistency problem associated with externally imposed perturbations.[47] It is unlikely that the higher order terms so neglected would suppress the existence of chaos, but in light of the exponentially small Melnikov integral, which is consistent with results for rapidly forced oscillators,[24,39,40] they could be important. (If it were not for the implicit form of the separatrix, the separatrix splitting induced by perturbations to a given order could perhaps be bounded using the method of Delshams and Seara;[25] nevertheless, this would not provide a definitive answer in our case, where there are infinitely many terms in the perturbation.) Another interesting aspect of the analysis is that it has been applied to a system with two degrees of freedom which cannot be expressed as a one degree-of-freedom system with a closed perturbation, as in previous studies of rapidly forced oscillators. It is also worth noting that while it has been proven that the Melnikov function is exponentially small for a class of rapidly forced second order equations, our analysis has been applied to a more general system, one whose basic state can include cross-terms, i.e., $\ddot{x}+f(x,\dot{x})=0$.

The numerical results of Sec. V have emphasized a geometrical interpretation of the dynamics through the use of two-dimensional Poincaré sections. Two-dimensional Poincaré sections, constructed at fixed energy and vortex orientation, simplify the visualization of the dynamics and facilitate a global view. From Poincaré sections and time series of the intercentroidal separation and aspect ratio, it is possible, for sufficiently small perturbation amplitude ϵ, to distinguish between predominantly vortex-vortex merger and predominantly vortex-shear merger. In particular, the importance of the relative orientation of the vortices in vortex-shear merger is highlighted. Considerations such as this are not captured by arguments based on energetics. The Poincaré sections also demonstrate that it may be preferable to define a merger criterion for noncircular vortices at fixed energy rather than fixed aspect ratio.

The exponentially small separatrix splitting suggested by the asymptotic Melnikov analysis has not been verified directly (which would in fact be rather difficult), but the numerical results for finite ϵ are consistent with it. Poincaré sections indicate that the separatrix splitting is indeed small. In our numerical examples of shear-induced merger of initially well-separated vortices, the aspect ratios were usually very small by the time the vortices reached separation distances comparable to the diameter of the separatrix. We do not know if this will be the case on Hamiltonian surfaces determined by criteria differing from those in our experiments.

Chaos in the interior is stronger than chaos around the separatrix. This indicates that even for finite ϵ, there exists, as assumed in the Melnikov analysis, a separation of time scales such that a higher order resonance is created near the separatrix—the order of the resonance increases away from the origin. The time-scale separation will eventually break down with increasing ϵ, but this has not been observed here because of the accompanying increase in vortex merger.

As for future work, it would be interesting to study the scattering behavior around the separatrix. Since the system is nonintegrable, chaotic scattering[48,49] is expected. Preliminary studies, however, indicate that this is a very weak effect, largely because of the background shear flow. Different background flows (we have restricted ourselves to a linear shear flow), and different Hamiltonian surfaces (H_0 has always been determined by a point on the unperturbed separatrix) should also be considered. This would, moreover, provide additional insight into the validity of the time-scale separation. A case for which the time-scale separation breaks down would provide an example in which the separatrix splitting has a direct effect on the interaction of the vortices and, perhaps, on vortex merger as well.

ACKNOWLEDGMENTS

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APPENDIX A: NONCANONICAL DYNAMICS AND HAMILTONIAN REDUCTION

Noncanonical methods have proven to be useful in several branches of continuum mechanics, notably geophysical fluid dynamics[50,51] and magnetohydrodynamics.[52,53] The noncanonical formalism readily lends itself to a systematic procedure for approximating the equations of motion. A simple noncanonical representation of the Hamiltonian structure of the general inviscid and nondiffusive fluid equations involves writing the equations of motion in the form[26,27]

$$\frac{\partial Z}{\partial t}=\{Z,H\}, \tag{A1}$$

where $Z(\mathbf{x},t)$ is the appropriate set of fluid variables (e.g., $\rho\mathbf{u},\rho,\ldots$), and $H[Z]$ is a Hamiltonian functional. In an Eulerian description, the noncanonical Poisson bracket, $\{,\}$, has the form

$$\{F,G\}[Z]=\left\langle Z,\left[\frac{\delta F}{\delta Z},\frac{\delta G}{\delta Z}\right]\right\rangle, \tag{A2}$$

where F and G are functionals, $\langle,\rangle$ is an integration over the volume corresponding to the spatial variable $\mathbf{x}$, and the functional derivative is defined by

$$\delta F[Z;\delta Z]=:\left\langle \delta Z,\frac{\delta F}{\delta Z}\right\rangle. \tag{A3}$$

The bracket of (A2) is a Lie algebra product for functionals, i.e., is bilinear, antisymmetric, and satisfies the Jacobi identity, $\{F,\{G,H\}\}+\{G,\{H,F\}\}+\{H,\{F,G\}\}=0$, provided the "inner bracket" $[,]$ is a Lie algebra product for functions. In the present context, the $[,]$ corresponds to the horizontal Jacobian. Brackets of the form of (A2) are called Lie-Poisson brackets.

In order to simplify the Poisson bracket, we confine our attention to a special subset of all admissible functionals F and G and apply a Hamiltonian reduction method. Specifically, for *functions*, f and g, of a finite set of linear functionals of Z, the Poisson bracket may be written as

$$\{f,g\}(z)=z^{\ell}c_{\ell}^{jk}\frac{\partial f}{\partial z^j}\frac{\partial g}{\partial z^k}, \quad j,k,\ell=1,2,\ldots M, \tag{A4}$$

where the z^j are the new "dynamical variables" and the quantities c_{ℓ}^{jk} are the structure constants of some Lie algebra. Repeated sum notation is used. The cosymplectic matrix, $J^{jk}:=z^{\ell}c_{\ell}^{jk}$, inherits the property of skew-symmetry and the reduced bracket automatically satisfies Jacobi's identity. Using (A1) and (A4) to obtain equations of motion for the dynamical variables,

$$\dot{z}^j=J^{jk}\frac{\partial H}{\partial z^k}. \tag{A5}$$

The idea of reduction has a long pedigree dating to Jacobi and Poincaré,[54,55] but for our purposes here it allows us to transform an infinite-dimensional system into a finite-dimensional one and simplifies the task of determining the cosymplectic matrix.

APPENDIX B: A PAIR OF POINT VORTICES IN SHEAR

The equations of motion for N point vortices in a steady background flow $\mathbf{u}(\mathbf{x}_i)$ are given by

$$\dot{\mathbf{x}}_i=\sum_{j=1}^{N}{}'\frac{\Gamma_j}{2\pi}\frac{\mathbf{k}\times(\mathbf{x}_i-\mathbf{x}_j)}{|\mathbf{x}_i-\mathbf{x}_j|^2}+\mathbf{u}(\mathbf{x}_i), \tag{B1}$$

where $\mathbf{k}\times(x_i,y_i)=(-y_i,x_i)$ and the prime denotes $j\neq i$. For $N=2$ vortices and a background flow given by (5), i.e.,

$$(u,v)=\left(-\frac{1}{2}(\omega-e)y,\frac{1}{2}(\omega+e)x\right), \tag{B2}$$

the equations of motion may be written in the form

$$\begin{aligned}
\dot{x}_1&=-\frac{\Gamma_2}{2\pi}\frac{y_1-y_2}{|\mathbf{x}_1-\mathbf{x}_2|^2}-\frac{1}{2}(\omega-e)y_1,\\
\dot{y}_1&=\frac{\Gamma_2}{2\pi}\frac{x_1-x_2}{|\mathbf{x}_1-\mathbf{x}_2|^2}+\frac{1}{2}(\omega+e)x_1,\\
\dot{x}_2&=-\frac{\Gamma_1}{2\pi}\frac{y_2-y_1}{|\mathbf{x}_1-\mathbf{x}_2|^2}-\frac{1}{2}(\omega-e)y_2,\\
\dot{y}_2&=\frac{\Gamma_1}{2\pi}\frac{x_2-x_1}{|\mathbf{x}_1-\mathbf{x}_2|^2}+\frac{1}{2}(\omega+e)x_2.
\end{aligned} \tag{B3}$$

Defining $X=x_1-x_2$, $Y=y_1-y_2$, and nondimensionalizing time by $(\Gamma_1+\Gamma_2)^{-1}$, the equations for the vortex separation are

$$\begin{aligned}
\dot{X}&=-\left(\frac{1}{2\pi}\frac{1}{X^2+Y^2}+\frac{1}{2}(\omega-e)\right)Y,\\
\dot{Y}&=\left(\frac{1}{2\pi}\frac{1}{X^2+Y^2}+\frac{1}{2}(\omega+e)\right)X,
\end{aligned} \tag{B4}$$

where now $\omega:=\omega/(\Gamma_1+\Gamma_2),e:=e/(\Gamma_1+\Gamma_2)$.

There are two types of fixed points for the preceding equations:

$$\begin{aligned}
I:&\quad X=0,\ Y=\pm\left(-\frac{1}{\pi}\frac{1}{\omega-e}\right)^{1/2},\\
II:&\quad X=\pm\left(-\frac{1}{\pi}\frac{1}{\omega+e}\right)^{1/2},\quad Y=0.
\end{aligned} \tag{B5}$$

Type I is present if $\omega-e<0$, and type II is present if $\omega+e<0$. Linearizing around the fixed points, it is easily shown that type Is are hyperbolic and type IIs are elliptic.

The hyperbolic fixed points are connected by a separatrix (see Fig. 1). The separatrix is defined implicitly by the Hamiltonian,

$$H=-\frac{1}{4\pi}\ln|X^2+Y^2|-\frac{1}{4}(\omega-e)Y^2-\frac{1}{4}(\omega+e)X^2. \tag{B6}$$

For $(e,\omega)=(1/\pi,-1/\pi)$, the hyperbolic fixed points are located at $(0,\pm1)$.

APPENDIX C: EQUATIONS OF MOTION FOR $N=2$

For reference we display the general equations of motion for $N=2$ vortices.

In terms of aspect ratios and orientations, the equations take the form

$$\begin{aligned}
\dot{R}&=\frac{1}{2}eR\sin 2\theta-\frac{\Gamma_0+\Gamma_1}{8\pi^2R^3}\left\{A_0\frac{1-\lambda_0^2}{\lambda_0}\sin 2(\theta-\phi_0)\right.\\
&\quad\left.+A_1\frac{1-\lambda_1^2}{\lambda_1}\sin 2(\theta-\phi_1)\right\},\\
\dot{\theta}&=\frac{\omega}{2}+\frac{e}{2}\cos 2\theta+\frac{\Gamma_0+\Gamma_1}{2\pi R^2}+\frac{\Gamma_0+\Gamma_1}{8\pi^2R^4}\left\{A_0\frac{1-\lambda_0^2}{\lambda_0}\right.\\
&\quad\left.\times\cos 2(\theta-\phi_0)+A_1\frac{1-\lambda_1^2}{\lambda_1}\cos 2(\theta-\phi_1)\right\},\\
\dot{\lambda}_0&=-\lambda_0\left\{\frac{\Gamma_1}{\pi R^2}\sin 2(\theta-\phi_0)+e\sin 2\phi_0\right\},\\
\dot{\lambda}_1&=-\lambda_1\left\{\frac{\Gamma_0}{\pi R^2}\sin 2(\theta-\phi_1)+e\sin 2\phi_1\right\},\\
\dot{\phi}_0&=\frac{q_0'\lambda_0}{(1+\lambda_0)^2}-\frac{1}{2}\frac{1+\lambda_0^2}{1-\lambda_0^2}\left\{\frac{\Gamma_1}{\pi R^2}\cos 2(\theta-\phi_0)\right.\\
&\quad\left.-e\cos 2\phi_0\right\}+\frac{\omega}{2},\\
\dot{\phi}_1&=\frac{q_1'\lambda_1}{(1+\lambda_1)^2}-\frac{1}{2}\frac{1+\lambda_1^2}{1-\lambda_1^2}\left\{\frac{\Gamma_0}{\pi R^2}\cos 2(\theta-\phi_1)\right.\\
&\quad\left.-e\cos 2\phi_1\right\}+\frac{\omega}{2};
\end{aligned} \tag{C1}$$

and the Hamiltonian is given by

$$
\begin{aligned}
-4\pi H=&\Gamma_0\Gamma_1 \ln R^2+\frac{1}{2}\left\{\Gamma_0^2 \ln\frac{(1+\lambda_0)^2}{\lambda_0}+\Gamma_1^2 \ln\frac{(1+\lambda_1)^2}{\lambda_1}\right\}+\frac{\Gamma_0+\Gamma_1}{4}\pi R^2(\omega+e\cos 2\theta)\\
&+\frac{\Gamma_0 A_0}{4}[\omega(\lambda_0+\lambda_0^{-1})+e(\lambda_0^{-1}-\lambda_0)\cos 2\phi_0]+\frac{\Gamma_1 A_1}{4}[\omega(\lambda_1+\lambda_1^{-1})+e(\lambda_1^{-1}-\lambda_1)\cos 2\phi_1]\\
&+\frac{\Gamma_0\Gamma_1}{4\pi R^2}[A_0(\lambda_0-\lambda_0^{-1})\cos 2(\theta-\phi_0)+A_1(\lambda_1-\lambda_1^{-1})\cos 2(\theta-\phi_1)]. \qquad \text{(C2)}
\end{aligned}
$$

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