Hamiltonian moment reduction for describing vortices in shear

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(Received 27 December 1996; accepted 4 March 1997)

This paper discusses a general method for approximating two-dimensional and quasigeostrophic three-dimensional fluid flows that are dominated by coherent lumps of vorticity. The method is based upon the noncanonical Hamiltonian structure of the ideal fluid and uses special functionals of the vorticity as dynamical variables. It permits the extraction of exact or approximate finite degree-of-freedom Hamiltonian systems from the partial differential equations that describe vortex dynamics. We give examples in which the functionals are chosen to be spatial moments of the vorticity. The method gives rise to constants of motion known as Casimir invariants and provides a classification scheme for the global phase space structure of the reduced finite systems, based upon Lie algebra theory. The method is illustrated by application to the Kida vortex [S. Kida, J. Phys. Soc. Jpn. **50**, 3517 (1981)] and to the problem of the quasigeostrophic evolution of an ellipsoid of uniform vorticity, embedded in a background flow containing horizontal and vertical shear [Meacham *et al.*, Dyn. Atmos. Oceans **14**, 333 (1994)]. The approach provides a simple way of visualizing the structure of the phase space of the Kida problem that allows one to easily classify the types of physical behavior that the vortex may undergo. The dynamics of the ellipsoidal vortex in shear are shown to be Hamiltonian and are represented, *without further approximation* beyond the assumption of quasigeostrophy, by a finite degree-of-freedom system in canonical variables. The derivation presented here is simpler and more complete than the previous derivation which led to a finite degree-of-freedom system that governs the semi-axes and orientation of the ellipsoid. Using the reduced Hamiltonian description, it is shown that one of the possible modes of evolution of the ellipsoidal vortex is chaotic. These chaotic solutions are noteworthy in that they are exact chaotic solutions of a continuum fluid governing equation, the quasigeostrophic potential vorticity equation.
 [S1070-6631(97)00608-9]

I. INTRODUCTION

An ubiquitous feature of fluid motion is the occurrence of long-lived localized vorticies. Notable naturally occuring examples are the recirculating vorticies generated at vorticity fronts associated with western boundary currents in the ocean, Meddies (saline lenses that originate near the Strait of Gibraltar), and Jupiter's Great Red Spot and White Ovals. Since such vortices can exhibit relatively uncomplicated behavior, several authors have developed low degree-of-freedom models to describe their dynamics.[1–5] Melander *et al.*[2,3] used moments of the vorticity as dynamical variables and showed that their reduced model was Hamiltonian. The model of Kida[1] (K hereafter) is also Hamiltonian, but Meacham *et al.*[5] (MPSZ hereafter) had some difficulty deciding whether their stratified quasigeostrophic (QG) model was Hamiltonian in the most general case. MPSZ used a classical Eulerian approach to determine a finite set of ordinary differential equations (ODEs) that exactly described the motion of a uniform ellipsoidal vortex in a shear flow given by a streamfunction that was quadratic in the spatial variables. For a restricted set of forms of this background shear, they were able to obtain an appropriate Hamiltonian by inspection of the ODEs. For the case of a general quadratic background streamfunction, the ODEs did not yield their secrets so gracefully. In the present paper, a systematic procedure, based upon the Hamiltonian structure of the ideal fluid equations, is given for obtaining exact or approximate moment reductions where the resulting finite degree-of-freedom model is manifestly Hamiltonian. We illustrate the procedure by applying it to the problems of K and MPSZ.

In our application of the Hamiltonian reduction technique to the Kida problem, Sec. III, we obtain a simple way of classifying the possible types of phase space trajectory. This consists of looking at different ways in which Hamiltonian and Casimir surfaces can intersect in the three-dimensional phase space. The different types of intersection, which are readily visualized, correspond to different types of physical behavior of the vortex (tumbling, nutation, and stretching). Useful choices of variables with which to represent the Kida problem are already known, e.g., Ref. 2; we will see that they arise naturally in the Hamiltonian approach once the problem has been couched in terms of normal coordinates. We are led to similarly simplified sets of variables in the more complicated problem of the quasigeostrophic ellipsoidal vortex in Sec. IV. The Hamiltonian reduction results in a much simpler set of equations for the three-dimensional problem than those derived by MPSZ. We go on

 1070-6631/97/9(8)/2310/19/$10.00

to use these equations to demonstrate that one type of behavior that the quasigeostrophic ellipsoid can undergo is a chaotic tumbling. Since solutions of the reduced Hamiltonian equations for the quasigeostrophic ellipsoidal vortex are exact solutions of the inviscid quasigeostrophic governing equations and since these latter equations are continuum equations, the chaotic solutions we observe correspond to chaotic behavior in a continuum model of a rotating stratified fluid.

The Hamiltonian form possessed by the ideal fluid equations arises in many guises, because of the various variables that are used to describe fluid motion. The natural Hamiltonian structure of the ideal fluid equations is most clearly seen when the fluid is represented in terms of Lagrangian variables. One describes the fluid as a continuum of fluid particles and it naturally inherits the Hamiltonian description of particle mechanics.[6,7] The Hamiltonian form of point vortex dynamics[8] and the Hamiltonian form of the Euler equations in terms of Clebsch potentials[9] can be shown to arise from this underlying structure. However, in terms of Eulerian variables, the Hamiltonian nature of ideal fluids is less immediately evident. The degenerate Lagrange bracket description,[10,11] the commutator description,[12] and the noncanonical Hamiltonian description in terms of a degenerate Poisson bracket[13] can also be shown to arise from the underlying Lagrangian form. The same is true for the noncanonical Hamiltonian description of vortex dynamics in three[13,14] and two[15–17] dimensions. It is this latter description that is the starting point of this paper. (For review see the works by Salmon[18] and Morrison.[19]) We now briefly sketch this noncanonical formalism.

The noncanonical Hamiltonian description amounts to writing the fluid equations in the form

$$\frac{\partial \chi}{\partial t} = \{\chi, H\}, \tag{1}$$

where $\chi(\mathbf{x},t)$ is a shorthand for the set of fluid variables, e.g., $\rho\mathbf{u}$, ρ,..., and $H[\chi]$ is the Hamiltonian functional. The noncanonical Poisson bracket, $\{,\}$, has the following form for Eulerian media fields:

$$\{F,G\} = \left\langle \chi, \left[\frac{\delta F}{\delta \chi}, \frac{\delta G}{\delta \chi} \right] \right\rangle, \tag{2}$$

where F and G are functionals, $\langle,\rangle$ is (for the purposes here) an integration over the volume corresponding to the spatial variable $\mathbf{x}$, and the functional derivative is defined by

$$\delta F[\chi;\delta\chi] =: \left\langle \delta\chi, \frac{\delta F}{\delta \chi} \right\rangle. \tag{3}$$

The bracket of (2) is a Lie algebra product for functionals, i.e., is bilinear, antisymmetric, and satisfies the Jacobi identity, $\{F,\{G,H\}\} + \{G,\{H,F\}\} + \{H,\{F,G\}\} = 0$, provided the ''inner bracket'' $[,]$ is a Lie algebra product for functions. Brackets of the form of (2) are called Lie–Poisson brackets. The analogous bracket in finite degree-of-freedom systems can be written out in coordinates as follows:

$$\{f,g\} = z^k c_k^{ij} \frac{\partial f}{\partial z^i} \frac{\partial g}{\partial z^j}, \quad i,j,k = 1,2,\ldots,N, \tag{4}$$

where f and g are functions of the dynamical variables z^i, which span the phase space, and the quantities c_k^{ij} are the structure constants for some Lie algebra. Repeated sum notation is used here (and henceforth).

Suppose now that a physically significant class of functionals F and G of (2) is comprised of those that depend on χ only as functions of a finite set of simpler functionals of χ. We will call these simpler functionals, ''moments of χ,'' and refer to the class as $\mathscr{F}$. What we mean by moments can be left fairly general, but we have in mind a procedure that involves integration over the spatial variables. The number of moment variables may be arbitrarily large. Since variations $\delta\chi$ induce variations in the moments, the chain rule can be applied to map the bracket of (2) to one on the moments. This procedure results in a bracket where the inner bracket is a *filtered* Lie algebra product.[20,21] Significantly, it is possible to obtain reduced descriptions in terms of a finite number of the moments where the Lie algebra product is closed. Details of the general mathematical structure will not be presented here, rather we will demonstrate this by specific examples.

The moment reduction described above does not give the whole story, since specification of the dynamics requires the Hamiltonian as well as the Poisson bracket. The above procedure is only of interest if the Hamiltonian belongs to $\mathscr{F}$ or can be sufficiently closely approximated by an element of $\mathscr{F}$, i.e., if the Hamiltonian can be written in terms of these variables. In general, this is not possible. However, for a restricted class of initial conditions it may be possible, which is the case for the examples presented here. Alternatively, there may exist an expansion in terms of a small parameter that renders the Hamiltonian a function of the moments. This is the case for the Hamiltonian structure in terms of moments given by Melander *et al.*,[2] which has been generalized to include background flow and worked out from first principles by the methods presented here.[22,23]

The paper is organized as follows. In Sec. II, we review the noncanonical Hamiltonian structure for a class of vorticity-like systems and sketch the general procedures of moment reduction. Then, in Sec. III, we illustrate this with the Kida[1] exact reduction. Kida obtained the equations of motion for an elliptical vortex patch in a background flow, where the dynamics involves time dependence of the ellipse aspect ratio and angle of orientation. Later, in an *ad hoc* manner, Melander *et al.*[2] and Meacham *et al.*[4] showed that Kida's equations were Hamiltonian. Here, we briefly review the Kida reduction and derive the Hamiltonian structure by projecting the noncanonical Poisson bracket for the two-dimensional Euler equation onto quadratic moments of the vorticity. Constants of motion are described and related to the underlying Lie algebra structure, where new and natural sets of canonical variables are obtained. A qualitative description of the motion is given by comparing the dynamics of the Kida vortex, which is shown to possess a phase space described by the Lie group SO(2,1), to that of the free rigid body, which posseses the phase space SO(3).

As a model for an intrathermocline vortex in a shear flow, Meacham *et al.*[5] considered a blob of uniform potential vorticity embedded in an unbounded, uniformly stratified, quasigeostrophic flow. The motivation for this work, which is a generalization of the Kida reduction to an ellipsoid in the quasigeostrophic flow, was to understand the conditions under which a shear flow might cause a vortex to break up. In MPSZ it was conjectured, but not shown, that the equations which describe the ellipsoid are Hamiltonian. In Sec. IV this is shown by beginning from the noncanonical Poisson bracket that describes continuously stratified quasigeostrophic flow and projecting onto moments. The resulting moment algebra is decomposed into the direct sum of semisimple and solvable components. The decomposition allows one to obtain the Casimir constants of motion and points to natural sets of variables which can be used to classify the dynamics. In the absence of vertical shear, the system is integrable. Using the equations of motion based on the natural variables, we consider the way in which phase trajectories are perturbed by the addition of weak vertical shear. We demonstrate empirically the presence of chaotic dynamics near homoclinic trajectories in the original system.

In Sec. V the paper is summarized, concluding remarks are given, and generalizations are suggested.

II. VORTEX DYNAMICS AND MOMENT REDUCTION

A. Review of the noncanonical Hamiltonian structure of vorticity-like systems

Consider a class of vorticity-like systems with dynamics governed by

$$\frac{\partial \tilde{q}}{\partial t}+[\tilde{\psi},\, \tilde{q}]=0, \tag{5}$$

where $\tilde{q}(x,y,z,t)$ is a vorticity-like variable,

$$[f,\, g]:=\frac{\partial f}{\partial x}\frac{\partial g}{\partial y}-\frac{\partial g}{\partial x}\frac{\partial f}{\partial y} \tag{6}$$

is the normal Jacobian or Poisson bracket, and $\tilde{\psi}$ is a ''streamfunction'' that is related to $\tilde{q}$ by means of $\tilde{q}=L\tilde{\psi}$, where the linear operator L is formally self-adjoint, i.e.,

$$\int_D fLg\, dx\, dy\, dz=\int_D gLf\, dx\, dy\, dz. \tag{7}$$

Here D, the domain of integration, can be taken to be $\mathbb{R}^2$ in the case of the two-dimensional (2-D) Euler equation. The conserved field $\tilde{q}(x,y,t)$ is the scalar vorticity, and $L:=\nabla^2 = \partial^2/\partial x^2+\partial^2/\partial y^2$ so that

$$\tilde{q}=\nabla^2\tilde{\psi}=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\tilde{\psi}. \tag{8}$$

For continuously stratified quasigeostrophic flow, the domain is $\mathbb{R}^3$, $\tilde{q}(x,y,z,t)$ is the potential vorticity, and

$$\tilde{q}=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial}{\partial z}\frac{\hat{f}^2}{N^2}\frac{\partial}{\partial z}\right)\tilde{\psi}, \tag{9}$$

where $\hat{f}$ is the Coriolis parameter and $N(z)$ is the Brunt–Väisälä frequency. In the case of uniform stratification, which we assume in Sec. IV, z can be scaled by $N/\hat{f}$ so that the potential vorticity relation becomes isotropic: $\tilde{q} = (\partial^2/\partial x^2+\partial^2/\partial y^2+\partial^2/\partial z^2)\tilde{\psi}$. Multi-layer quasigeostrophic potential vorticity dynamics and other systems are given by different choices for L.

The noncanonical Poisson bracket for this class of system[15,24] is given by

$$\{F,G\}=\int_D \tilde{q}\left[\frac{\delta F}{\delta\tilde{q}},\frac{\delta G}{\delta\tilde{q}}\right] dx\, dy\, dz, \tag{10}$$

from which (5) is obtained in the form

$$\frac{\partial\tilde{q}}{\partial t}=\{\tilde{q},\tilde{H}\}, \tag{11}$$

with the Hamiltonian functional given by

$$\tilde{H}[\tilde{q}]=-\frac{1}{2}\int_D \tilde{q}\tilde{\psi}\, dx\, dy\, dz. \tag{12}$$

The evolution equation (11) can be verified by observing that $\delta\tilde{H}/\delta\tilde{q}=-\tilde{\psi}$, making use of the identity

$$\int_D f[g,\, h]\, dx\, dy\, dz=-\int_D g[f,\, h]\, dx\, dy\, dz, \tag{13}$$

from integration by parts and the neglect of surface terms (which is justifiable in the case of interest here where f has compact support), and by using the relation

$$\frac{\delta\tilde{q}(x',y',z',t)}{\delta\tilde{q}(x,y,z,t)}=\delta(x-x')\,\delta(y-y')\,\delta(z-z'),$$

which follows from (3).

In the examples considered below, we wish to include stationary background flows with horizontally uniform vorticity, $\overline{q(z)}$ and streamfunction, $\overline{\psi}$. The uniformity of $\overline{q}$ means that integrals such as those in (7) may not formally converge. This is easily remedied as follows. We introduce the decomposition

$$\tilde{\psi}=\overline{\psi(x,y,z)}+\psi(x,y,z,t),$$

$$\tilde{q}=\overline{q(z)}+q(x,y,z,t),\qquad \tilde{H}=\overline{H}+H[q],$$

where $q=L\psi$ and

$$H=-\int q\left(\overline{\psi}+\frac{1}{2}\,\psi\right). \tag{12$'$}$$

We will make the restriction that the perturbation vorticity, q, has compact support, although this could be relaxed a little. Then, using the self-adjoint property of L, we have that

$$\frac{\delta H}{\delta q}=-(\overline{\psi}+\psi).$$

The perturbation vorticity satisfies an evolution equation similar to (5):

$$\frac{\partial q}{\partial t}+[\overline{\psi}+\psi,q]=0. \tag{5$'$}$$

Defining a new Poisson bracket

$$\{F,G\}=\int_D q\left[\frac{\delta F}{\delta q},\,\frac{\delta G}{\delta q}\right] dx\, dy\, dz, \tag{10$'$}$$

this becomes

$$\frac{\partial q}{\partial t}=\{q,H\}. \tag{11$'$}$$

In addition to the modified Hamiltonian (12′) (the ‘‘excess energy’’), the system (5) conserves the Casimir invariant

$$C[q]:=\int_D \mathscr{C}(q)\ dx\ dy\ dz, \tag{14}$$

where $\mathscr{C}$ is an arbitrary function. Casimir invariants are defined by

$$\{C,F\}=0, \tag{15}$$

for all functionals F. This type of invariant is a property of the noncanonical Poisson bracket and should be distinguished from invariants that depend upon the particular form of the Hamiltonian, namely,

$$\{P,H\}=0. \tag{16}$$

Of course, $\{P\}\supset\{C\}$. The following linear momenta are examples of this latter type of invariant:

$$P_y=-\int_D xq\ dx\ dy\ dz, \quad P_x=\int_D yq\ dx\ dy\ dz, \tag{17}$$

and follow if H has spatial symmetries (invariance with respect to translations in y and x).

B. Reduction

There are physical situations in which parts of the fluid are behaving coherently so that the number of ‘‘interesting’’ degrees of freedom is finite, at least over some limited time. Examples include the evolution of a vortex blob of finite size in an external shear flow and the interactions of multiple blobs of vorticity. In the latter case, though, the vortex blobs may eventually filament in a complicated way; if they are not too close together, in the initial phase of their evolution the blobs will behave qualitatively like point vortices. In this phase, the many internal degrees of freedom that correspond to rearrangement of fluid parcels within each vortex may be relatively unimportant. We are interested in obtaining a kinematic reduction that allows us to focus on the degrees of freedom that dominate the dynamics when the vorticity field is distributed in coherent lumps. This amounts to finding a particular set of reduced variables for describing the dynamics that contain less information than $q(x,y,z,t)$. In general, this approach will yield low-order *approximations* to the full equations of motion. However, there are special cases for which the reduced equations are an exact representation of the flow dynamics. We provide examples of exact reduction in Secs. III and IV. Since we would like the set of reduced variables to inherit a Hamiltonian structure, we begin with the Poisson bracket of (10′). In actuality, we are seeking a Lie subalgebra associated with this bracket; this amounts to expressing the Poisson bracket in terms of projections of q, which will be seen to be an exercise in the chain rule for functional derivatives.

Suppose we have a set of functions $m^j(x,y,z)$ and define the projections of q on them

$$a^j=\int_D m^j q\ dx\ dy\ dz. \tag{18}$$

If the set of functions m^j is not complete, the transformation between q and the a's is not invertible; however, the chain rule can still be effected in ‘‘one direction.’’ To this end, suppose that the functionals of q that we choose to deal with are restricted so that their dependence on q occurs only through functions, f, of the moments a, i.e.,

$$F[q]=f(a), \tag{19}$$

and consider variations in a that are induced by arbitrary variations in q:

$$\delta a^i=\int_D m^i\,\delta q\ dx\ dy\ dz. \tag{20}$$

Variations in F and f are thus related according to

$$\delta F[q;\delta q]=\int_D \frac{\delta F}{\delta q}\ \delta q\ dx\ dy\ dz$$
$$=\delta f(a;\delta a)=\frac{\partial f}{\partial a^i}\ \delta a^i=\frac{\partial f}{\partial a^i}\int_D m^i\delta q\ dx\ dy\ dz. \tag{21}$$

Since δq is assumed to be arbitrary, comparison of the second and last terms of (21) results in

$$\frac{\delta F}{\delta q}=\frac{\partial f}{\partial a^i}\ m^i. \tag{22}$$

Substitution of (22) and a counterpart for the functional G into (10′) yields

$$\{F,G\}=\frac{\partial f}{\partial a^i}\frac{\partial g}{\partial a^j}\int_D q[m^i,\ m^j]\ dx\ dy\ dz=:\{f,g\} \tag{23}$$

with

$$\{f,g\}=:\frac{\partial f}{\partial a^i}\ J^{ij}\ \frac{\partial g}{\partial a^j}, \tag{24}$$

where the matrix J, the cosymplectic form, is given by

$$J^{ij}=\int_D q[m^i,\ m^j]\ dx\ dy\ dz. \tag{25}$$

The crucial closure property necessary for reduction is evident from (25), namely that J can be expressed in terms of the reduced variables, a. The moment reduction used below is a special case of a more general situation where reduction leads to Lie–Poisson form: if

$$[m^i,\ m^j]=c_k^{ij}m^k, \tag{26}$$

the cosymplectic form becomes

$$J^{ij}=c_k^{ij}a^k \tag{27}$$

and the Poisson bracket takes the form (4) with the a^i variables serving as coordinates.

III. QUADRATIC MOMENT REDUCTION—THE KIDA PROBLEM

A. Kida review

The Kida reduction presupposes a two-dimensional (z-independent) velocity field composed of an elliptical patch of uniform vorticity in a background shear flow. It is assumed that the elliptical vortex patch has unit vorticity, and that the background flow is given by a quadratic streamfunction,

$$\overline{\psi}=\tfrac{1}{4}\omega(x^2+y^2)+\tfrac{1}{4}e(x^2-y^2), \tag{28}$$

where ω is the background vorticity and the principal rates of strain in the directions $y=\pm x$ are $\pm\tfrac{1}{2}e$. Kida[1] showed that the subsequent evolution of the vortex patch maintains the elliptical shape, though the semi-major and semi-minor axes of the ellipse, a and b respectively, and the ellipse orientation, ϕ, are time dependent and governed by

$$\begin{aligned}\dot{a}&=\frac{a}{2}\,e\,\sin 2\phi,\\ \dot{b}&=-\frac{b}{2}\,e\,\sin 2\phi,\\ \dot{\phi}&=\frac{ab}{(a+b)^2}+\frac{1}{2}\,\omega+\frac{1}{2}\,e\,\frac{a^2+b^2}{a^2-b^2}\cos 2\phi.\end{aligned} \tag{29}$$

Incorporating the constraint of area preservation, these equations may be simplified,

$$\begin{aligned}\dot{\lambda}&=-e\lambda\,\sin 2\phi\\ \dot{\phi}&=\frac{\lambda}{(1+\lambda)^2}+\frac{\omega}{2}+\frac{e}{2}\,\frac{1+\lambda^2}{1-\lambda^2}\cos 2\phi,\end{aligned} \tag{30}$$

where $\lambda:=b/a$ is the aspect ratio of the ellipse.

The above equations can be expressed as a simple Hamiltonian system with one degree of freedom:[1,2,4,25]

$$\begin{aligned}\dot{\lambda}&=\frac{1}{2}\left(\frac{\lambda^2}{1-\lambda^2}\right)\frac{\partial H}{\partial\phi},\\ \dot{\phi}&=-\frac{1}{2}\left(\frac{\lambda^2}{1-\lambda^2}\right)\frac{\partial H}{\partial\lambda},\end{aligned} \tag{31}$$

where the Hamiltonian is given by

$$H=e\,\frac{1-\lambda^2}{\lambda}\cos 2\phi+\omega\,\frac{1+\lambda^2}{\lambda}+2\ln\frac{(1+\lambda)^2}{\lambda}. \tag{32}$$

Equations (31) are Hamiltonian and canonical up to the prefactor $\tfrac{1}{2}\lambda^2/(1-\lambda^2)$, which is easily transformed away. Other canonical variables are discussed in Sec. III H.

B. Bracket quadratic moment reduction

For ellipses, the quadratic moments completely characterize the orientation and aspect ratio. Therefore we shall examine the projection of q onto the functions

$$m^1=x^2,\quad m^2=xy,\quad m^3=y^2, \tag{33}$$

with the moments given by

$$a^i=\int_D q m^i\,dx\,dy,\quad i=1,2,3. \tag{34}$$

The closure property necessary for reduction, (26), follows by examining the product $[m^i,\,m^j]$:

$$\begin{aligned}&[m^1,\,m^2]=2x^2=2m^1,\quad [m^1,\,m^3]=4xy=4m^2,\\ &[m^2,\,m^3]=2y^2=2m^3.\end{aligned} \tag{35}$$

Therefore, the matrix J can be written in terms of the moments as follows:

$$J=\begin{pmatrix}0 & 2a^1 & 4a^2\\ -2a^1 & 0 & 2a^3\\ -4a^2 & -2a^3 & 0\end{pmatrix}. \tag{36}$$

Since J is proportional to a, this has the Lie–Poisson form, c.f. (4). In Sec. III D we will discuss the corresponding Lie algebra. Consequences of the form of J are discussed in Secs. III C and III D below. We postpone a consideration of H until Sec. III E.

The closure property observed above occurs for quadratic and lower moments, but in general fails for collections containing higher moments. However, there do exist special sets of higher moments that result in closure.

C. Casimir invariant of reduced system

As observed in Sec. II, associated with noncanonical Poisson brackets are special invariants known as Casimir invariants, which for the finite-dimensional bracket obtained above satisfiy

$$\{f,C\}=\frac{\partial f}{\partial a^i}\,J^{ij}\,\frac{\partial C}{\partial a^j}=0, \tag{37}$$

where f is an arbitrary function. Since f is an arbitrary function, the phase space gradient of a Casimir invariant corresponds to a null eigenvector of J. Since

$$\begin{pmatrix}0 & 2a^1 & 4a^2\\ -2a^1 & 0 & 2a^3\\ -4a^2 & -2a^3 & 0\end{pmatrix}\begin{pmatrix}a^3\\ -2a^2\\ a^1\end{pmatrix}=0, \tag{38}$$

it is seen that

$$C=a^1a^3-(a^2)^2 \tag{39}$$

is a Casimir and hence a constant of the motion.

In terms of the vorticity, q,

$$C=\left[\int_D x^2q\,dxdy\right]\left[\int_D y^2q\,dxdy\right]-\left[\int_D xyq\,dxdy\right]^2. \tag{40}$$

Observe that by Schwarz's inequality, $C\geqslant 0$ when q is uniform. C has a simple physical interpretation when q is uniform within an elliptical area centered on the origin (Kirchhoff's elliptical vortex). Then,

$$C=\frac{q^2}{16\pi^2}\,(\text{Area})^4. \tag{41}$$

In this case, constancy of C is equivalent to constancy of the vortex area.

We can make a related interpretation of C in the case of a spatially varying vorticity distribution with a Gaussian profile,

$$q(x,y)=Q\,\frac{\sqrt{\lambda_1\lambda_2}}{2\pi}\,e^{-(1/2)(\lambda_1 x^2+\lambda_2 y^2)}. \tag{42}$$

Contours of constant q are ellipses with semi-major and semi-minor axes in the ratio $(\lambda_1/\lambda_2)^{1/2}$. The area within the contour

$$x^2\lambda_1+y^2\lambda_2=1$$

is

$$A_{\mathrm{SD}}=\frac{\pi}{\sqrt{\lambda_1\lambda_2}},$$

and the value of the Casimir is

$$C=Q^2\,\frac{\pi^2}{16}\,\frac{1}{\lambda_1^2\lambda_2^2}.$$

Thus

$$C=\frac{Q^2}{16\pi^2}\,(A_{\mathrm{SD}})^4. \tag{43}$$

Equation (43) remains true even when the orientation of the elliptical Gaussian is rotated around the origin. Again we see that C is related to the fourth power of the area inside a particular vorticity contour of the vorticity distribution. This should remain true for any smooth vorticity distribution that contains vorticity of only a single sign and has a single extremum. However, it is unclear how to generalize this interpretation of C when the vorticity distribution is more complicated.

D. Lie algebra normal coordinates

The matrix (36), being linear in the dynamical variables, is of Lie–Poisson form and can be written as

$$J^{ij}=c_k^{ij}a^k, \tag{44}$$

where, as noted above, c_k^{ij} are the structure constants for some Lie algebra. Since the indices range over 1,2,3, this Lie algebra is of dimension three. It is known[26] that all Lie algebras of dimension three belong to one of nine equivalence classes, where equivalence is defined by identification under real coordinate transformations. It remains to determine which algebra is associated with (36). This is an easy task, which can be based upon a quantity called the Killing form, and leads to natural sets of coordinates, both for the algebra and for the dynamics of the Kida problem.

The Killing form, for the purposes here, is defined by

$$g^{ij}:=c_k^{it}c_t^{jk}. \tag{45}$$

Since g^{ij} is symmetric under the interchange of i and j, it possesses three real eigenvalues. If none of these eigenvalues vanish, i.e., g^{ij} is nondegenerate, then the algebra is called *semi-simple*. This is the case for the algebra associated with (36) for which the Killing form is

$$-8\begin{pmatrix} 0 & 0 & 2\\ 0 & 1 & 0\\ 2 & 0 & 0\end{pmatrix}. \tag{46}$$

The eigenvalues of (46) are 8 and ± 16. There are two semi-simple Lie algebras of dimension three, which are distinguished by the signature of the eigenvalues: either all the eigenvalues have the same sign or one sign is different. The first case is so(3), the Lie algebra associated with the Lie group SO(3), the group of rotations, while the second case, which applies to the algebra of (3.9) that was obtained here by reduction of the noncanonical bracket, is so(2,1), the algebra associated with the group SO(2,1), where the arguments indicate the number of eigenvalues with positive and negative signs, respectively.

In terms of the Killing form, the Casimir invariant for semi-simple algebras can be written as follows:

$$C=g_{ij}a^ia^j, \tag{47}$$

where g_{ij} is the inverse of g^{ij}. For the case here, (47) is equivalent to (39). In order for the expression (47) to be a Casimir it must satisfy

$$J^{ij}\,\frac{\partial C}{\partial a^j}=2c_k^{ij}g_{jt}a^ta^k=0. \tag{48}$$

With $a^t=:g^{ta}a_a$ and $a^k=:g^{kb}a_b$, (48) is equivalent to

$$c_k^{ij}g_{jt}g^{ta}a_ag^{kb}a_b=c_k^{ia}g^{kb}a_ba_a=c_k^{ia}c_t^{kl}c_l^{bt}a_ba_a=0, \tag{49}$$

where the first equality follows from $g_{it}g^{ta}=\delta_i^{\ a}$ and the second from (45), the definiton of g^{kb}. To establish the last equality we use the Jacobi identity for the structure constants,

$$c_k^{ia}c_t^{kl}+c_k^{al}c_t^{ki}+c_k^{li}c_t^{ka}=0, \tag{50}$$

which results in

$$c_k^{ia}c_t^{kl}c_l^{bt}a_ba_a=-c_t^{ki}(c_l^{bt}c_k^{al}-c_l^{at}c_k^{bl})a_ba_a=0, \tag{51}$$

where the last equality is now evident because of the antisymmetry in a and b of the term in parentheses.

One can define *normal coordinates* as those in which the Killing form is diagonal. For our present system normal coordinates can be obtained by the following orthogonal transformation:

$$z^1=(a^1+a^3)/\sqrt{2},\quad z^2=a^2,\quad z^3=(a^1-a^3)/\sqrt{2},$$

or equivalently

$$z=Aa,\quad A=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1\\ 0 & \sqrt{2} & 0\\ 1 & 0 & -1\end{pmatrix}.$$

The associated cosymplectic matrix transforms according to

$$\widetilde{J}=AJA^T=\begin{pmatrix} 0 & 2z^3 & -4z^2\\ -2z^3 & 0 & -2z^1\\ 4z^2 & 2z^1 & 0\end{pmatrix}$$

and can be expressed as $\widetilde{J}^{ij}=:\widetilde{c}_{\ k}^{\,ij}z^k$, which defines the structure constants in terms of the normal coordinates.

In what follows, we will use a nonorthogonal transformation to an alternative set of normal coordinates which has the advantage of making the Casimir symmetric with respect to z^2 and z^3 and simplifying the cosymplectic matrix, i.e.,

$$z^1=(a^1+a^3)/4, \quad z^2=a^2/2, \quad z^3=(a^1-a^3)/4, \tag{52}$$

or equivalently

$$z=Aa, \quad A=\frac{1}{4}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & -1 \end{pmatrix}. \tag{53}$$

The corresponding cosymplectic matrix transforms as

$$\widetilde{J}=AJA^T=\begin{pmatrix} 0 & z^3 & -z^2 \\ -z^3 & 0 & -z^1 \\ z^2 & z^1 & 0 \end{pmatrix} \tag{54}$$

and can be expressed as $\widetilde{J}^{\,ij}=:\widetilde{c}^{\,ij}_{k}z^k$, which defines the structure constants in terms of the normal coordinates.

In the normal coordinates, the Casimir invariant possesses the following diagonal form:

$$C=4[(z^1)^2-(z^3)^2-(z^2)^2]. \tag{55}$$

The surfaces of constant Casimir are hyperboloids of revolution with the Oz^1 axis the axis of symmetry. Since these surfaces extend to infinity, the algebra so(2,1), although semi-simple, is not compact. We note that C is a homogeneous polynomial in z^i. We can rescale any positive value of C simply by applying a uniform rescaling to the z^i without affecting the nature of the kinematical constraint imposed by the Casimir. Similarly we can rescale any negative C into any other negative C. Whether or not this rescaling affects the *dynamics* of the motion will depend on how the Hamiltonian is affected by the rescaling. In the particular case of the Kida ellipse, we know[27] that the dynamics are insensitive to the area of the vortex and so inspection of (41) tells us that the way in which trajectories on any single positive Casimir surface vary as e and ω are varied (and so as the positions of constant H surfaces vary) will provide a representative picture of all of the possible behavior of the elliptical vortex in shear. In Sec. III E, we will choose to fix the area of the elliptical vortex at π which means that $C=\pi^2/16$.

E. Hamiltonian moment reduction

Now we return to the remaining task of reduction, writing the Hamiltonian (32) in terms of the moments. It is at this point of the reduction process that we introduce the assumption that the initial condition for the vortex dynamics is an elliptical vortex patch. Since, as Kida[1] has shown, an initially elliptical vortex remains elliptical in background flows of the form of (28), the reduction is exact. The crucial reason for this is that the Hamiltonian can be written exactly in terms of the quadratic moments, a, which in turn determine the semi-axes and orientation of the ellipse.

The centroid position is determined by the linear moments, which together with the quadratic moments form a closed algebra. However, for the Kida problem these moments are not needed, since in the background flow (28) the vortex centroid remains fixed. In the case of two or more vortex patches with dynamics as described in Refs. 2, 22, and 23, the linear moments possess time dependence.

Relative to the fixed coordinate frame Oxy, the principal axes of the ellipse are determined by the time-dependent orientation $\phi(t)$, as described in Sec. III A. We define coordinates $\widetilde{\boldsymbol{x}}:=(\widetilde{x},\widetilde{y})$ in the frame instantaneously co-rotating with the ellipse:

$$\widetilde{\boldsymbol{x}}=\mathscr{M}^T\boldsymbol{x}, \tag{56}$$

where

$$\mathscr{M}^T=\begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}. \tag{57}$$

Using (56), moments in the co-rotating frame can be related to those relative to the nonrotating frame.

The Hamiltonian is the excess energy for the system, i.e., with the logarithmic divergence subtracted off, and has two parts, one associated with the background flow (28) and a contribution due to the elliptical vortex patch:

$$H=-\int_D q\left(\overline{\psi}+\frac{1}{2}\,\psi\right)\,dx\,dy=:\mathscr{H}+H', \tag{58}$$

where $q=\nabla^2\psi$ is unity inside the ellipse and zero outside, and H' describes the self-interaction.

From (28) we have

$$\overline{\psi}=\tfrac{1}{4}(\omega+e)m^1+\tfrac{1}{4}(\omega-e)m^3. \tag{59}$$

Recall that ω is the background vorticity and e is the background strain. From (59) the first term of the Hamiltonian is readily calculated:

$$\mathscr{H}=-\tfrac{1}{4}(\omega+e)a^1-\tfrac{1}{4}(\omega-e)a^3.$$

To evaluate the self-interaction term we use the expression for the streamfunction due to an elliptical vortex patch, which can be found in the work by Lamb (art. 159),[27]

$$\psi=\begin{cases} \dfrac{1}{2}\,\dfrac{\lambda\widetilde{x}^{\,2}+\widetilde{y}^{\,2}}{1+\lambda}-\dfrac{1}{4}+\psi_0, & \xi<\xi_0, \\[2ex] \dfrac{1}{2}\,(\xi-\xi_0)-\dfrac{1}{4}\,\dfrac{\lambda-1}{\lambda+1}\,e^{-2(\xi-\xi_0)}\cos 2\phi+\psi_0, & \xi>\xi_0, \end{cases} \tag{60}$$

where ξ is an elliptical coordinate,

$$\begin{aligned} \widetilde{x}&=\sqrt{(1-\lambda^2)/\lambda}\,\cosh\xi\,\cos\phi, \\ \widetilde{y}&=\sqrt{(1-\lambda^2)/\lambda}\,\sinh\xi\,\sin\phi, \end{aligned} \tag{61}$$

and $\tanh\xi_0=\lambda$. Here we have normalized the area of the ellipse to π so that the semi-major and semi-minor axes are $\lambda^{-1/2}$ and $\lambda^{1/2}$, respectively. Note that this fixes the value of the Casimir surface, on which the motion lies, to be $C=\pi^2/16$. The quantity ψ_0 is necessary for obtaining the correct behavior at infinity.[28] This quantity is not trivial since it depends upon the time-dependent ellipse aspect ratio, λ. To see this, we write ψ in terms of the Green's function as follows:

$$\psi=\int_D G(|\mathbf{x}-\mathbf{x}'|)q(\mathbf{x}')\,dx\,dy, \tag{62}$$

and observe that this expression asymptotes to

$$\psi \sim G(r) \int_D q\, dx\, dy - \frac{1}{r^2} G'(r) \mathbf{x} \cdot \int_D \mathbf{x}' q\, dx' dy'. \quad (63)$$

The terms of (63) are, respectively, the monopole and dipole terms of the two-dimensional multipole expansion. The important point here is that this representation of the streamfunction has only a ln(r) term, with no constant term. Requiring the same of expression (60) selects ψ_0. Since

$$\xi - \xi_0 \sim \ln(r) + \tfrac{1}{2} \ln(\lambda) - \ln(1+\lambda) + \ln(2), \quad (64)$$

the asymptotic form of ψ, according to (60), is

$$\psi \sim \tfrac{1}{2} \ln(r) + \tfrac{1}{4} [\ln \lambda - 2 \ln(1+\lambda)] + \tfrac{1}{2} \ln(2) + \psi_0. \quad (65)$$

Thus

$$\psi_0 = -\frac{1}{4} \ln \frac{\lambda}{(1+\lambda)^2} + \text{const} = -\frac{1}{4} \ln \Omega + \text{const}, \quad (66)$$

where $\Omega := \lambda/(1+\lambda)^2$ is the natural rotation rate of the Kirchhoff ellipse. The constant terms do not depend on the time-dependent ellipse parameters and therefore can be dropped from the Hamiltonian.

We can now evaluate the self-interaction energy. First, the part of the streamfunction that depends upon $\tilde{x}$ and $\tilde{y}$ yields

$$-\frac{1}{4} \int_D \frac{\lambda \tilde{x}^2 + \tilde{y}^2}{1+\lambda} dx\, dy = -\frac{\pi}{16} \frac{\lambda+1}{1+\lambda} = -\frac{\pi}{16}, \quad (67)$$

which is constant and can be dropped. The integral of the $-\frac{1}{4}$ term likewise is not important. This leaves only the contribution from ψ_0. Using Eq. (67) and the fact that the area of the vortex has been set to π gives

$$H' = \frac{\pi}{8} \ln \Omega. \quad (68)$$

The complete Hamiltonian is thus

$$H = -\frac{1}{4} (\omega + e) a^1 - \frac{1}{4} (\omega - e) a^3 + \frac{\pi}{8} \ln \Omega. \quad (69)$$

In (69) H still depends upon λ in addition to the moments a^1 and a^3. It remains for us to express the rotation frequency $\Omega(\lambda)$ in terms of the moments. The moments are seen to be

$$\begin{aligned} a^1 &= \frac{\pi}{4} (\lambda^{-1} \cos^2 \phi + \lambda \sin^2 \phi), \\ a^2 &= \frac{\pi}{4} (\lambda^{-1} - \lambda) \sin \phi \cos \phi, \\ a^3 &= \frac{\pi}{4} (\lambda^{-1} \sin^2 \phi + \lambda \cos^2 \phi), \end{aligned} \quad (70)$$

Equations (70) imply

$$\begin{aligned} a^1 + a^3 &= \frac{\pi}{4} \left(\frac{1}{\lambda} + \lambda \right), \\ a^1 - a^3 &= \frac{\pi}{4} \left(\frac{1}{\lambda} - \lambda \right) \cos 2\phi, \\ a^1 a^3 - (a^2)^2 &= \frac{\pi^2}{16}, \\ \frac{4}{\pi} \left(a^1 + a^3 + \frac{\pi}{2} \right) &= \frac{(1+\lambda)^2}{\lambda}. \end{aligned} \quad (71)$$

Using the last of Eqs. (71), Ω can be expressed in terms of the moments as follows:

$$\Omega^{-1} = \frac{4}{\pi} \left(a^1 + a^3 + \frac{\pi}{2} \right). \quad (72)$$

The remaining equations of (71) are recorded for later use. The Hamiltonian, H, is then given by

$$\begin{aligned} H(a) = &-\frac{1}{4} (\omega + e) a^1 - \frac{1}{4} (\omega - e) a^3 \\ &- \frac{\pi}{8} \ln \left[a^1 + a^3 + \frac{\pi}{2} \right], \end{aligned} \quad (73)$$

where we have dropped a constant term. Making use of

$$a^1 = 2(z^1 + z^3), \quad a^3 = 2(z^1 - z^3), \quad a^2 = 2z^2, \quad (74)$$

the Hamiltonian in terms of the coordinates z of Sec. III D becomes

$$H(z) = -\omega z^1 - e z^3 - \frac{\pi}{8} \ln \left[z^1 + \frac{\pi}{8} \right] + \text{const} \quad (75)$$

and is $-\pi/16$ times the quantity (32).

F. Equations of motion

The equations of motion, either in terms of a or z, are given in a straightforward manner. Using

$$\dot{a}^i = \{a^i, H\} = J^{ij} \frac{\partial H}{\partial a^j}, \quad (76)$$

with (36) implies

$$\begin{aligned} \dot{a}^1 &= 2a^1 \frac{\partial H}{\partial a^2} + 4a^2 \frac{\partial H}{\partial a^3}, \\ \dot{a}^2 &= 2a^3 \frac{\partial H}{\partial a^3} - 2a^1 \frac{\partial H}{\partial a^1}, \\ \dot{a}^3 &= -4a^2 \frac{\partial H}{\partial a^1} - 2a^3 \frac{\partial H}{\partial a^2}, \end{aligned} \quad (77)$$

where, with (73), we arrive at the noncanonical Hamiltonian system

$$\dot{a}^1 = -a^2 \left\{ \omega - e + \frac{\frac{\pi}{2}}{a^1 + a^3 + \frac{\pi}{2}} \right\},$$

$$\dot{a}^2=\frac{1}{2}\left\{ e(a^1+a^3)+\omega(a^1-a^3)+\frac{\frac{\pi}{2}(a^1-a^3)}{a^1+a^3+\frac{\pi}{2}}\right\}, \tag{78}$$

$$\dot{a}^3=a^2\left\{ \omega+e+\frac{\frac{\pi}{2}}{a^1+a^3+\frac{\pi}{2}}\right\}.$$

Similarly, using (54) and (75) in

$$\dot{z}^i=\{z^i,H\}=\widetilde{J}^{\,ij}\,\frac{\partial H}{\partial z^j} \tag{79}$$

yields

$$\begin{aligned}
\dot{z}^1&=ez^2,\\
\dot{z}^2&=ez^1+z^3\left\{\omega+\frac{\pi/8}{z^1+\pi/8}\right\},\\
\dot{z}^3&=-z^2\left\{\omega+\frac{\pi/8}{z^1+\pi/8}\right\}.
\end{aligned} \tag{80}$$

As a check, we show that Eqs. (78) imply (30), the equations derived by Kida. Differentiating (71) yields

$$\begin{aligned}
\dot{\lambda}(1-\lambda^{-2})\frac{\pi}{4}&=(\dot{a}^1+\dot{a}^3)=2ea^2\\
&=-\frac{\pi}{4}\,e\lambda(1-\lambda^{-2})\sin 2\phi,
\end{aligned} \tag{81}$$

where the last equality follows from (70). Therefore

$$\dot{\lambda}=-e\lambda\ \sin 2\phi. \tag{82}$$

Similarly, from (71),

$$\begin{aligned}
(\dot{a}^1-\dot{a}^3)\ \frac{4}{\pi}&=\frac{d}{dt}\ (\lambda^{-1}-\lambda)\ \cos 2\phi\\
&=-\dot{\lambda}(\lambda^{-2}+1)\cos 2\phi\\
&\quad-2(\lambda^{-1}-\lambda)\sin 2\phi\dot{\phi},
\end{aligned} \tag{83}$$

while from (78) and (70)

$$(\dot{a}^1-\dot{a}^3)\ \frac{4}{\pi}=-(\lambda^{-1}-\lambda)\sin 2\phi\left\{\omega+\frac{2\lambda}{(1+\lambda)^2}\right\}. \tag{84}$$

Equating the terms of (83) to (84) and making use of (82) yields

$$\dot{\phi}=\frac{\lambda}{(1+\lambda)^2}+\frac{\omega}{2}+\frac{e}{2}\,\frac{1+\lambda^2}{1-\lambda^2}\cos 2\phi. \tag{85}$$

Equations (82) and (85) are the equations of Kida.

G. A geometric characterization of dynamics: Comparison to rigid body

From the preceding sections it is evident that in the Hamiltonian description of a dynamical system, one can distinguish two aspects: the ''dynamics'' as embodied in the form of the Hamiltonian and the ''kinematics'' represented by the algebraic properties of the underlying cosymplectic structure. In the case of three-dimensional systems like that of the Kida problem, the kinematics implies that the system is integrable, i.e., that one can use the Hamiltonian and the Casimir invariant to write down a quadrature that determines the dynamics. For systems of this type there is a geometrical way to understand the qualitative nature of the solutions. To demonstrate this, we now compare the Kida problem to the free rigid body. (This should be compared to the characterizations given by Meacham *et al.*[4] and Bayly *et al.*[25]).

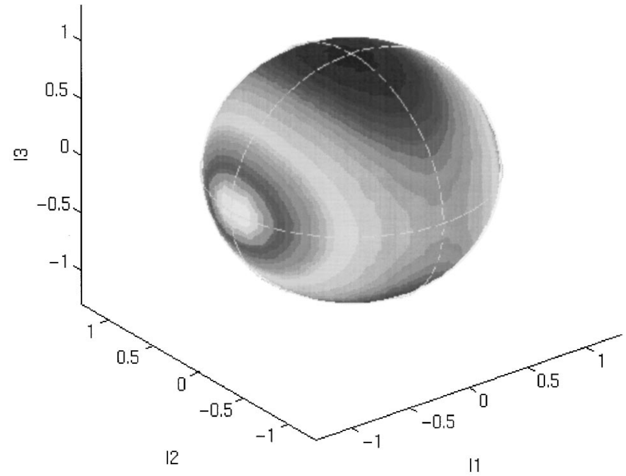

FIG. 1. An isosurface of the Casimir, C, for the rigid body problem, (86)–(88). The value of the Hamiltonian on this surface is shown by the shading. Trajectories are constrained to follow lines of constant shading (Hamiltonian) on the Casimir surface.

The free rigid body is governed by Euler's equations, which is the statement of zero torque in the rotating principal axes frame of reference. They can be written as follows:

$$\dot{\ell}_i=\{\ell_i,H\}=-\epsilon_{ijk}\ell_k\frac{\partial H}{\partial \ell_j}, \tag{86}$$

where ℓ is the angular momentum, and

$$H(\ell)=\frac{1}{2}\left(\frac{\ell_1^2}{I_1}+\frac{\ell_2^2}{I_2}+\frac{\ell_3^2}{I_3}\right), \tag{87}$$

with I_1, I_2, and I_3 being the three principal moments of inertia. The structure constants, ϵ_{ijk}, are represented by the Levi–Civita symbol for the completely antisymmetric tensor. (Note, since the structure constant is completely antisymmetric all the indices have been written in the down position. Repeated indices are still summed.) The algebra associated with the cosymplectic form in this case is so(3) and the Casimir invariant is the square of the magnitude of the angular momentum,

$$C(\ell)=\ell_1^2+\ell_2^2+\ell_3^2. \tag{88}$$

Conventionally, the qualitative description of the rigid body dynamics is given by examining the intersection of the Casimir sphere with the Hamiltonian ellipsoid. This is depicted in Fig. 1, where we have selected a value for C and then used a grey scale to show the values of H on the Casimir surface. Lines of constant shading correspond to the curves along which the Hamiltonian ellipsoids intersect the Casimir sphere. The principal moments of inertia are assumed to be distinct. As H increases, we first observe a point of tangency, which corresponds to the equilibrium point of rotation about the axis of the dominant principal moment of inertia. The nearby ellipses of intersection, for larger values of H, indicate that this equilibrium point is stable. As H is increased further the point of tangency corresponding to the equilibrium point of rotation about the intermediate principal axis is observed. Nearby locally hyperbolic intersections indicate that this equilibrium point is unstable. Finally, for still larger values of H the point of tangency corresponding to stable rotations about the smallest principal axis is seen. Hence, an examination of the intersection has characterized the equilibrium points and qualitative nature of the solutions of this system.

For the Kida problem in normal coordinates, z, the Casimir surfaces are hyperboloids of revolution and the Oz^1 axis is an axis of symmetry. From (41), we see that, on physical grounds, we are restricted to the sheets $C>0$. These fall into two groups—those wholly above the plane $z_1=0$ and those wholly below. Since $a_1 + a_3$ must be $\geqslant 0$, we have that $z_1 \geqslant |z_3|$ and so we are restricted to the sheets in $z_1>0$. These surfaces are depicted in Fig. 2. A simplification occurs in the Kida problem because, in terms of the z coordinates, the Hamiltonian has a symmetry direction; i.e., it is independent of z^2. Surfaces of constant H are curved sheets with symmetry in the z^2 direction and these sheets can intersect the Casimir hyperboloid in various ways depending on the parameters ω and e. Because of the symmetry we need only examine these intersections in the z^1z^3 plane in order to understand the motion. In Fig. 2 the various kinds of intersections are depicted. Case (a) of the figure shows intersections of the Hamiltonian surface with the Casimir hyperboloid that correspond to two types of trajectory. One is a curve that extends to infinity and is topologically equivalent to a hyperbola. (The reader must imagine the continuation of the intersection in the z^2 coordinate.) This type of intersection represents a continual elongation of the elliptical vortex patch; it typically occurs when the background strain e is large. The second corresponds to an intersection that is topologically circular but does not enclose the z^1 axis. The motion in this case corresponds to nutation of the elliptical vortex patch. Case (b) of the figure represents an intersection that corresponds to a closed curve, topologically equivalent to a circle that *does* enclose the z^1 axis. This type of intersection represents a rotation of the elliptical vortex patch with periodic dependence in the aspect ratio, λ. From Eqs. (52), (70), and (71) we see that motion around the circle is related to rotation of the patch according to

$$\tan \overline{\phi} := z^2/z^3 = \tan 2\phi, \tag{89}$$

and similarly since $z^1 = (\pi/16)(1/\lambda + \lambda)$, excursion in the z^1 coordinate corresponds to variation in the aspect ratio. Warping these coordinates yields action-angle variables. A further case (not shown) relies on the effect of the logarithm in the Hamiltonian to produce two regions of nutation delineated by a ''figure-8'' separatrix. (For details, see Refs. 4 and 25.)

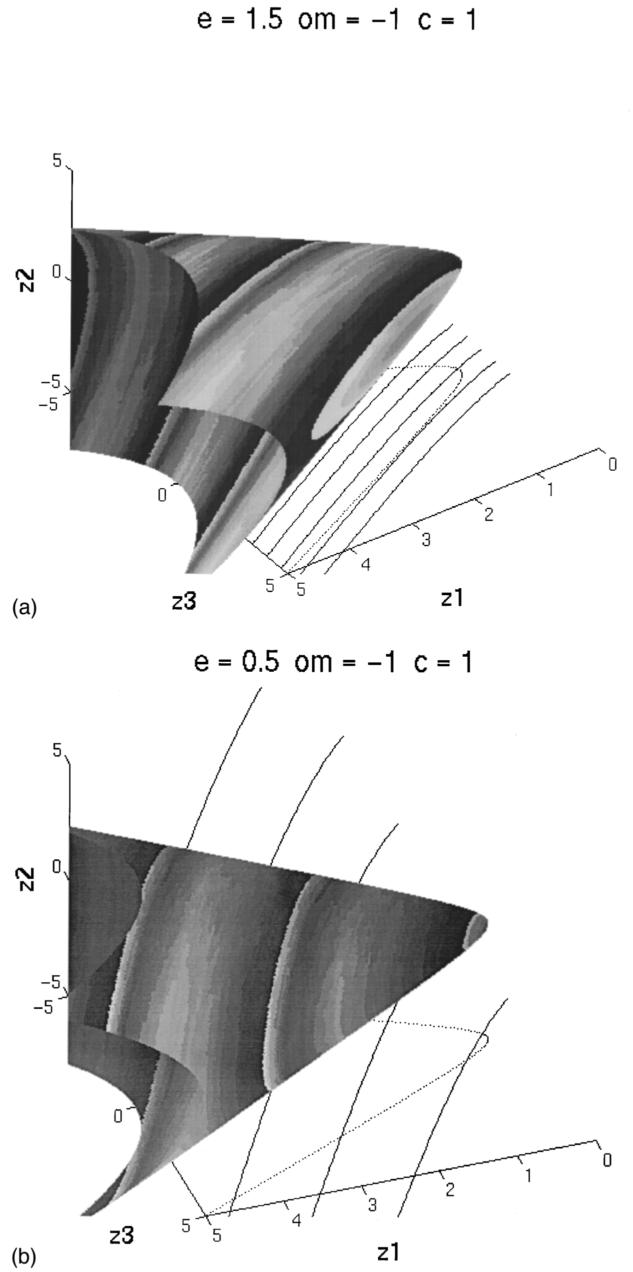

FIG. 2. Each plot shows an isosurface of the Casimir, C, for the Kida vortex in shear. The shading corresponds to values of the Hamiltonian, H, at different points on the Casimir surface. Trajectories are constrained to follow intersections of the constant H and C surfaces. (Because of the z^2 independence of H, surfaces of constant H are sheets parallel to Oz^2.) The projection of the Casimir surface on the z^1z^3 plane is indicated by the dotted curves; the intersections of various sheets of constant H with the z^1z^3 plane are shown with solid curves. (a) Case: $C=1$, $e=1.5$, $\omega=-1$ includes two types of trajectory: (i) open (hyperbola-like) and (ii) closed trajectories that do not circle the Oz^1 axis. (b) Case: $C=1$, $e=0.5$, $\omega=-1$ exhibits closed trajectories that circle the Oz^1 axis.

H. Reduction of order using a Casimir—canonical coordinates

We can use the Casimir $C=4[(z^1)^2-(z^3)^2-(z^2)^2]$ [see (55)] to reduce the system (80) as follows. We introduce coordinates b defined by

$$b^1=z^2, \quad b^2=z^3, \quad b^3=C=4[(z^1)^2-(z^3)^2-(z^2)^2]. \tag{90}$$

Then

$$\frac{\partial H}{\partial z^1}=8[C/4+(b^1)^2+(b^2)^2]^{1/2}\frac{\partial H}{\partial b^3},$$

$$\frac{\partial H}{\partial z^2}=\frac{\partial H}{\partial b^1}-8b^1\frac{\partial H}{\partial b^3},\quad \frac{\partial H}{\partial z^3}=\frac{\partial H}{\partial b^2}-8b^2\frac{\partial H}{\partial b^3},$$

where, (80) becomes

$$\begin{aligned}\dot{b}^1&=-8[C/4+(b^1)^2+(b^2)^2]^{1/2}\frac{\partial H}{\partial b^2},\\ \dot{b}^2&=8[C/4+(b^1)^2+(b^2)^2]^{1/2}\frac{\partial H}{\partial b^1},\\ \dot{b}^3&=0.\end{aligned}\tag{91}$$

Thus we reduce the problem to a Hamiltonian system with one degree of freedom [which is therefore integrable—phase trajectories are just contours of H over the (b^1,b^2) plane]. The use of the Casimir as a coordinate brought about two simplifications that follow directly from the defining property of a Casimir, $[F, C]=0$ for arbitrary functionals F: (i) one coordinate, b^3—the Casimir, is a constant, (ii) $\partial H/\partial b^3$ does not appear on the right-hand side. We will employ a similar technique to reduce the ellipsoidal vortex problem in Sec. IV.

Given the normal form of the algebra asociated with the Poisson bracket, we can deduce two natural families of canonical variables that are near to action-angle variables. The first set of variables, which is appropriate for bounded motion, is given by $(z^1,\overline{\phi},C)$, where

$$\overline{\phi}=\tan^{-1}(z^2/z^3).\tag{92}$$

Here z^1 is the coordinate along the symmetry axis, while $\overline{\phi}$ is the angle around the closed curve defined by the intersection of the plane $z^1=\text{const}$ with the hyperboloid $C=\text{const}$. Action-angle variables would be obtained by warping these coordinates so that the intersection is a circle.

The second set, which is appropriate for motion that asymptotes, is given by $(\psi,\tilde{z}^2,C)$, where

$$\psi=\tanh^{-1}(z^1/\tilde{z}^3).\tag{93}$$

Here the $O\tilde{z}^2$ and $O\tilde{z}^3$ axes are given by a rotation of the Oz^2 and Oz^3 axes through an arbitrary fixed angle around Oz^1. Now $O\tilde{z}^2$ is a coordinate direction normal to the symmetry axis and lying in a plane that includes the symmetry axis. ψ is a pseudo-angle denoting position along one of the two hyperbolae that result from the intersection of the plane perpendicular to $O\tilde{z}^2$ that includes the symmetry axis, and the hyperboloid C = const.

IV. ELLIPSOIDAL VORTEX IN CONTINUOUSLY STRATIFIED QUASIGEOSTROPHIC FLOW

A. MPSZ review

The intrathermocline vortex model of MPSZ considered an ellipsoidal blob of uniform potential vorticity embedded in an unbounded, uniformly stratified, quasigeostrophic flow. The background flow of this model is given by a streamfunction of the form

$$\overline{\psi}=\tfrac{1}{4}\omega(x^2+y^2)+\tfrac{1}{4}e(x^2-y^2)-\tau yz.\tag{94}$$

In MPSZ it was shown that an initially ellipsoidal blob of potential vorticity will remain ellipsoidal for all future times, which is clearly a generalization of the Kida result of Sec. III A.

In the MPSZ model, the motion of the ellipsoid is described by the three variables that describe the shape of the ellipsoid—the semiaxis lengths, $a(t)$, $b(t)$, $c(t)$—and three that describe its orientation— the Euler angles $\phi(t)$, $\theta(t)$, $\psi(t)$. The equations that govern these variables are given in the Appendix. These equations are rather complicated, a fact which limits their utility and makes it difficult to classify all of the modes of behavior of the vortex. It was noted in MPSZ that the equations (A1) and (A2) possess conserved quantities: vortex volume, particle height, and excess energy. Volume conservation can be exploited quite readily to reduce the system from sixth order to fifth order, but it is cumbersome, without the insight afforded by the Hamiltonian structure, to achieve any further reduction of order by using the other integrals of motion. The Hamiltonian moment approach leads to a considerably simpler formulation of the problem.

B. Moment reduction

The state (shape and orientation) of an ellipsoid is uniquely determined by the values of its six quadratic moments defined by

$$m^1=x^2,\quad m^2=xy,\quad m^3=y^2,\quad m^4=yz,$$
$$m^5=zx,\quad m^6=z^2\tag{95}$$

and

$$a^i=\int_D qm^i\,dx\,dy\,dz.\tag{96}$$

(Expressions for the a's in terms of axis lengths and Euler angles can be found in the Appendix.) For this selection of the m's, closure is achieved; in light of the above and Sec. II B, the cosymplectic matrix is seen to be

$$J=\begin{pmatrix}0&2a^1&4a^2&2a^5&0&0\\-2a^1&0&2a^3&a^4&-a^5&0\\-4a^2&-2a^3&0&0&-2a^4&0\\-2a^5&-a^4&0&0&-a^6&0\\0&a^5&2a^4&a^6&0&0\\0&0&0&0&0&0\end{pmatrix},\tag{97}$$

which, being linear in the a's is of Lie–Poisson form.

C. Casimirs of reduced system

We search for Casimirs as in Sec. III C, which amounts to finding null eigenvectors of the cosymplectic matrix J. The characteristic equation of this matrix has the form

$$\lambda^2(\lambda^4+A\lambda^2+B)=0,$$

where λ is an eigenvalue; since there are only two zero roots, there are only two independent Casimirs. The first Casimir is seen immediately to be

$$C_1=a^6, \tag{98}$$

while the second Casimir, which is found by calculating the second null eigenvector and then integrating, is the quantity

$$\begin{aligned} C_2=4[&2a^2a^4a^5+a^1a^3a^6-a^1(a^4)^2-a^3(a^5)^2\\ &-a^6(a^2)^2]. \end{aligned} \tag{99}$$

A discussion similar to that of Sec. III C reveals that these invariants correspond to an effective height and volume of the ellipsoid. For the case of a uniform blob of vorticity it was shown in MPSZ that C_1 corresponds to the conservation of particle height in the quasigeostrophic system and C_2 was seen to be proportional to the fifth power of the ellipsoidal volume. (In quasigeostrophic flows, fluid parcels maintain their z coordinate, even though horizontal velocities are z dependent. Vertical velocity is relegated to a higher order in the quasigeostrophic approximation.[29])

D. Lie algebra splitting—normal coordinates

It is well known that Lie algebras can be split into the sum of a semi-simple part plus a part that is called *solvable* (see, e.g., Ref. 26). We will not go into the details of how to effect this in the general case, but simply present the following transformation to normal coordinates:

$$\begin{aligned} z^1&=\frac{1}{4}\left(a^1+a^3-\frac{(a^4)^2}{a^6}-\frac{(a^5)^2}{a^6}\right),\\ z^2&=\frac{1}{2}\left(a^2-\frac{a^4a^5}{a^6}\right),\\ z^3&=\frac{1}{4}\left(a^1-a^3+\frac{(a^4)^2}{a^6}-\frac{(a^5)^2}{a^6}\right),\\ z^4&=a^4,\quad z^5=a^5,\quad z^6=a^6, \end{aligned} \tag{100}$$

which has the inverse transformation

$$\begin{aligned} a^1&=2(z^1+z^3)+\frac{(z^5)^2}{z^6},\\ a^2&=2z^2+\frac{z^5z^4}{z^6},\\ a^3&=2(z^1-z^3)+\frac{(z^4)^2}{z^6},\\ a^4&=z^4,\quad a^5=z^5,\quad a^6=z^6. \end{aligned} \tag{101}$$

The cosymplectic form in the normal coordinates is conveniently obtained by calculating $\tilde{J}^{ij}=\{z^i,z^j\}$, e.g.,

$$\tilde{J}^{12}=\{z^1,z^2\}=z^3,$$

whence

$$\tilde{J}=\begin{pmatrix} 0 & z^3 & -z^2 & 0 & 0 & 0\\ -z^3 & 0 & -z^1 & 0 & 0 & 0\\ z^2 & z^1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & -z^6 & 0\\ 0 & 0 & 0 & z^6 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{102}$$

The transformation $J\to\tilde{J}$ has evidently split the algebra into two blocks, i.e., into the direct sum of two 3×3 algebras. The algebra of the upper diagonal block is identical to the semi-simple algebra of Sec. III, while that of the lower diagonal block is *solvable*. An algebra is solvable if its sequence of derived algebras, i.e., the algebras of products, eventually reduces to $\{0\}$. In this case

$$\begin{aligned} \mathscr{L}'&=\{\{z^4,z^5\},\{z^4,z^6\},\{z^5,z^6\}\}=\{-z^6,0,0\},\\ \mathscr{L}''&=\{0\}. \end{aligned} \tag{103}$$

The Casimir for the upper algebra is clearly the same as that of Sec. III,

$$C_U=4[(z^1)^2-(z^2)^2-(z^3)^2], \tag{104}$$

while that of the lower algebra is

$$C_L=z^6.$$

To see that (104) is equivalent to (99) we substitute the transformation for the a's into (104), and obtain

$$\begin{aligned} C_U=\frac{1}{a^6}\,(&2a^2a^4a^5+a^1a^3a^6-a^1(a^4)^2-a^3(a^5)^2\\ &-(a^2)^2a^6). \end{aligned} \tag{105}$$

The normal coordinates have a relatively simple form when expressed in terms of spatial integrals:

$$\begin{aligned} C_Lz^1&=\frac{1}{4}\left\{\int z^2\int (x^2+y^2)-\left(\int yz\right)^2-\left(\int zx\right)^2\right\},\\ C_Lz^2&=\frac{1}{2}\left\{\int z^2\int xy-\int yz\int zx\right\},\\ C_Lz^3&=\frac{1}{4}\left\{\int z^2\int (x^2-y^2)+\left(\int yz\right)^2-\left(\int zx\right)^2\right\}. \end{aligned}$$

E. Moment Hamiltonian

In a manner analogous to Sec. III E, we turn to the task of writing the excess energy, the Hamiltonian, in terms of the moments. Since MPSZ have shown that an initially ellipsoidal vortex remains ellipsoidal in the background flow of (94), the reduction is exact; the Hamiltonian can be written exactly in terms of the moments, which uniquely determine the shape and orientation of the ellipsoid.

As in Sec. III E, the excess energy is again the Hamiltonian and is given by

$$H=-\int_D q\left(\overline{\psi}+\frac{1}{2}\,\psi\right)\,dx\;dy\;dz=:\mathscr{H}+H', \tag{106}$$

but now $\overline{\psi}$ is the contribution of the background flow, as given by (94), ψ is due to the uniform ellipsoidal vortex, and $q=(\partial^2/\partial x^2+\partial^2/\partial y^2+\partial^2/\partial z^2)\psi$ is unity inside the ellipsoid and zero outside. The integral involving the background flow is readily seen to be

$$\mathscr{H}=-\tfrac{1}{4}(\omega+e)a^1-\tfrac{1}{4}(\omega-e)a^3+\tau a^4. \tag{107}$$

From Chandrasekhar[30] or Ref. 5,

$$H'=\frac{2\pi}{15}(abc)^2\int_0^\infty ds\ K(s),$$

$$K(s)=[(a^2+s)(b^2+s)(c^2+s)]^{-1/2}, \tag{108}$$

where a, b, and c are the principal axes lengths and the quantity abc is proportional to the volume and is fixed.

To write this in terms of moments, we will use two Cartesian coordinate systems: $Oxyz$, which is fixed with respect to the underlying f-plane and $O\tilde{x}\,\tilde{y}\,\tilde{z}$, which moves with the principal axes of the ellipsoid. In both cases, the origin coincides with the center of the ellipsoid. The transformation between the fixed and co-rotating reference frames is given by the following expression in term of the Euler-angles:

$$\tilde{\mathbf{x}}=\mathscr{M}^T\mathbf{x} \tag{109}$$

where

$$\mathscr{M}^T=\begin{pmatrix}\cos\psi & \sin\psi & 0\\ -\sin\psi & \cos\psi & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\cos\theta & 0 & -\sin\theta\\ 0 & 1 & 0\\ \sin\theta & 0 & \cos\theta\end{pmatrix}\times\begin{pmatrix}\cos\phi & \sin\phi & 0\\ -\sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{pmatrix} \tag{110}$$

Equation (109) will be used to relate moments in the co-rotating frame to those in the nonrotating frame for evaluation of the Hamiltonian.

First define a matrix of moments

$$\mathscr{A}_{ij}=\int_D qx_ix_j=\begin{pmatrix}a^1 & a^2 & a^5\\ a^2 & a^3 & a^4\\ a^5 & a^4 & a^6\end{pmatrix}$$

Now,

$$\mathscr{A}_{ij}=\mathscr{M}_{ip}\mathscr{M}_{jq}\tilde{\mathscr{A}}_{pq},$$

where

$$\tilde{\mathscr{A}}_{pq}=\int_D q\tilde{x}_p\tilde{x}_q=\begin{pmatrix}Va^2 & 0 & 0\\ 0 & Vb^2 & 0\\ 0 & 0 & Vc^2\end{pmatrix},$$

where $V=4\pi abc/15$ is proportional to the volume of the ellipsoid and will be normalized to unity. The eigenvalues of $\mathscr{A}$ and $\tilde{\mathscr{A}}$ will be the same so we may identify the coefficients in the respective characteristic equations

$$\begin{aligned}\mathscr{A}:\quad & \lambda^3-[a^1+a^3+a^6]\lambda^2+[a^3a^6+a^6a^1+a^1a^3\\ & -\{(a^2)^2+(a^4)^2+(a^5)^2\}]\lambda-[2a^2a^4a^5+a^1a^3a^6\\ & -a^1(a^4)^2-a^3(a^5)^2-a^6(a^2)^2]=0,\\ \tilde{\mathscr{A}}:\quad & \lambda^3-V[a^2+b^2+c^2]\lambda^2+V^2[b^2c^2+c^2a^2\\ & +a^2b^2]\lambda-V^3[a^2b^2c^2]=0.\end{aligned} \tag{111}$$

This allows us to evaluate the coefficients of the various powers of s in the expression (108) for $K(s)$

In terms of the moments, the vortex Hamiltonian is

$$H'=\frac{1}{2}(C_LC_U)^{1/2}\int_0^\infty ds\ \tilde{K}(s), \tag{112}$$

where

$$\begin{aligned}\tilde{K}(s)^{-2}=s^3&+[a^1+a^3+a^6]s^2+[a^3a^6+a^6a^1+a^1a^3\\ &-\{(a^2)^2+(a^4)^2+(a^5)^2\}]s+[2a^2a^4a^5\\ &+a^1a^3a^6-a^1(a^4)^2-a^3(a^5)^2-a^6(a^2)^2].\end{aligned}$$

In terms of the normal coordinates of Sec. IV D the Hamiltonian takes the form

$$\begin{aligned}H=&-\omega\left[z^1+\frac{1}{4}\left(\frac{(z^4)^2+(z^5)^2}{z^6}\right)\right]\\ &-e\left[z^3-\frac{1}{4}\left(\frac{(z^4)^2-(z^5)^2}{z^6}\right)\right]+\tau z^4\\ &+\frac{1}{2}(C_LC_U)^{1/2}\int_0^\infty ds\ \hat{K}(s)\end{aligned} \tag{113}$$

where

$$\hat{K}(s)=[s^3+p_2s^2+p_1s+p_0]^{-1/2}$$

and

$$p_2=[4z^1z^6+(z^4)^2+(z^5)^2+(z^6)^2]/z^6,$$

$$\begin{aligned}p_1=\frac{2}{z^6}&[z^1\{2(z^6)^2+(z^4)^2+(z^5)^2\}+z^3\{(z^4)^2-(z^5)^2\}\\ &-2z^2z^4z^5]+4[(z^1)^2-(z^2)^2-(z^3)^2],\end{aligned}$$

$$p_0=4z^6[(z^1)^2-(z^2)^2-(z^3)^2].$$

F. Moment equations of motion

As in Sec. III, the equations of motion are given by

$$\dot{a}^i=\{a^i,H\}=J^{ij}\frac{\partial H}{\partial a^j}.$$

As noted above, we use the chain rule to evaluate the equations of motion, which means we need to calculate $\partial H'/\partial a^j$. First note that

$$\frac{\partial H'}{\partial[(a)^2]}=H_2'+(b^2+c^2)H_1'+b^2c^2H_0', \tag{114}$$

where

$$H_j' = -\frac{1}{4}(C_L C_U)^{1/2} \int_0^\infty s^j K^3(s)\, ds$$

and $\partial H'/\partial b^2$ and $\partial H'/\partial c^2$ are given by expressions similar to (114) with $\{a^2, b^2, c^2\}$ cyclically permuted. Thus

$$\frac{\partial H'}{\partial a^i} = H_2' \frac{\partial}{\partial a^i}(a^2+b^2+c^2) + H_1' \times \frac{\partial}{\partial a^i}(b^2c^2+c^2a^2+a^2b^2) + H_0' \frac{\partial}{\partial a^i}(a^2b^2c^2).$$

The partial derivatives occurring on the right-hand side can be evaluated fairly simply from (111). The resulting equations of motion are

$$\begin{aligned}
\dot a^1 &= -(w-e)a^2 + 2\tau a^5 - 2\{[a^2a^6 - a^4a^5]H_1' + a^2 H_2'\},\\
\dot a^2 &= \tfrac{1}{2}(\omega+e)a^1 - \tfrac{1}{2}(\omega-e)a^3 + \tau a^4 + \{[a^6(a^1-a^3)\\
&\quad + [(a^4)^2-(a^5)^2]]H_1' + (a^1-a^3)H_2'\},\\
\dot a^3 &= (\omega+e)a^2 + 2\{[a^2a^6 - a^4a^5]H_1' + a^2H_2'\},\\
\dot a^4 &= \tfrac{1}{2}(\omega+e)a^5 + \{[a^3a^5 - a^4a^2]H_1' + a^5H_2'\},\\
\dot a^5 &= -\tfrac{1}{2}(\omega-e)a^4 + \tau a^6 - \{[a^1a^4 - a^5a^2]H_1' + a^4H_2'\},\\
\dot a^6 &= 0.
\end{aligned} \tag{115}$$

These equations can be verified by using the expressions for $\dot\phi$, $\dot\psi$, $\dot\theta$, $(\dot{a}^2)$, $(\dot{b}^2)$, $(\dot{c}^2)$ obtained by Meacham *et al.*[5] (see the Appendix) to independently compute $\dot a^i$ as a ''function'' of e, ω, τ, H_1', and H_2'.

In terms of the normal coordinates, we can take advantage of the constancy of the Casimirs and obtain the following fourth-order system:

$$\begin{aligned}
\dot z^2 &= e\left[\frac{C_U}{4} + (z^2)^2 + (z^3)^2\right]^{1/2}\\
&\quad + z^3\left\{\omega + \left(2C_L + \frac{[(z^4)^2+(z^5)^2]}{C_L}\right)H_1' + 2H_2'\right\}\\
&\quad + \left[\frac{C_U}{4} + (z^2)^2 + (z^3)^2\right]^{1/2} \frac{[(z^4)^2-(z^5)^2]}{C_L} H_1',\\
\dot z^3 &= -z^2\left\{\omega + \left(2C_L + \frac{[(z^4)^2+(z^5)^2]}{C_L}\right)H_1' + 2H_2'\right\}\\
&\quad + 2\left[\frac{C_U}{4} + (z^2)^2 + (z^3)^2\right]^{1/2} \frac{z^4z^5}{C_L} H_1',\\
\dot z^4 &= \left\{\frac{1}{2}(e+\omega) + H_2' + 2\left[\frac{C_U}{4} + (z^2)^2 + (z^3)^2\right]^{1/2}\right\} z^5\\
&\quad - 2(z^2z^4 + z^3z^5)H_1',\\
\dot z^5 &= \left\{\frac{1}{2}(e-\omega) - H_2' - 2\left[\frac{C_U}{4} + (z^2)^2 + (z^3)^2\right]^{1/2}\right\} z^4\\
&\quad + 2(z^2z^5 - z^3z^4)H_1' + \tau C_L.
\end{aligned} \tag{116}$$

This form has several appealing features. It is a simple set of equations to integrate numerically and, unlike (118) below, it does not contain any artificial singularities. The four-dimensional space, (z^2,z^3,z^4,z^5), decomposes relatively naturally into a product of two two-dimensional spaces, (z^2,z^3) and (z^4,z^5). The latter two are associated with the upper and lower algebras, respectively, in the decomposition of the algebra associated with the Poisson bracket. When there is no vertical shear, $\tau=0$, the space (z^2,z^3) is an invariant subset of the full phase space in that if both z^4 and z^5 are initially zero (one axis of the ellipsoid is initially vertical), they remain zero. The structure of the (z^2,z^3) phase space is then similar to the phase space of the Kida problem, as proven explicitly in the next section.

G. Canonical coordinates

One of the a coordinates, $a^6 = C_L$, is already a Casimir. We can reduce the dimension of the active dynamical variables to four if we change to a new set of variables in which the second Casimir, C_U, is also used as a variable. Further simplification ensues if we use variables deduced from the normal coordinates. We therefore use the variables R, α, S, β, C_U, C_L defined by

$$\begin{aligned}
&z^1 = \tfrac{1}{2}R, \quad z^2 = \tfrac{1}{2}(R^2 - C_U)^{1/2}\sin 2\alpha,\\
&z^3 = \tfrac{1}{2}(R^2 - C_U)^{1/2}\cos 2\alpha,\\
&z^4 = (2C_LS)^{1/2}\sin\beta, \quad z^5 = (2C_LS)^{1/2}\cos\beta,\\
&z^6 = C_L.
\end{aligned} \tag{117}$$

In the new variables, the Hamiltonian takes the form

$$\begin{aligned}
H &= -\frac{\omega}{2}\{R+S\} - \frac{e}{2}\{W\cos 2\alpha + S\cos 2\beta\}\\
&\quad + \tau(2C_LS)^{1/2}\sin\beta + \frac{1}{2}(C_LC_U)^{1/2}\\
&\quad \times \int_0^\infty \{s^3 + [2R+2S+C_L]s^2 + [2R(C_L+S)\\
&\quad - 2WS\cos 2(\alpha-\beta) + C_U]s + C_LC_U\}^{-1/2},
\end{aligned}$$

where $W = (R^2 - C_U)^{1/2}$. Note that for an ellipsoid enclosing uniform vorticity, $R^2 > C_U$. The equations of motion in the new variables are

$$\begin{aligned}
&\frac{dR}{dt} = \frac{\partial H}{\partial\alpha}, \quad \frac{d\alpha}{dt} = -\frac{\partial H}{\partial R},\\
&\frac{dS}{dt} = \frac{\partial H}{\partial\beta}, \quad \frac{d\beta}{dt} = -\frac{\partial H}{\partial S},\\
&\frac{dC_U}{dt} = 0, \quad \frac{dC_L}{dt} = 0.
\end{aligned} \tag{118}$$

The new variables constitute a set of canonical coordinates for the system, with a form similar to action-angle variables. Two of these, α and β, are effectively angles. When there is no vertical shear in the background flow ($\tau=0$), the system depends on α and β modulo π. When vertical shear is present, the system depends on β modulo 2π.

When one axis of the ellipsoid is vertical, $S=0$ ($\int zx$ and $\int yz$ vanish). If there is no vertical shear, then S will remain zero and one of the axes will remain vertical. The system then reduces to a simple two-dimensional form

$$\frac{dR}{dt}=\frac{\partial H}{\partial \alpha},\quad \frac{d\alpha}{dt}=-\frac{\partial H}{\partial R}$$

and

$$H=-\frac{\omega}{2}R-\frac{e}{2}(R^2-C_U)^{1/2}\cos 2\alpha$$
$$+\frac{1}{2}(C_LC_U)^{1/2}\int_0^\infty\{s^3+[2R+C_L]s^2$$
$$+[2RC_L+C_U]s+C_LC_U\}^{-1/2}.$$

If we write

$$\Omega(R)=\frac{1}{2}(C_LC_U)^{1/2}\int_0^\infty(s^2+C_Ls)\{s^3+[2R+C_L]s^2$$
$$+[2RC_L+C_U]s+C_LC_U\}^{-3/2},$$

then

$$H=-\frac{\omega}{2}R-\frac{e}{2}(R^2-C_U)^{1/2}\cos 2\alpha-\int^R\Omega(R')\,dR'$$

and

$$\frac{dR}{dt}=e(R^2-C_U)^{1/2}\sin 2\alpha,$$
$$\frac{d\alpha}{dt}=\left(\frac{\omega}{2}+\Omega(R)\right)+\frac{e}{2}\frac{R}{(R^2-C_U)^{1/2}}\cos 2\alpha. \tag{119}$$

Here $\Omega(R)$ is the rate at which the ellipsoid rotates around the vertical axis when no background flow is present, and in the presence of horizontal shear and strain the system behaves like the Kida ellipse with the rotation rate of the 2-D ellipse, $\lambda/(1+\lambda)^2$, replaced by Ω.

To see this explicitly, we normalize the volume of the ellipsoid to be $4\pi/3$ (the dynamics are independent of the volume) and note that when one axis of the ellipsoid is vertical, say the c axis, then the normal coordinates, z_1, z_2, z_3, become

$$z_1=\rho(a^2+b^2),$$
$$z_2=\rho(a^2-b^2)\sin 2(\phi+\psi),$$
$$z_3=\rho(a^2-b^2)\cos 2(\phi+\psi),$$

where $\rho=\pi/15$. Defining a horizontal aspect ratio $\lambda=b/a$,

$$z_1=\frac{\rho}{c}(\lambda^{-1}+\lambda),$$

$$z_2=\frac{\rho}{c}(\lambda^{-1}-\lambda)\sin 2(\phi+\psi),$$

$$z_3=\frac{\rho}{c}(\lambda^{-1}-\lambda)\cos 2(\phi+\psi).$$

In Euler angles, when $\theta=0$ (the c axis vertical), $\tilde{\phi}:=\phi+\psi$ is just the total angle through which the coordinate frame has rotated about the vertical axis. Thus,

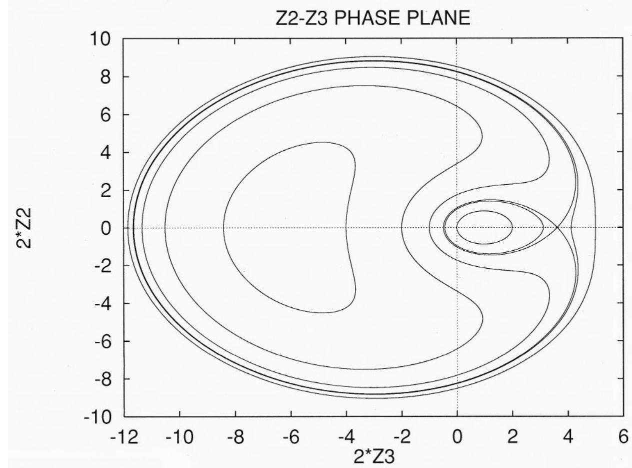

FIG. 3. Contours of the Hamiltonian for the QG ellipsoidal vortex with $z^4=z^5=0$, in a background flow with $\tau=0$, on the $\{z^2,z^3\}$ plane for the case $\omega=-0.1$, $e=-0.01$, $C_L=0.25$, and $C_U=12.0$. (Contours are unequally spaced.)

$$R=\frac{2\rho}{c}(\lambda^{-1}+\lambda),\quad \alpha=\tilde{\phi},\quad \text{and}\ C_U=\frac{8\rho^2}{c^2}\lambda,$$

and (119) reduces to

$$\frac{d\lambda}{dt}=-e\lambda\sin 2\alpha,$$

$$\frac{d\alpha}{dt}=\left(\frac{\omega}{2}+\Omega\right)+\frac{e}{2}\frac{1+\lambda^2}{1-\lambda^2}\cos 2\alpha,$$

which should be compared to (30).

H. Chaotic motion

The system (116) is four-dimensional and, with the exception of the Hamiltonian itself, we have used all of the invariants that we may anticipate on physical grounds in order to reduce the dimension of the system to four. This suggests that the system is likely to be nonintegrable and we anticipate the occurrence of chaotic solutions. This is intriguing because the system (116) is an *exact* reduction of the original quasigeostrophic problem. Solutions of it are exact solutions of the continuum quasigeostrophic equations of motion.

We can use the form of the equations of motion given in (116) to verify nonintegrability by looking for an example of chaotic motion in this system. We first note that when $\tau=0$, (116) admits an invariant manifold, $z_4=z_5=0$. Physically this corresponds to one of the principal axes of the ellipsoid being aligned with the Oz axis. In the absence of any vertical variation in the background flow, such an axis, if initially vertical, will remain so. Note also that, when $\tau=0$, there is a negative threshold value of ω, ω_c say, a function of $|e/\omega|$, such that when $|e/\omega|<1$ and $0>\omega>\omega_c$, this invariant plane, with coordinates $\{z_2,z_3\}$, contains a homoclinic trajectory that begins and ends on a saddle point and encloses a neutral fixed point.[31] An example is given in Fig. 3 which

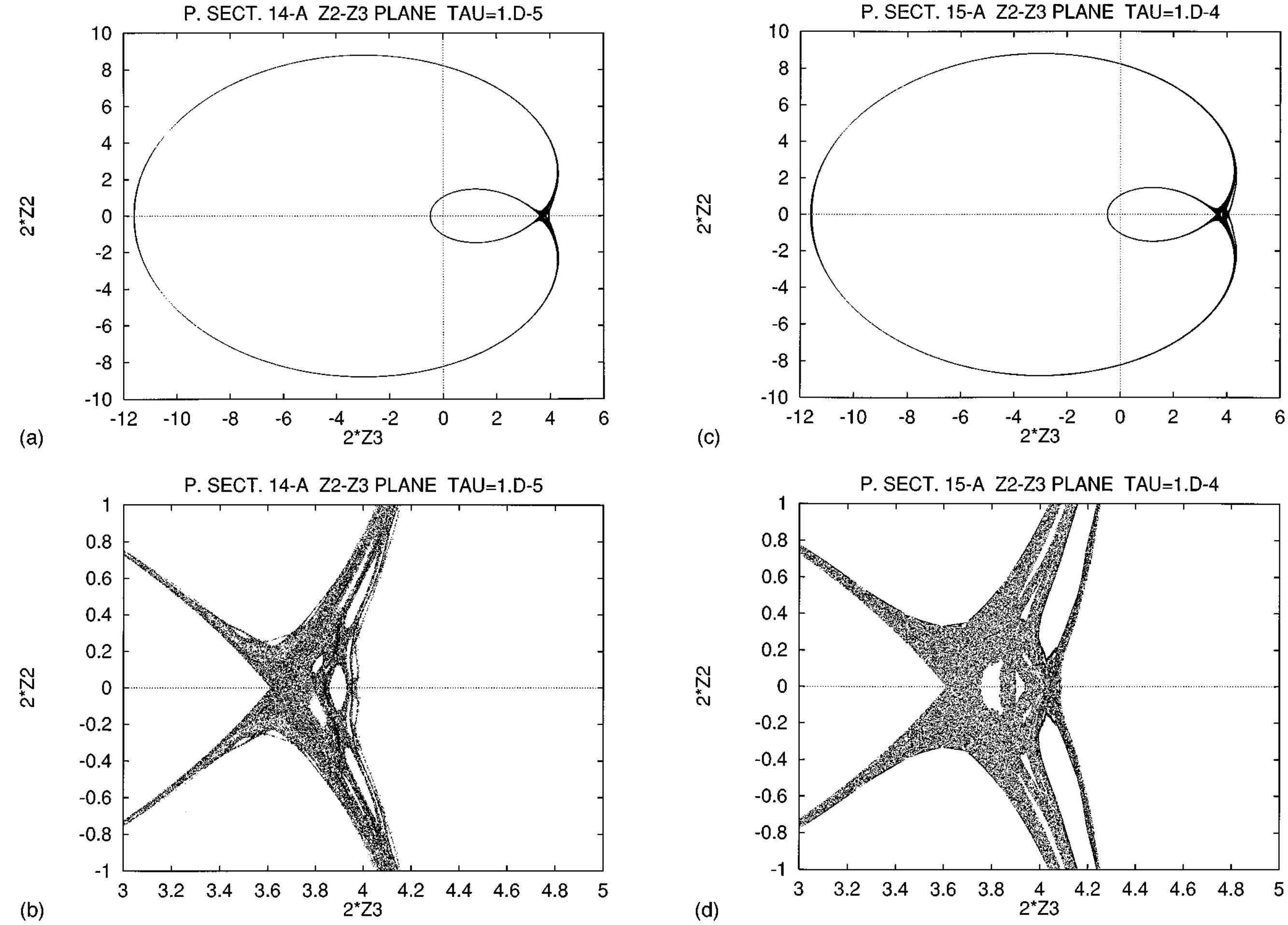

FIG. 4. Poincaré sections for trajectories started close to the hyperbolic point in Fig. 3. Values of ω, e, C_L, and C_U are as in Fig. 3. (a) $\tau=10^{-5}$, (b) $\tau=10^{-5}$—expanded view near the hyperbolic point, (c) $\tau=10^{-4}$, (d) $\tau=10^{-4}$—expanded view near the hyperbolic point.

shows the contours of the Hamiltonian: $H(z_2,z_3;z_4=0,z_5=0)$ over the coordinate plane $\{z_2,z_3\}$ for the case $\omega=-0.1$, $e=-0.01$, $C_L=0.25$, and $C_U=12.0$.

A second result that guides our thinking is the observation that when the background flow is absent, an ellipsoidal vortex with one axis vertical is not unstable to small perturbations that tilt that axis slightly away from the vertical provided the axis in question is not the axis of intermediate length. The near-vertical axis just wobbles around the vertical. When a background, horizontally sheared flow with $|e/\omega|<1$ and $0>\omega>\omega_c$ is present, we anticipate similar wobbles when the vertical axis is tilted slightly (provided it is the longest or shortest axis). These wobbles produce oscillations on the right-hand side of the dz_2/dt and dz_3/dt equations. We wish to see if these oscillations can produce chaotic motion. (One possibile way in which this might occur is if the wobbles remain small and simply produce splitting and transverse intersections of the stable and unstable manifolds of the saddle point on the $\{z_2,z_3\}$ plane.)

The formal Melnikov analysis of the perturbed system is difficult so instead we resort to direct numerical simulation. Picking initial conditions close to the saddle point in Fig. 3, we follow the resulting trajectories for small nonzero values of the vertical sliear. Poincaré sections are made by first recording the successive points of intersection of the trajectories with the hyperplane $z_5=0$ (z_5 increasing) and then projecting these points onto the $\{z_2,z_3\}$ plane. Fig. 4(a) shows the Poincaré section for $\tau=10^{-5}$. An expanded view of the region around the location of the original hyperbolic point [Fig. 4(b)] shows the island structure characteristic of invariant tori delimiting chaotic regions. This remains true when $\tau=10^{-4}$ [Figs. 4(c) and 4(d)]. Further evidence for the chaotic nature of the solution can be seen in the power spectrum of a time series of the variable $z_2(t)$ taken along a trajectory when $\tau=10^{-4}$ (Fig. 5). There are several dominant peaks but broad-band noise is also present at a level several orders of magnitude greater than the noise associated with the numerical integrator. This allows us to distinguish the time series as chaotic rather than quasiperiodic.

V. CONCLUDING REMARKS

In the work we have presented above, we have been guided by the observation[1] that the two-dimensional elliptical vortex in shear is a finite-dimensional Hamiltonian system together with the observation[5] that the three-dimensional, quasigeostrophic ellipsoidal vortex in horizontal shear is Hamiltonian and the conjecture that this

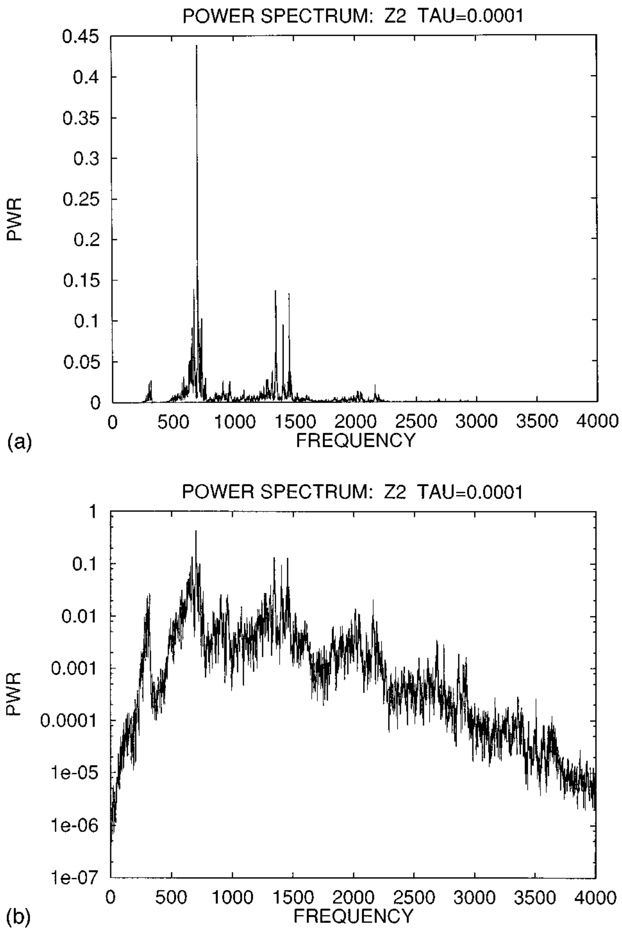

FIG. 5. Power spectrum of long time series of $z^2(t)$ from a trajectory started near the hyperbolic point in Fig. 3: $\omega=-0.1$, $e=-0.01$, $\tau=10^{-4}$, $C_L=0.25$, and $C_U=12.0$. (a) Linear–linear plot and (b) log–linear plot.

remains true when vertical shear is added. In the two-dimensional case, an explicit Hamiltonian form can be readily obtained from the standard Eulerian equations of motion, but for the three-dimensional problem the task is much more difficult. The equations of motion for the ellipsoidal vortex derived by MPSZ are unwieldy. In this paper we have shown that, by beginning from a description of the continuous Euler and quasigeostrophic equations *in noncanonical Hamiltonian form*, one can obtain a much simpler set of equations for the motion of the three-dimensional vortex.

We have then used the equations of motion for the ellipsoidal vortex to show that the ellipsoidal vortex in shear can exhibit chaotic motion. This is unusual for the following reason. The majority of mathematical examples of chaos in fluid flow involve approximations in the form of finite truncations. Though our final system, e.g., (114), is of finite dimension we have not made any truncation; our solutions are *exact* solutions of the inviscid quasigeostrophic equations for a stratified fluid. However, one must recall that our solutions are ''weak solutions'' of the QG problem in the sense that the vorticity field contains discontinuities.

We note, in closing, some generalizations. The technique we have described can also be used as a method for *approximating* the interaction between more than one vortex. This idea is in the spirit of Melander *et al.*[2] and has been applied successfully by Ngan *et al.*[23] in the study of two interacting vortices in a shear flow.

Also, in the stratified algebra of Sec. IV B, the variable z plays the role of a constant. Setting $z=1$ gives the algebra of Ngan.[22,23] In the two-dimensional setting examined by Ngan *et al.*,[22,23] the transformation (100) is equivalent to, first, transforming the quadratic moments based on a fixed origin to their counterparts based on the centroid of a vortex patch, and then, applying the transformation to normal coordinates used in the Kida problem (52). Note that z can be replaced by a set of functions of z, $f_1(z),\dots,f_N(z)$, and one may still obtain closure. Consider

$$m^1=x^2,\quad m^2=xy,\quad m^3=y^2,$$

together with

$$m^{4,i}=yf_i(z),\quad m^{5,i}=xf_i(z),\quad 1\leqslant i\leqslant N,$$

and

$$m^{6,i,j}=f_i(z)f_j(z),\quad 1\leqslant i,j\leqslant N.$$

Observe that

$$\{m^1,m^{4,i}\}=2xf_i=2m^{5,i},$$

$$\{m^{4,i},m^{5,j}\}=f_if_j=m^{6,i,j},\quad \text{etc.}$$

Therefore, it is possible to generalize the approach described in this paper to different classes of moments which may prove more appropriate in different situations. For example, one can represent three-dimensional quasigeostrophic vortices in which the potential vorticity is horizontally uniform, but varies with z in an arbitrary manner, by considering a set of f_i that form a basis.

ACKNOWLEDGMENTS

This paper had its genesis at the GFD Summer Program in Woods Hole in 1993 and much of the work herein has been continued over succeeding summers at the GFD Progam. The authors would like to thank the Woods Hole Oceanographic Institution and the National Science Foundation for making this interaction possible. S.P.M. has been supported by NSF Contract No. OCE 94-01977, P.J.M. by the Department of Energy under Contract No. DE-FG05-80ET-53088. S.P.M. and P.J.M. would like to thank the Institute for Fusion Studies at the University of Texas (Austin) and the Geophysical Fluid Dynamics Institute at Florida State University, respectively, for their hospitality and support of visits by S.P.M. to Austin and P.J.M. to Tallahassee.

APPENDIX

An ellipsoid can be described by the lengths of its three principal semi-axes, $a(t)$, $b(t)$, $c(t)$ and three Euler angles, $\phi(t)$, $\theta(t)$, $\psi(t)$, that specify its orientation relative to a fixed reference frame. For an ellipsoidal vortex in a shear flow such as that given by (94), the equations of motion for these variables are

$$(\dot{a^2}) = ea^2\{\cos\theta\sin 2\psi\cos 2\phi + [\cos 2\psi - \sin^2\theta\cos^2\psi]\sin 2\phi\} + \tau a^2\sin\theta\{\sin 2\psi\sin\phi - 2\cos\theta\cos^2\psi\cos\phi\},$$

$$(\dot{b^2}) = -eb^2\{\cos\theta\sin 2\psi\cos 2\phi + [\cos 2\psi + \sin^2\theta\sin^2\psi]\sin 2\phi\} - \tau b^2\sin\theta\{\sin 2\psi\sin\phi + 2\cos\theta\sin^2\psi\cos\phi\}, \tag{A1}$$

$$(\dot{c^2}) = c^2\sin\theta\{e\sin\theta\sin 2\phi + 2\tau\cos\theta\cos\phi\},$$

and

$$\dot\phi = \frac{1}{2}(\Omega_1+\Omega_2) + \frac{1}{2}(\Omega_1-\Omega_2)\cos 2\psi + \frac{1}{2}\omega - \frac{1}{2}e\left\{\left[1+\left(\frac{c^2}{a^2-c^2}+\frac{c^2}{b^2-c^2}\right)\right]\cos 2\phi - \left(\frac{c^2}{a^2-c^2}-\frac{c^2}{b^2-c^2}\right)\times(\cos 2\psi\cos 2\phi - \cos\theta\sin 2\psi\sin 2\phi)\right\} + \frac{1}{2}\frac{\tau}{\sin\theta}\left\{\left(\frac{c^2}{a^2-c^2}+\frac{c^2}{b^2-c^2}\right)\cos\theta\sin\phi - \left(\frac{c^2}{a^2-c^2}-\frac{c^2}{b^2-c^2}\right)(\cos\theta\sin\phi\cos 2\psi + \cos 2\theta\sin 2\psi\cos\phi)\right\},$$

$$\dot\theta = \frac{1}{2}\sin\theta\sin 2\psi(\Omega_2-\Omega_1) - \frac{1}{2}e\sin\theta\left\{\left[1+\left(\frac{c^2}{a^2-c^2}+\frac{c^2}{b^2-c^2}\right)\right]\cos\theta\sin 2\phi + \left(\frac{c^2}{a^2-c^2}-\frac{c^2}{b^2-c^2}\right)\times(\cos\theta\sin 2\phi\cos 2\psi + \sin 2\psi\cos 2\phi)\right\} + \frac{1}{2}\tau\left\{\left[2\sin^2\theta - \cos 2\theta\left(\frac{c^2}{a^2-c^2}+\frac{c^2}{b^2-c^2}\right)\right]\cos\phi + \left(\frac{c^2}{a^2-c^2}-\frac{c^2}{b^2-c^2}\right)(\cos\theta\sin 2\psi\sin\phi - \cos 2\theta\cos 2\psi\cos\phi)\right\}, \tag{A2}$$

$$\dot\psi = \cos\theta\left\{\Omega_3 - \frac{1}{2}(\Omega_1+\Omega_2) - \frac{1}{2}(\Omega_1-\Omega_2)\cos 2\psi\right\} + \frac{1}{2}e\left\{\left[1+\left(\frac{c^2}{a^2-c^2}+\frac{c^2}{b^2-c^2}\right)\right]\cos\theta\cos 2\phi + \left(\frac{c^2}{a^2-c^2}-\frac{c^2}{b^2-c^2}\right)(\cos^2\theta\sin 2\psi\sin 2\phi - \cos\theta\cos 2\psi\cos 2\phi) + \frac{a^2+b^2}{a^2-b^2}\left(\cos\theta\cos 2\psi\cos 2\phi - \frac{1}{2}(1+\cos^2\theta)\sin 2\psi\sin 2\phi\right)\right\} - \frac{1}{2}\frac{\tau}{\sin\theta}\left\{\left(\frac{c^2}{a^2-c^2}+\frac{c^2}{b^2-c^2}\right)\times\cos^2\theta\sin\phi - \left(\frac{c^2}{a^2-c^2}-\frac{c^2}{b^2-c^2}\right)(\cos^2\theta\cos 2\psi\sin\phi + \cos\theta\cos 2\theta\sin 2\psi\cos\phi) - \sin^2\theta\sin\phi - \frac{a^2+b^2}{a^2-b^2}\left(\sin^2\theta\cos 2\psi\sin\phi + \frac{1}{2}\sin\theta\sin 2\theta\sin 2\psi\cos\phi\right)\right\}.$$

The Ωs are the principal rotation rates of the ellipsoid, i.e., the rate at which it would rotate around a given principal axis, in the absence of any background flow, if that axis were vertical. These are elliptic functions of the lengths of the semi-axes:

$$\Omega_1 = a^2 I_1 + I_2, \quad \Omega_2 = \beta^2 I_1 + I_2, \quad \Omega_3 = I_1 + I_2 \tag{A3}$$

$$I_j = \frac{1}{2}abc\int_0^\infty s^j\{(a^2+s)(b^2+s)(c^2+s)\}^{-3/2}ds, \quad j=1,2.$$

The moments can be expressed in terms of the semi-axis lengths and the Euler angles as follows:

$$a^1 = \cos^2\phi[\cos^2\theta(a^2\cos^2\psi + b^2\sin^2\psi) + c^2\sin^2\theta] + \sin^2\phi(a^2\sin^2\psi + b^2\cos^2\psi) - \tfrac{1}{2}(a^2-b^2)\cos\theta\sin 2\phi\sin 2\psi,$$

$$a^2 = \tfrac{1}{2}\sin 2\phi[\cos^2\theta(a^2\cos^2\psi + b^2\sin^2\psi) + c^2\sin^2\theta - (a^2\sin^2\psi + b^2\cos^2\psi)] + \tfrac{1}{2}(a^2-b^2)\times\cos\theta\sin 2\psi\sin 2\phi,$$

$$a^3 = \sin^2\phi[\cos^2\theta(a^2\cos^2\psi + b^2\sin^2\psi) + c^2\sin^2\theta] + \cos^2\phi(a^2\sin^2\psi + b^2\cos^2\psi) + \tfrac{1}{2}(a^2-b^2)\times\cos\theta\sin 2\phi\sin 2\psi, \tag{A4}$$

$$a^4 = -\sin\theta[\sin\phi\cos\theta(a^2\cos^2\psi + b^2\sin^2\psi - c^2) + \tfrac{1}{2}(a^2 - b^2)\cos\phi\sin 2\psi],$$

$$a^5 = -\sin\theta[\cos\phi\cos\theta(a^2\cos^2\psi + b^2\sin^2\psi - c^2) - \tfrac{1}{2}(a^2 - b^2)\sin\phi\sin 2\psi],$$

$$a^6 = (a^2\cos^2\psi + b^2\sin^2\psi)\sin^2\theta + c^2\cos^2\theta.$$

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