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Editors
S Benkadda
F Doveil
Y Elskens
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Spectral reduction for two-dimensional turbulence

John C. Bowman

B. A. Shadwick

P. J. Morrison

Institute for Fusion Studies, The University of Texas, Austin, TX 78712

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Abstract

A new method for predicting the statistical properties of fluid turbulence, called Spectral Reduction, is described. Collections of Fourier wavenumbers are represented by certain nonuniformly spaced sample modes that interact *via* enhanced coupling coefficients. The approximation reduces to the exact Navier–Stokes equation as a control parameter, the number of fundamental wavenumbers associated with each sample mode, tends to unity. Even at large values of this parameter, the time-averaged predictions of the theory can be shown to recover the statistics of the exact dynamics to high accuracy. Preliminary results from the numerical implementation of Spectral Reduction for two-dimensional homogeneous turbulence are very encouraging; for example, the method is used to illustrate a recent modification to Kraichnan's logarithmically corrected two-dimensional enstrophy cascade.

I. INTRODUCTION

Much of the computational effort in conventional turbulence simulations is devoted to resolving the high wavenumbers. The fact that most of the modes are concentrated at the

smallest scales seems incongruous with the observation that the energy balances at the large scales are often of greater interest in many applications than are the precise details of the dissipation dynamics.

For the purpose of computing transport coefficients, it may be necessary to resolve only the low wavenumbers accurately. It is possible that the high wavenumbers could be represented by certain nonuniformly spaced sample Fourier modes that interact *via* enhanced coupling coefficients. In other words, it would seem that finite computational resources could be best utilized by judiciously distributing the evolved modes throughout the wavenumber spectrum, concentrating them in the regions of physical interest. It is the goal of this work to introduce a general wavenumber reduction scheme that enjoys precisely this kind of freedom.

In the next two sections we discuss the method we have developed. In IV, we proceed to apply it in two dimensions to the enstrophy inertial range. We conclude with a discussion of the new technique and compare it to existing methods in V.

II. SPECTRAL REDUCTION

Although the ideas in this work can be generalized to more realistic situations, we restrict our focus to homogeneous and isotropic incompressible turbulence in two dimensions. The idealization of homogeneous turbulence strictly makes sense only in the limit of an infinite domain, where there are no walls to interfere with translational invariance. The appropriate spectral transform in this limit is the integral Fourier transform, under which the two-dimensional Navier–Stokes vorticity equation takes the form

$$\frac{\partial}{\partial t}\omega_{\boldsymbol{k}} + \nu_{\boldsymbol{k}}\omega_{\boldsymbol{k}} = \int_{\mathcal{D}} d\boldsymbol{p} \int_{\mathcal{D}} d\boldsymbol{q}\, \frac{1}{p^2}\, \epsilon_{\boldsymbol{kpq}}\, \omega^*_{\boldsymbol{p}} \omega^*_{\boldsymbol{q}}, \tag{1}$$

where $\nu_{\boldsymbol{k}} := \nu_0 k^2$ represents time-independent linear dissipation (or forcing *via* linear instabilities) and the interaction coefficient

$$\epsilon_{\boldsymbol{kpq}} := (\hat{z} \cdot \boldsymbol{p} \times \boldsymbol{q})\, \delta(\boldsymbol{k} + \boldsymbol{p} + \boldsymbol{q}) \tag{2}$$

is antisymmetric under permutation of any two indices. (A stirring force could also be added to the right-hand side of Eq. (1) without changing any of the following discussion.) We restrict the integration to a finite wavenumber domain $\mathcal{D}$ that excludes a neighbourhood of both the origin and infinity. This truncation, which is invariably required by spectral methods, preserves the lowest two (quadratic) nonlinear invariants, energy and enstrophy, but not the higher-order Casimir[1] invariants of the inviscid dynamics. The effect of this truncation will not be discussed here further since it is widely believed that only the energy and enstrophy play fundamental roles in the turbulent cascade dynamics.

Let us introduce some coarse-grained grid on $\mathcal{D}$ that partitions the wavenumber space into connected regions that we will call *bins*. The (coarse-grained) bins will be labeled by capital letters (such as $\boldsymbol{K}$) to distinguish them from the continuum wavenumbers, which we are representing by small letters (such as $\boldsymbol{k}$).

To this grid, let us associate new variables

$$\Omega_{\boldsymbol{K}} := \frac{1}{\Delta_{\boldsymbol{K}}} \int_{\Delta_{\boldsymbol{K}}} \omega_{\boldsymbol{k}} \, d\boldsymbol{k} =: \langle \omega_{\boldsymbol{k}} \rangle_{\boldsymbol{K}} \,, \tag{3}$$

where $\Delta_{\boldsymbol{K}}$ is the k-space area of bin $\boldsymbol{K}$. If the true vorticity is a continuous function of wavenumber, the Mean Value Theorem for integrals implies that there exists a wavenumber $\boldsymbol{k}'$ in bin $\boldsymbol{K}$ such that $\Omega_{\boldsymbol{K}} = \omega(\boldsymbol{k}')$. That is, the bin-averaged vorticity $\Omega_{\boldsymbol{K}}$ will always be the vorticity of some mode in the bin.

The exact evolution of $\Omega_{\boldsymbol{K}}$ is given by

$$\frac{\partial}{\partial t}\Omega_{\boldsymbol{K}} + \langle \nu_{\boldsymbol{k}} \omega_{\boldsymbol{k}} \rangle_{\boldsymbol{K}} = \sum_{\boldsymbol{P},\boldsymbol{Q}} \Delta_{\boldsymbol{P}} \Delta_{\boldsymbol{Q}} \left\langle \frac{1}{p^2} \, \epsilon_{\boldsymbol{kpq}} \, \omega^*_{\boldsymbol{p}} \omega^*_{\boldsymbol{q}} \right\rangle_{\boldsymbol{KPQ}} , \tag{4}$$

where

$$\langle f \rangle_{\boldsymbol{KPQ}} = \frac{1}{\Delta_{\boldsymbol{K}} \Delta_{\boldsymbol{P}} \Delta_{\boldsymbol{Q}}} \int_{\Delta_{\boldsymbol{K}}} d\boldsymbol{k} \int_{\Delta_{\boldsymbol{P}}} d\boldsymbol{p} \int_{\Delta_{\boldsymbol{Q}}} d\boldsymbol{q} \, f \tag{5}$$

is a six-dimensional integral that depends only on the geometry and the mode-coupling coefficients f. Although the computation of the weight factors $\langle f \rangle_{KPQ}$ is typically quite challenging, this calculation has already been performed for a variety of geometries, using a

combination of analytical and computational techniques.[2–4] Since this is a *time-independent* calculation, it can be carried out as part of the initial computational overhead. These factors do not depend on the initial conditions, so they need only be computed once for each new wavenumber partition.

Unfortunately, the resulting system of equations, Eq. (4), is not closed. To proceed further, some type of approximation is clearly required. One rather obvious solution would be to approximate $\omega_{\boldsymbol{k}}$, $\omega_{\boldsymbol{p}}$, and $\omega_{\boldsymbol{q}}$ by their bin-averaged values $\Omega_{\boldsymbol{K}}$, $\Omega_{\boldsymbol{P}}$, and $\Omega_{\boldsymbol{Q}}$, respectively. This would lead to the following closed set of equations for the bin-averaged vorticities:

$$\frac{\partial}{\partial t}\Omega_{\boldsymbol{K}} + \nu_{\boldsymbol{K}}\Omega_{\boldsymbol{K}} = \sum_{\boldsymbol{P},\boldsymbol{Q}} \Delta_{\boldsymbol{P}}\Delta_{\boldsymbol{Q}} \left\langle \frac{1}{p^2}\epsilon_{\boldsymbol{kpq}} \right\rangle_{\boldsymbol{KPQ}} \Omega^*_{\boldsymbol{P}}\Omega^*_{\boldsymbol{Q}}, \tag{6}$$

where $\nu_{\boldsymbol{K}} \coloneqq \langle \nu_{\boldsymbol{k}} \rangle_{\boldsymbol{K}}$ is the bin-averaged viscosity. It could be argued that this approximation should be reasonable if the bins are small enough so that within each bin the modal vorticities vary slowly with respect to wavenumber. However, for a wavenumber reduction scheme to be useful, presumably at least some bins must be quite large, encompassing many fundamental modes. It might be unclear to the reader what relevance Eq. (6) could possibly have to the bin-averaged dynamics of a highly turbulent flow, where the individual vorticities within each bin deviate wildly from their bin-averaged values. Nevertheless, there is a sense in which equations similar to Eq. (6) can be justified, as we will discuss shortly. For now, let us proceed to consider the conservation properties of Eq. (6), comforted by the knowledge that at least in the limit of small bin size (e.g., in a discrete version of equation Eq. (1), the limit where there is exactly one mode per bin), it reduces to the exact dynamical equation, (1).

It is widely held that inertial-range behaviour is intimately connected to the conservation properties of the nonlinear terms in the Navier–Stokes equation. The dimensional arguments of Kolmogorov and Kraichnan suggest that the quadratic nonlinear invariants, energy and enstrophy, are fundamental to the phenomenology of the turbulent cascade. Consequently, it would seem advisable that any proposed reduction of the full equations preserve all of the quadratic invariants.

In the absence of forcing or dissipation, Eq. (1) conserves the energy E and enstrophy Z,

$$E := \tfrac{1}{2}\int d\boldsymbol{k}\,\frac{1}{k^2}|\omega_{\boldsymbol{k}}|^2, \qquad Z := \tfrac{1}{2}\int d\boldsymbol{k}\,|\omega_{\boldsymbol{k}}|^2, \tag{7}$$

as a consequence of the following symmetries of $\epsilon_{\boldsymbol{kpq}}$, respectively:

$$\begin{aligned} \frac{1}{k^2}\frac{1}{p^2}\,\epsilon_{\boldsymbol{kpq}} \qquad & \text{antisymmetric in} \qquad \boldsymbol{k} \leftrightarrow \boldsymbol{p}, \\ \frac{1}{p^2}\,\epsilon_{\boldsymbol{kpq}} \qquad & \text{antisymmetric in} \qquad \boldsymbol{k} \leftrightarrow \boldsymbol{q}. \end{aligned} \tag{8}$$

A problem with closure is again encountered when we try to express these invariants on the coarse grid. We define the coarse-grained energy $\bar{E}$ and enstrophy $\bar{Z}$ to be

$$\bar{E} := \tfrac{1}{2}\sum_{\boldsymbol{K}}\frac{1}{K^2}|\Omega_{\boldsymbol{K}}|^2, \qquad \bar{Z} := \tfrac{1}{2}\sum_{\boldsymbol{K}}|\Omega_{\boldsymbol{K}}|^2. \tag{9}$$

As it stands, there is some arbitrariness in these formulae, particularly in the choice of the characteristic wavenumber in each bin $\boldsymbol{K}$, which (without any ambiguity) we also label $\boldsymbol{K}$. We leave the actual definition of $\boldsymbol{K}$ quite arbitrary, except that we require $\boldsymbol{K}$ to be the value of some wavenumber from bin $\boldsymbol{K}$. There are many definitions, such as $\Delta_{\boldsymbol{K}}^{-1}\int_{\Delta_{\boldsymbol{K}}} k\,d\boldsymbol{k}$ and $\Delta_{\boldsymbol{K}}^{-1}\int_{\Delta_{\boldsymbol{K}}} k^2\,d\boldsymbol{k}$ that satisfy this restriction, as can be seen from the Mean Value Theorem for integrals. This condition guarantees that the coarse-grained invariants reduce to the corresponding continuum invariants in the limit of small bin size.

Recognizing that there is no general time-independent definition of the wavenumber $\boldsymbol{K}$ for which the coarse-grained quantities reduce to the corresponding continuum values for all time, we would be content with a definition that leads to exact conservation of $\bar{E}$ and $\bar{Z}$. However, we now show that even this is not possible.

It is readily seen that Eq. (6) conserves enstrophy since

$$\left\langle \frac{1}{p^2}\epsilon_{\boldsymbol{kpq}} \right\rangle_{\boldsymbol{KPQ}} \tag{10}$$

is antisymmetric in $\boldsymbol{K} \leftrightarrow \boldsymbol{Q}$. However, energy conservation has been lost since

$$\frac{1}{K^2}\left\langle \frac{1}{p^2}\epsilon_{\boldsymbol{kpq}} \right\rangle_{\boldsymbol{KPQ}} \tag{11}$$

is *not* antisymmetric in $\boldsymbol{K} \leftrightarrow \boldsymbol{P}$. The origin of the problem lies in the fact that $K\,\langle p\rangle_{\boldsymbol{KPQ}} \neq P\,\langle k\rangle_{\boldsymbol{KPQ}}$. The resulting violation of energy conservation is really a consequence of the assumption that the vorticity varies slowly over a bin.

However, both of the desired energy and enstrophy symmetries can be reinstated by replacing the weight factor

$$\left\langle \frac{1}{p^2}\epsilon_{\boldsymbol{kpq}} \right\rangle_{\boldsymbol{KPQ}} \tag{12}$$

in Eq. (6) with the slightly modified coefficient

$$K^2Q^2 \left\langle \frac{1}{k^2p^2q^2}\epsilon_{\boldsymbol{kpq}} \right\rangle_{\boldsymbol{KPQ}}. \tag{13}$$

This modification introduces negligible error in the limit of small bin size, so that the resulting equation

$$\frac{\partial}{\partial t}\Omega_{\boldsymbol{K}} + \nu_{\boldsymbol{K}}\Omega_{\boldsymbol{K}} = \sum_{\boldsymbol{P},\boldsymbol{Q}} \Delta_{\boldsymbol{P}}\Delta_{\boldsymbol{Q}}\, K^2Q^2 \left\langle \frac{1}{k^2p^2q^2}\epsilon_{\boldsymbol{kpq}} \right\rangle_{\boldsymbol{KPQ}} \Omega^*_{\boldsymbol{P}}\Omega^*_{\boldsymbol{Q}}. \tag{14}$$

is a more acceptable alternative to Eq. (6) as a closure of Eq. (4): not only does it reduce to the exact dynamics in this limit, but it conserves both energy and enstrophy even when the bins are large. The final modification leading to Eq. (14) may be viewed as a partial compensation for the error introduced by the slowly varying approximation. The resulting reduced dynamical equation has the same general structure and symmetries as Eq. (1) and in this sense may be thought of as a renormalization of the original equation.

III. TIME-AVERAGED JUSTIFICATION

Let us now examine the validity of Eq. (14) as an approximation to the exact turbulent dynamics. If the bins are large, the true vorticity will vary rapidly with wavenumber within each bin and it is unlikely that Eq. (14) could yield a reasonable description of the instantaneous dynamics. However, on a time-averaged basis the situation is much more encouraging; the time-averaged (or ensemble-averaged) spectrum predicted by Eq. (14) corresponds very closely to the exact bin-averaged statistics, as we now show.

Upon time averaging the bin-averaged enstrophy equation derived from Eq. (1), one obtains

$$\overline{\frac{\partial}{\partial t}\left\langle|\omega_{\boldsymbol{k}}|^2\right\rangle_{\boldsymbol{K}}}+\left\langle\nu_{\boldsymbol{k}}\overline{|\omega_{\boldsymbol{k}}|^2}\right\rangle_{\boldsymbol{K}}=\sum_{\boldsymbol{P},\boldsymbol{Q}}\Delta_{\boldsymbol{P}}\Delta_{\boldsymbol{Q}}\left\langle\frac{1}{p^2}\,\epsilon_{\boldsymbol{kpq}}\,\overline{\omega_{\boldsymbol{k}}^*\omega_{\boldsymbol{p}}^*\omega_{\boldsymbol{q}}^*}\right\rangle_{\boldsymbol{KPQ}},\tag{15}$$

where the over-bar denotes a time average. (The first term will vanish upon time-averaging far into a statistically steady state; the saturated turbulent state will then reflect a balance between linear forcing/dissipation and nonlinear equilibration.)

Time-averaged quantities such as $\overline{|\omega_{\boldsymbol{k}}|^2}$ and $\overline{\omega_{\boldsymbol{k}}^*\omega_{\boldsymbol{p}}^*\omega_{\boldsymbol{q}}^*}$ are generally smooth functions of the wavenumbers $\boldsymbol{k}$, $\boldsymbol{p}$, $\boldsymbol{q}$. Strictly speaking, for homogeneous turbulence, the triplet correlation $\overline{\omega_{\boldsymbol{k}}^*\omega_{\boldsymbol{p}}^*\omega_{\boldsymbol{q}}^*}$ is smooth in these variables only on the four-dimensional surface defined by the triad condition $\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=0$; however, the integration in Eq. (5) is already confined to this surface by the $\delta(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q})$ factor in $\epsilon_{\boldsymbol{kpq}}$. To good accuracy these statistical averages may therefore be evaluated at the characteristic wavenumbers $\boldsymbol{K}$, $\boldsymbol{P}$, $\boldsymbol{Q}$ of each bin, yielding

$$\overline{\frac{\partial}{\partial t}|\Omega_{\boldsymbol{K}}|^2}+\langle\nu_{\boldsymbol{k}}\rangle_{\boldsymbol{K}}\,\overline{|\Omega_{\boldsymbol{K}}|^2}=\sum_{\boldsymbol{P},\boldsymbol{Q}}\Delta_{\boldsymbol{P}}\Delta_{\boldsymbol{Q}}\left\langle\frac{1}{p^2}\,\epsilon_{\boldsymbol{kpq}}\right\rangle_{\boldsymbol{KPQ}}\overline{\Omega_{\boldsymbol{K}}^*\Omega_{\boldsymbol{P}}^*\Omega_{\boldsymbol{Q}}^*}.\tag{16}$$

Moreover, to the extent that the (nonstochastic) wavenumber magnitudes k, p, and q vary slowly over a bin, Eq. (15) may equally well be reduced to the conservative approximation

$$\overline{\frac{\partial}{\partial t}|\Omega_{\boldsymbol{K}}|^2}+\langle\nu_{\boldsymbol{k}}\rangle_{\boldsymbol{K}}\,\overline{|\Omega_{\boldsymbol{K}}|^2}=\sum_{\boldsymbol{P},\boldsymbol{Q}}\Delta_{\boldsymbol{P}}\Delta_{\boldsymbol{Q}}\,K^2Q^2\left\langle\frac{1}{k^2p^2q^2}\,\epsilon_{\boldsymbol{kpq}}\right\rangle_{\boldsymbol{KPQ}}\overline{\Omega_{\boldsymbol{K}}^*\Omega_{\boldsymbol{P}}^*\Omega_{\boldsymbol{Q}}^*}.\tag{17}$$

This is precisely the equation for the time-averaged enstrophy that is obtained from Eq. (14). Thus, if we solve Eq. (14) and compute the mean-squared vorticity (enstrophy) spectrum, the result should be close to the spectrum computed from Eq. (1). Furthermore, as the wavenumber partition is refined, one expects the solutions of Eq. (17) to converge to the those of Eq. (15). The same arguments apply to the computation of the mean-squared velocity (energy) spectrum.

It thus appears that the technique underlying Eq. (14), which we call Spectral Reduction,* can provide an accurate statistical description of turbulence, even in the case

*The name Spectral Reduction reflects the fact that the method does not rely on specific properties of the Fourier transform.

where each bin contains many statistically independent modes. In the next section, we present preliminary results in support of this conclusion.

IV. TWO-DIMENSIONAL ENSTROPHY RANGE

Let us apply Spectral Reduction to illustrate the Kolmogorov-Kraichnan scaling for the two-dimensional enstrophy range, including some recently derived modifications to Kraichnan's result.

Based on Kolmogorov's[5] idea of self-similar energy transfer in the inertial range, Kraichnan[6] proposed an asymptotic form for the energy spectrum of the enstrophy inertial range,

$$E(k) \sim k^{-3} \left[\ln \left(\frac{k}{k_1} \right) \right]^{-1/3} \qquad (k \gg k_1), \tag{18}$$

where k_1 is the smallest wavenumber in the inertial range. The logarithmic correction in this result prevents the formation of an infinite amount of enstrophy in the inertial range.

Recently, Eq. (18) was extended to the entire inertial range ($k \geq k_1$):[7]

$$E(k) \sim k^{-3} \chi^{-1/3}(k) \qquad (k \geq k_1), \tag{19}$$

where

$$\chi(k) \coloneqq \ln \left(\frac{k}{k_1} \right) + \chi_1. \tag{20}$$

The new positive constant χ_1, which removes the divergence from Kraichnan's expression at the injection wavenumber k_1, is set by the large-scale dynamics and cannot be determined by dimensional reasoning.

To demonstrate Eq. (19), we applied Eq. (14) using a polar partition with a logarithmically spaced radial grid spanning $k = 1$ to $k = 1024$ and a uniformly spaced angular grid. We evolved a thermal equilibrium spectrum (with both inverse temperatures set to unity) to a statistically steady state for 16×8 (radial $\times$ angular) and 32×8 geometries. The linear forcing and dissipation were given by

$$\nu_k \coloneqq \nu_L H(k_0 - k) k^m + \nu_H H(k - kd) k^n + \begin{cases} -\dfrac{\gamma_f}{\Delta_f} & \text{if } k_f - \frac{1}{2}\Delta_f < k < k_f + \frac{1}{2}\Delta_f, \\ 0 & \text{otherwise.} \end{cases} \tag{21}$$

The Heaviside function H was introduced in this pedagogical study to allow the formation of a pristine inertial range. The parameters used were

$$\nu_L = 0.2, \quad k_0 = 2.378, \quad m = 0, \tag{22}$$

$$\nu_H = 1.0 \times 10^{-17}, \quad k_d = 664, \quad n = 6, \tag{23}$$

$$\gamma_f = 1.492, \quad \Delta_f = 1.290, \quad k_f = 3.023. \tag{24}$$

The saturated energy spectra for the two partitions are compared in Fig. 1. We see that the convergence of the spectra as the partition is refined, while not perfect, is quite good even at this low resolution. The evolution of $\bar{E}$, $\bar{Z}$, and the palinstrophy $\bar{P} \coloneqq \frac{1}{2}\sum_{\boldsymbol{K}} K^2 |\Omega_{\boldsymbol{K}}|^2$ to this statistically stationary state is depicted in Fig. 2 for the 32×8 geometry.

We illustrate Eq. (19) by graphing the logarithmic slope $\ln E(k)/\ln k$ in Fig. 3. We verify in Fig. 4 the linear behaviour of $[k^3 E(k)]^{-3}$ with respect to $\ln(k/k_1)$ as predicted by Eq. (19), using the values $k_1 = 14.3$ and $\chi_1 = 0.39$ determined by a least-squares fit (for details, see Ref. 7). Note the excellent agreement with Eq. (19); the average energies at 14 radial wavenumbers (out of 32) are perfectly described by the logarithmic correction, even after dividing out the overall k^{-3} dependence.

To obtain these results, we employed a new integration algorithm that in the inviscid limit conserves the energy and enstrophy to *all orders* in the time step.[8] This algorithm was extended to solve for the evolution on the linear time scale exactly, allowing it to be applied to forced-dissipative turbulence, which possesses both linear and nonlinear time scales. The resulting algorithm ensures that the energy evolves in a manner consistent with the equations of motion.

V. DISCUSSION

In this work we proposed a new technique for computational fluid dynamics that dramatically decreases the number of degrees of freedom required to simulate turbulent flow. For example, a problem that would conventionally require 2048×2048 modes was accurately modelled with only 32×8 bins. One of the advantages of using such a reduced description is that one can then easily evolve the system for thousands of eddy turnover times to obtain extremely smooth energy spectra and transport coefficients, which can then be compared with known theoretical results.

Over the past forty years, much research effort has been spent trying to develop a satisfactory statistical theory of turbulence. Until relatively recently, statistical closures were widely studied as approximate but quantitative descriptions of turbulence. These analytical theories attempt to find closed expressions for the unknown triplet correlation function appearing in Eq. (15). A familiar example of a statistical closure is Kraichnan's direct-interaction approximation (DIA).[9–13] Unfortunately, there is considerable arbitrariness in the formulation of statistical closures and it is generally believed that low-order statistical theories are simply not capable of capturing the effects of coherent structures.[14,15] But probably the greatest weakness of these methods is that there exists neither an error estimate nor a control parameter that can be varied to increase the accuracy of the solution.

The existence of a control parameter (bin size) is an important feature of Spectral Reduction that thus distinguishes it from other statistical theories. Moreover, it does not make a closure assumption on the (time-averaged) triplet correlation appearing in Eq. (17). It circumvents the closure problem entirely by *reducing* the number of triplet correlations to a tractable number, rather than by eliminating them in favour of lower order statistical variables. Unlike statistical closures, Spectral Reduction does not destroy the phase information embodied in the triple product $\overline{\Omega^*_{\boldsymbol{K}}\Omega^*_{\boldsymbol{P}}\Omega^*_{\boldsymbol{Q}}}$ in Eq. (17).

There is one minor drawback to the method. It suffers from being formulated entirely in a spectral domain, with a nonlinearity more complicated than a simple convolution, so that a

pseudospectral (collocation) technique is no longer applicable. Consequently, there can be a tradeoff between the discrete version of Spectral Reduction and conventional pseudospectral methods: if there are only one or two discrete modes per bin, Spectral Reduction will surely lose in any competition with pseudospectral methods. However, this handicap is quickly offset as soon as the number of modes per bin increases.

The idea of wavenumber reduction is certainly not new. Constrained decimation[16–18] consists of a reduction in which the effect of the deleted modes on the retained modes is accounted for through the introduction of an additive stochastic forcing to the right-hand side of Eq. (1). She and Jackson[19] have proposed an alternative reduction scheme in which the linearity in Eq. (1) is enhanced to account partially for the implicitly evolved modes. In Spectral Reduction, a third alternative is chosen: it is the nonlinearity that is enhanced to account for the effect of the discarded modes. There have been a few other more direct attempts at wavenumber reduction,[20–23] none of which appear to account for the effect of the neglected modes on the retained dynamics in a systematic manner. Moreover, these methods typically neglect very nonlocal triad interactions, which play an important role in two-dimensional turbulence.

It is also worthwhile to compare Spectral Reduction to the renormalization group (RNG)[24] theory. This approach retains only the large-scale modes, but attempts to express the effect of the small-scale modes on the large scales using a self-similarity *ansatz*. In contrast, Spectral Reduction effectively retains certain modes from all scales and discards other modes from these same scales. Moreover, the generality of the formulation allows one to refine the partition wherever necessary.

In this work, Spectral Reduction was used to verify the logarithmically corrected two-dimensional enstrophy law to a remarkably high accuracy. This approximation appears to be a promising candidate as a statistical description of turbulence. However, while it provides an accurate technique for computing turbulent statistics, it does not provide further insight into the underlying dynamical processes. Consequently, it should be thought of more as a computational tool than as a real analytical theory of turbulence. The latter is a challenge

that still awaits us.

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FIGURES

FIG. 1. Saturated energy spectrum predicted by Spectral Reduction.

FIG. 2. Evolution of the nonlinear invariants for the 32×8-bin geometry.

FIG. 3. Logarithmic slope of the energy spectrum in Fig. 1.

FIG. 4. Linearity of $[k^3 E(k)]^{-3}$ with respect to $\ln(k/k_1)$ for $k \geq k_1 = 14.3$. The solid triangles are the predictions of Spectral Reduction.

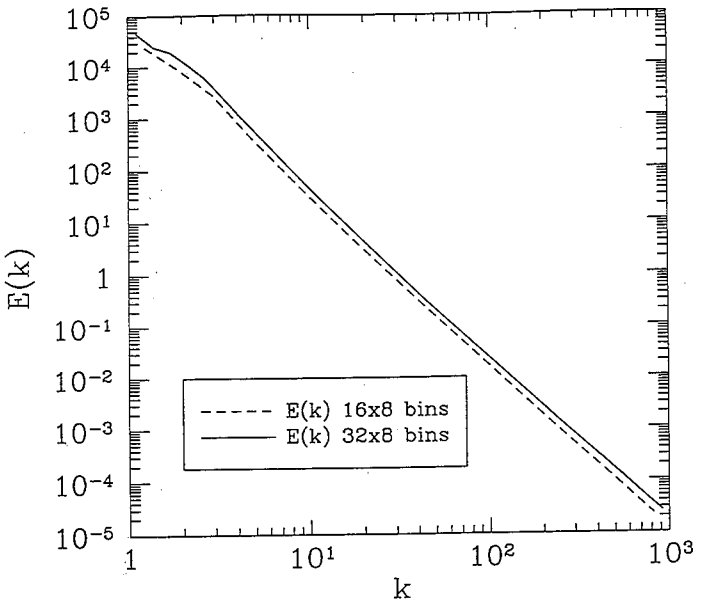

FIG. 1. Saturated energy spectrum predicted by Spectral Reduction.

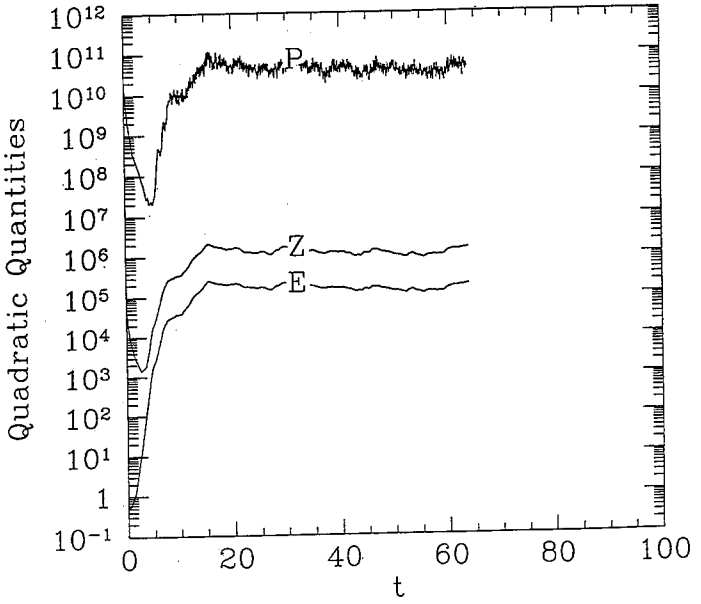

FIG. 2. Evolution of the nonlinear invariants for the 32 × 8-bin geometry.

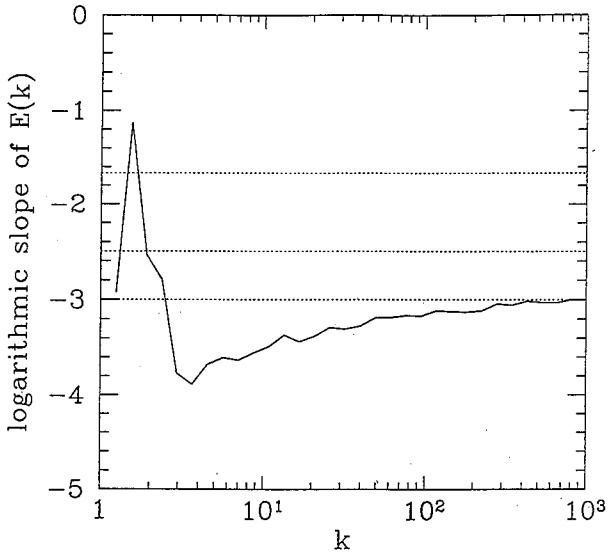

FIG. 3. Logarithmic slope of the energy spectrum in Fig. 1.

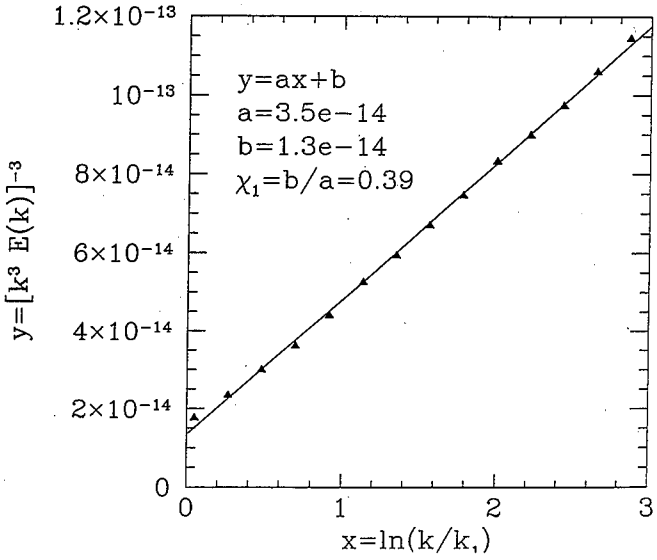

FIG. 4. Linearity of $[k^3E(k)]^{-3}$ with respect to $\ln(k/k_1)$ for $k \geq k_1 = 14.3$. The solid triangles are the predictions of Spectral Reduction.