Negative Energy Modes and Gravitational Instability of Interpenetrating Fluids

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> "A negative energy electron will have less energy the faster it moves and will have to absorb energy in order to be brought to rest. No particles of this nature have ever been observed." — P. A. M. Dirac, 1930

ABSTRACT: We study the longitudinal instabilities of two interpenetrating fluids interacting only through gravity. When one of the constituents is of relatively low density, it is possible to have a band of unstable wavenumbers well separated from those involved in the usual Jeans instability. If the initial streaming is large enough, and there is no linear instability, the indefinite sign of the free energy has the possible consequence of explosive interactions between positive and negative energy modes in the nonlinear regime. The effect of dissipation on the negative energy modes is also examined.

1. BACKGROUND

It used to be thought that the stellar components of two spiral galaxies would pass right through each other in the event of a collision and that only the gaseous components would merge. However, simulations over the past twenty years or so [1], [2] have shown that the macroscopic energy of such a collision is quickly converted into internal energy and that merger of the stellar systems is a common natural outcome of a collision. How is this conversion effected?

An answer to this may lie in the physics of streaming instabilities. In the context of plasma physics, interpenetrating electron beams produce the two-stream instability [3] whose gravitational analog has long been recognized, beginning with the investigations of Sweet [4] and Lynden-Bell [5]. Of course, in the case of galaxy collisions, which occur quickly, the conventional two-stream instability may operate too slowly to be effective. However, it is known that even when two streams of interacting particles are not linearly unstable, they may collectively produce negative energy modes that lead to an explosive nonlinear growth of perturbations for arbitrarily small disturbances. There are well-developed criteria for the occurrence of this explosive growth in plasma physics [6] and, as Lovelace *et al.* [7] have suggested

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in their analysis of counterrotating galaxies, we may expect something analogous in the gravitational setting. Our aim here is to briefly develop this topic of negative energy modes for the case of gravitational interaction in the expectation that the phenomena involved will be found significant in a variety of astrophysical processes.

Even in the event of linear instability, the case of counterstreaming populations is significantly different from the standard gravitational instability, which occurs only for perturbation scales greater than the Jeans length [8]. When there are two interpenetrating fluids such as stars and gas, modes of arbitrary wavelength can be rendered unstable. Numerous investigators have reported on these issues. Most of them (such as Ikeuchi *et al.* [9], Fridman and Polyachenko [10], and Araki [11]) focused primarily on the symmetric situation of identical fluids in counterstreaming motion. In that case one finds that the spectrum of any instabilities arising from the relative motion is wholly contained within the Jeans instability band, and this blurs the distinction between the two processes. This need not be true when this symmetry is broken, and indeed not all authors restricted themselves entirely to the symmetric case. The present venture into the asymmetric problem is intended to focus on the possibility of well-separated instability bands, which has not been elucidated in the gravitational context, as far as we are aware.

2. EQUATIONS OF MOTION

To see the problem in its simplest version, it is useful to have a uniform medium as the unperturbed state. Rather than formulate the problem inconsistently to achieve this end, as Jeans did, we prefer the Einstein device of introducing a cosmological repulsion term. In the Newtonian setting we readily see how to redefine the gravitational potential so that, instead of introducing such a repulsion term, we fill space with a fluid of *negative* gravitational mass of density ρ_Λ. As in the one-fluid plasma model, we treat this density as a constant since its purpose is to allow a gravitationally neutral background state. One may also contemplate the analog of the two-fluid plasma model in which this background antigravitational fluid has its own dynamics, but we do not do that here. The two dynamically active fluids we consider are gravitationally ordinary and polytropic. Thus, the Poisson equation is written here as,

$$\nabla^2 V = 4\pi G(\rho_1 + \rho_2 - \rho_\Lambda), \tag{1}$$

where V is the gravitational potential, ρ_1 and ρ_2 are the source densities of the conventional fluids, and $-\rho_\Lambda$ is the cosmological background density.

The equations of motion for the two fluids ($j = 1, 2$) are,

$$\rho_j(\partial_t + \mathbf{u}_j \cdot \nabla)\mathbf{u}_j = -\nabla p_j - \rho_j \nabla V \tag{2}$$

$$\partial_t \rho_j + \nabla \cdot (\rho_j \mathbf{u}_j) = 0, \tag{3}$$

where we do not sum over repeated indices. Each fluid has a sound speed, $c_j^2 = \partial p_j / \partial \rho_j$, and a Jeans wavenumber, $k_{Jj}^2 = 8\pi G \hat{\rho}_j / c_j^2$, where the hat signifies the equilibrium value and an uncustomary factor of 2 appears in the definition of the Jeans wavenumbers.

We shall use natural units with k_{J1}^{-1} as the length scale and $(k_{J1}c_1)^{-1}$ as the time scale. We further simplify the description by considering only longitudinal motions in one-dimension, so each single-component velocity field may be expressed as the gradient of a velocity potential: $u_j = \partial_x\phi_j$. The fundamental equations (1)–(3) take the dimensionless form,

$$M_1\partial_t\phi_1 + \frac{1}{2}M_1^2(\partial_x\phi_1)^2 + \frac{\rho_1^{\gamma_1-1}}{\gamma_1-1} + V = \beta_1 \tag{4}$$

$$cM_2\partial_t\phi_2 + \frac{1}{2}c^2M_2^2(\partial_x\phi_2)^2 + \frac{c^2\rho_2^{\gamma_2-1}}{\gamma_2-1} + V = \beta_2 \tag{5}$$

$$\partial_t\rho_1 + M_1\partial_x(\rho_1\partial_x\phi_1) = 0 \tag{6}$$

$$\partial_t\rho_2 + cM_2\partial_x(\rho_2\partial_x\phi_2) = 0 \tag{7}$$

$$\partial_x^2V - \frac{1}{2}\rho_1 - \frac{1}{2}\beta\rho_2 + \frac{\rho_\Lambda}{2\hat{\rho}_1} = 0 \tag{8}$$

where the $M_j = \mathcal{U}_j/c_j$ are Mach numbers, $\mathcal{U}_j$ measures the initial streaming velocities, $\beta = \hat{\rho}_2/\hat{\rho}_1$, $c = c_2/c_1$, and the Bernoulli constants B_j are chosen to balance the basic state.

3. LINEAR THEORY

We perturb from the state of uniform densities and constant velocities by setting $\phi_j = (-1)^{j+1}x + \delta\phi_j$, $\rho_j = 1 + \delta\rho_j$, and $V = \hat{V} + \delta V$. The density terms of the Poisson equation (8) combine to vanish and $\hat{V}$ is a constant. Since the linearized equations are separable we may decompose the perturbations into normal modes proportional to $\exp(i\omega t - ikx)$ to find the dispersion relation,

$$\Gamma(\omega, k) = 1 + \frac{1}{2[(\omega - kM_1)^2 - k^2]} + \frac{\beta}{2[(\omega + ckM_2)^2 - c^2k^2]}$$

$$\equiv 1 + \Gamma_1 + \Gamma_2 = 0. \tag{9}$$

The quantity Γ, which we call the diagravic function by analogy with the dielectric function of electrodynamics, measures the collective response of the fluid to disturbances in the gravitational field and will serve to indicate the energy signature of any normal mode (Section 4).

For real k the solutions of (9) with complex ω correspond to instability; if ω is real, then solutions of (9) with complex k can give rise to wave amplification instability. Here we analyze only the case of real k. However, we have to deal with both the traditional Jeans instability as well as the two-stream instability, the latter of which involves a sympathetic bunching of particles and is effective for creating instability when the phase speed of the disturbance conspires to create a resonance between different modes.

TABLE 1[a]

Mach Range	Mode Type	k_{crit}^2	$\lim_{M\to 1} k_{crit}^2$	$\lim_{M\to\infty} k_{crit}^2$
$0 \le M < 1$	Jeans	$\dfrac{1}{1-M^2}$	∞	Not applicable
$1 \le M$	Two-Stream	$\dfrac{\sqrt{M^2-1}}{4M\left(M^2-1+M\sqrt{M^2-1}\right)}$	$\dfrac{1}{4}$	0
$1 \le M$	Jeans	$\dfrac{\sqrt{M^2-1}}{4M\left(1-M^2+M\sqrt{M^2-1}\right)}$	$\dfrac{1}{4}$	$\dfrac{1}{2}$

[a]The value of k_{crit}^2 for the two-stream modes with $M \ge 1$ more accurately refers to the k value at which the unstable two-stream and Jeans branches merge.

3.1. Symmetric Case

If both fluids have the same basic properties ($c = 1$ and $\beta = 1$), a frame exists in which $M_1 = M_2 = M$. The dispersion relation then simplifies into a manageable bi-quadratic with solutions,

$$\omega_\pm{}^2 = -\frac{1}{2} + k^2(M^2+1) \pm \sqrt{\frac{1}{4} - 2k^2M^2 + 4M^2k^4}\,. \tag{10}$$

For $M = 0$, we recover a simple version of the previously studied two-fluid Jeans problem [10], [12], [13]. We find $\omega_+{}^2 = k^2$, corresponding to sound waves at all k, and $\omega_-{}^2 = k^2 - 1$, which is the conventional Jeans dispersion relation. The new acoustic modes arise because the aggregate fluid now allows motions unaffected by the gravitational field; for these modes the perturbed gravitational potential is zero.

With relative velocity in the subsonic regime ($0 < M < 1$), there is only a single unstable mode that branches continuously from the Jeans mode at $M = 0$. This mode is unstable for all wavenumbers below a critical value that approaches infinity as M tends to unity from below (see TABLE 1). To study this limit, we let $M^2 = 1 - \alpha/k^2$ with $0 < \alpha < 1$. As $k \to \infty$ we find the approximate solution $\omega_-{}^2 \sim -\alpha(1-\alpha)/4k^2$, which reveals a weak instability at large k. Thus, weak relative streaming allows gravitational instability at arbitrarily small wavelengths. These large-k instabilities do not arise for Maxwellian velocity distributions within the context of the Vlasov equation [9].

For supersonic motion ($M > 1$) the large-k gravitational instability is no longer present, but a new instability emerges that we call a two-stream instability since it owes its presence to the energy contained in the initial streaming motion. As M ranges from 1 to ∞, the critical wavenumber for instability increases from $k_{\mathrm{crit}} < 1/2$ to $k_{\mathrm{crit}} < \sqrt{2}/2$.

The upper half of FIGURE 1 shows that near $k = 0$ the two-stream modes are wholly contained within the Jeans band. This fact coupled with the larger growth rates of the Jeans modes has led some to believe that the two-stream instability is swamped by the Jeans instability and is essentially unimportant [11]. As k increases the two-stream and Jeans modes collapse upon each other and together bifurcate into grow-

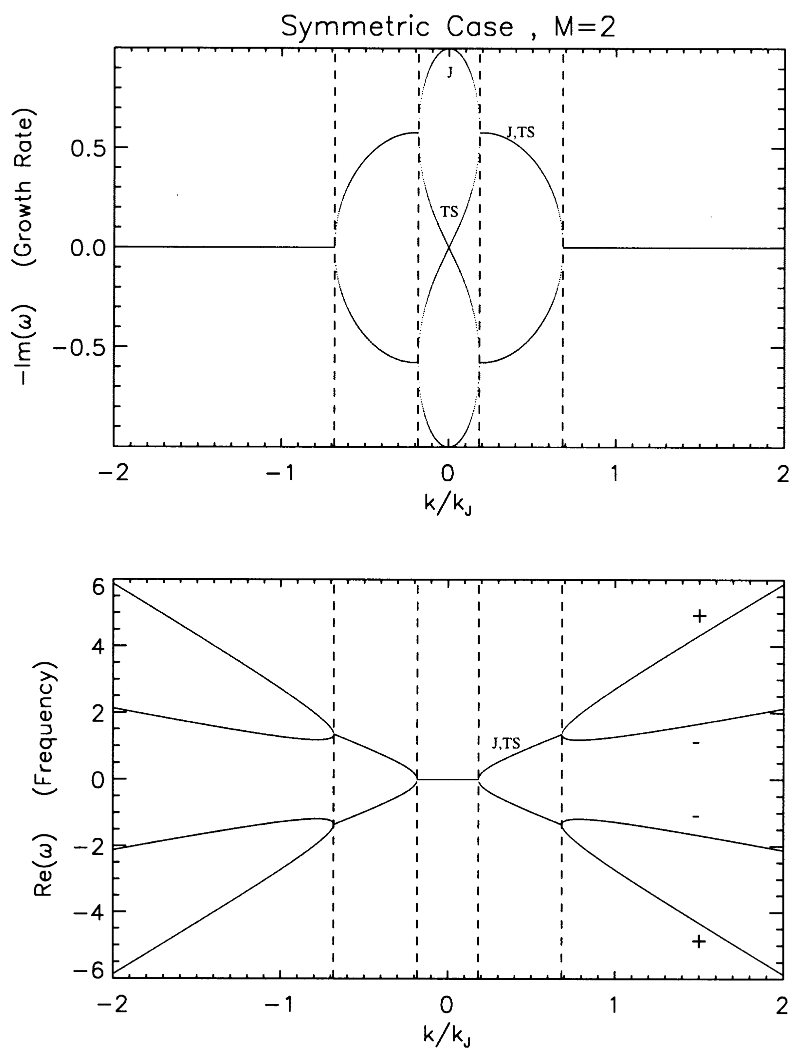

FIGURE 1. Dispersion curves for supersonic motion, $M = 2$. The Jeans modes are indicated with a "J" and the two-stream modes with a "TS." The growth rates of the J and TS modes merge at the onset of oscillations shown in the lower panel. Dashed vertical lines demarcate the critical wavenumbers for growth of the Jeans and two-stream modes. The "+" and "–" signs in the frequency plot denote the energy signature discussed in Section 4.

ing and damped oscillations. At still larger wavenumbers all motions are stable, propagating waves. The critical wavenumbers below which growth is possible at any Mach number are shown in TABLE 1.

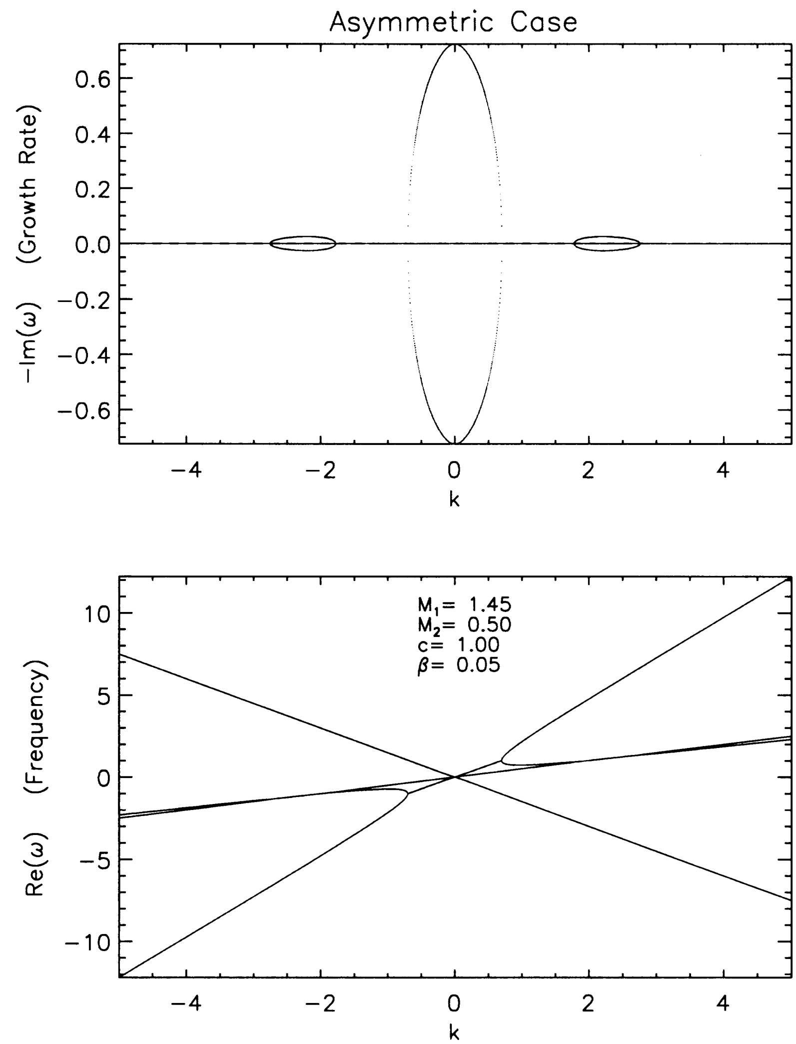

FIGURE 2. Asymmetric case: $\beta = .05$, $c = 1$, $M_1 + M_2 = 1.95$. Illustration of the pinched two-stream bubbles. The frequencies corresponding to the two-stream modes are degenerate along the entire k-band of the bubble. At the edge of the bubble the frequencies once again separate.

3.2. Asymmetric Case

When we relax the constraint of identical conditions in the two fluids, one of the more interesting consequences is the possibility of large wavenumber two-stream instability bands well-separated from the Jeans instabilities clustered at small k. For illustration we consider the effect of changing the initial relative streaming $M_1 + cM_2$ for fixed β and c. In fact, β turns out to be the crucial parameter in achieving the spectral separation; variations in the sound speed ratio c widens both bands together. The distancing of a bubble of two-stream modes from the Jeans band is illustrated in FIGURE 2.

The large-k two-stream modes can be explained qualitatively by examining the separate pieces of the dispersion relation (9). Since the streaming instability is related to resonant motions, it is revealing to examine the solutions to $1 + \Gamma_1(\omega, k) = 1 + \Gamma_2(\omega, k) = 0$ in isolation and see where the curves intersect. The frequencies of these noninteracting modes are given by,

$$\omega_1 = kM_1 \pm \sqrt{k^2 - \frac{1}{2}} \tag{11}$$

$$\omega_2 = -ckM_2 \pm \sqrt{c^2k^2 - \frac{\beta}{2}}. \tag{12}$$

When $\omega_1 = \omega_2$, the assumed independent frequencies match one another for wavenumbers satisfying,

$$k(M_1 + cM_2) = \begin{cases} \pm\left(\sqrt{k^2 - \frac{1}{2}} + \sqrt{c^2k^2 - \frac{\beta}{2}}\right) \\ \pm\left(\sqrt{k^2 - \frac{1}{2}} - \sqrt{c^2k^2 - \frac{\beta}{2}}\right). \end{cases} \tag{13}$$

In the symmetric case where $M_1 = M_2 \equiv M$, $c = \beta = 1$, we see that two of the resonances are lost except in the irrelevant cases $M = 0$ and $k = 0$. The other pair of possibilities, $kM = \pm(\sqrt{k^2 - \frac{1}{2}})$, just restate the critical wavenumber condition for what we know to be the modified Jeans instability when $M < 1$. In the general case, we can expect another pair of intersections that account for the two-stream bubbles of FIGURE 2.

4. NONLINEAR THEORY

4.1. Hamiltonian Formulation and Energy Signature

The dynamical equations (4)–(8) derive from a variational principle and a conserved Hamiltonian functional. The variational formulation has the advantage of shedding light on the relation between the energy content of the disturbances and nonlinear stability. Here we present results for the symmetric case, though the formalism follows through for the asymmetric case as well.

The Hamiltonian and associated equations are,

$$H = \frac{1}{M}\sum_{j=1}^{2}\int_0^L dx\left(\frac{M^2}{2}\rho_j\phi_{jx}^2 + \rho_j U_j - V_x^2\right), \tag{14}$$

$$\partial_t\phi_j = \{\phi_j, H\} = -\frac{\delta H}{\delta\rho_j}, \tag{15}$$

$$\partial_t\rho_j = \{\rho_j, H\} = -\frac{\delta H}{\delta\phi_j}, \tag{16}$$

where $U_j(\rho_j) = \rho_j^{\gamma-1}/\gamma(\gamma - 1)$ is the internal energy for the jth fluid and the Poisson bracket is defined by,

$$\{F, G\} = \sum_{j=1}^{2}\int_0^L dx'\left(\frac{\delta G(x)}{\delta\phi_j(x')}\frac{\delta F(x)}{\delta\rho_j(x')} - \frac{\delta F(x)}{\delta\phi_j(x')}\frac{\delta G(x)}{\delta\rho_j(x')}\right), \tag{17}$$

For definiteness we have chosen a box-geometry of length L.

The energy content of a particular mode when the perturbation amplitude is small is given by the second variation of H evaluated at equilibrium (the free energy). After some calculation this is seen to be,

$$\delta^2 H = \frac{1}{2M}\int_0^L dx(M^2[\delta\phi_{1x}^2 + \delta\phi_{2x}^2] + 2M^2[\delta\rho_1\delta\phi_{1x} - \delta\rho_2\delta\phi_{2x}] + \delta\rho_1^2 + \delta\rho_1^2 - 2\delta V_x^2) \quad . \tag{18}$$

It may be verified that this functional is conserved by the equations of motion. Suppose we now insert into (18) an eigenfunction corresponding to a single stable mode with Im(ω) = 0. Employing overbars to denote eigenvector components and * for complex conjugation, we write,

$$\delta V = \bar{V}e^{i(\omega t - kx)} + \bar{V}_j^* e^{-i(\omega^* t - kx)}, \tag{19}$$

and similarly for the other perturbation variables. Upon effecting the integrations, we can make use of the dispersion relation,

$$\Gamma(\omega, k) = 1 + \frac{1}{2[(\omega - kM)^2 - k^2]} + \frac{1}{2[(\omega + kM)^2 - k^2]} = 0, \tag{20}$$

to express the modal free energy in the compact form,

$$\delta^2 H = -2Lk^2|\overline{V}|^2\omega\frac{\partial\Gamma}{\partial\omega}. \tag{21}$$

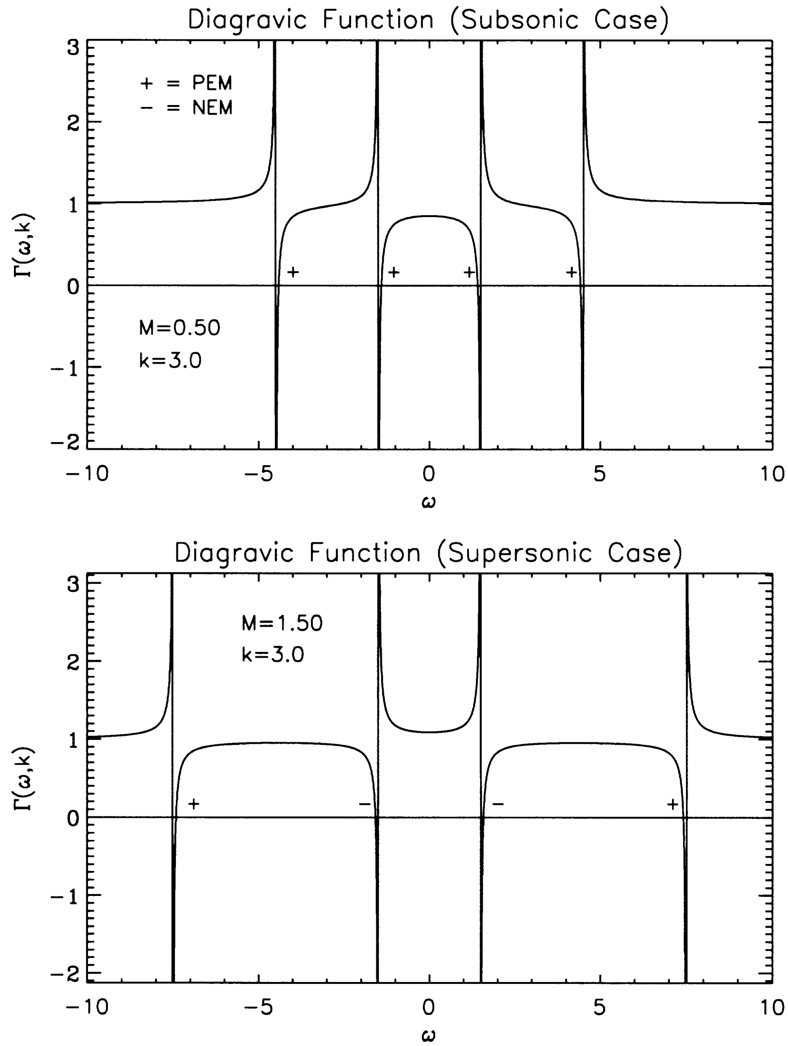

FIGURE 3. Diagravic function in the symmetric case for linearly stable modes with $k = 3$. A stable mode exists wherever Γ crosses the ω-axis. The positive energy modes are indicated with + and the negative energy modes with −.

Wherever $\omega(\partial\Gamma/\partial\omega) < 0$ a positive energy mode (PEM) is implied by (21), while the condition $\omega(\partial\Gamma/\partial\omega) > 0$ defines a negative energy mode (NEM). This possibility of modes of either signature has been elucidated in the plasma physics literature [14] and a gravitational analog was suggested by Lovelace *et al.* [7] in the context of thin, counterrotating stellar disks.

FIGURE 3 shows the diagravic function for both a subsonic and a supersonic case of stable modes with $k = 3$. In the subsonic regime, we find the Hamiltonian to be positive definite near equilibrium since $\omega(\partial\Gamma/\partial\omega) < 0$ at every crossing of Γ on the ω-axis. Right at the border of supersonic streaming ($M = 1$), concomitantly with the appearance of the two-stream instability, the Γ curves undergo a topological transition that allows the coexistence of positive and negative energy modes. The NEMs are *slow modes* in that they have smaller frequencies than their PEM counterparts. This is the typical situation; ω must pass through zero if the energy signature changes [15]. We expect from the precedents of plasma physics [6] that the simultaneous presence of positive and negative energy modes has dramatic consequences on the nonlinear stability of the system.

4.2. Reduction to Action Angle Variables

In the rest of this section we will concentrate on the nonlinear interactions between linearly stable modes in the symmetric problem (see FIGURE 3). From a physical standpoint, attention is focused on situations where the disturbances are of sufficiently small scale so that the Jeans instability can be ignored, though there are no compelling reasons why this ought to be the case. We will further assume supersonic motion in order to examine the interaction of positive and negative energy modes, a situation we expect to be the most interesting. Under these assumptions the equations of motion achieve their simplest form in action-angle coordinates that we now develop.

First we Fourier transform the field variables:

$$\rho_j = \sum_{m=-\infty}^{\infty} \rho_m^{(j)}(t) e^{ik_m x}, \quad \rho_{-m}^{(j)} = \rho_m^{(j)*}, \quad k_m = \frac{2\pi m}{L}. \tag{22}$$

We can then write the free energy in terms of real variables as,

$$\delta^2 H = \frac{1}{2} \sum_{m=1}^{\infty} (\mathbf{q}^T \mathbf{A} \mathbf{q} + \mathbf{p}^T \mathbf{B} \mathbf{p}), \tag{23}$$

where $\mathbf{q} \equiv (q_1, q_2, q_3, q_4)^T$, $\mathbf{p} \equiv (p_1, p_2, p_3, p_4)^T$, are linear combinations of the complex modal amplitudes, $\mathbf{A}$ and $\mathbf{B}$ are symmetric matrices (given in Casti [16]), and the "T" indicates transpose.

Defining the configuration variables $\mathbf{z} = (q_1, \ldots, p_4)^T$, we recast the linearized equations in the form,

$$\frac{d\mathbf{z}}{dt} = \mathbf{J}\nabla_{\mathbf{z}}\delta^2 H \equiv \mathbf{L}\mathbf{z}, \quad \mathbf{L} = \begin{pmatrix} 0 & \mathbf{B} \\ -\mathbf{A} & 0 \end{pmatrix}, \quad \mathbf{J} = \begin{pmatrix} 0 & \mathbf{I} \\ -\mathbf{I} & 0 \end{pmatrix}, \tag{24}$$

where $\mathbf{J}$ is the canonical 8×8 cosymplectic form. The next order of business is to construct a symplectic transformation that puts $\delta^2 H$ in its normal form [17], [18]. This can be achieved by writing $\mathbf{z} = \mathbf{S}\mathbf{Z}$, where the matrix $\mathbf{S}$ consists of suitably ordered eigenvectors of $\mathbf{L}$ satisfying the symplectic condition, $\mathbf{S}^T\mathbf{J}\mathbf{S} = \mathbf{J}$. After a final transformation to action-angle coordinates, the free energy expression (23) becomes a superposition of harmonic oscillators,

$$\delta^2 H = \sum_{m=1}^{\infty} (\omega_+ J_1 + \omega_+ J_2 - \omega_- J_3 - \omega_- J_4)\,. \tag{25}$$

The free energy is thus manifestly composed of two pairs each of positive and negative energy modes.

4.3. Three-Wave Resonance and Explosive Growth

Energy conservation forbids nonlinear runaway growth if H is definite (Dirichlet's theorem), as is the case here for subsonic motion. When the relative streaming is supersonic, however, interacting PEMs and NEMs can circumvent this restriction since they contribute energy of opposite sign.

We demonstrate the possibility of explosive growth with a three-wave resonant interaction between two NEMs and one PEM. Since the energy signature of a mode is not Galilean invariant, the existence of a reference frame in which all three modes have the same signature implies nonlinear stability. It may be shown that there is no reference frame in which all three modes have the same energy signature if the highest frequency wave has opposite signature to that of the other two [19]. This provides a criterion for three-wave interaction leading to instability.

The third-order resonance conditions for a triplet of modes are,

$$\begin{aligned} m_1 k_1 + m_2 k_2 + m_3 k_3 &= 0 \\ m_1 \omega_1 \pm m_2 \omega_2 \pm m_3 \omega_3 &= 0 \\ |m_1| + |m_2| + |m_3| &= 3 \quad (m_1, m_2, m_3 \text{ integers}), \end{aligned} \tag{26}$$

which here may be satisfied by $(m_1, m_2, m_3) = (1, 1, -1)$, $(k_1, k_2, k_3) = (k_m, k_m, 2k_m)$, and $(\omega_1, \omega_2, \omega_3) = (\omega_+, \omega_-, \omega_-)$, where the ω_j are taken to be positive. One can see from FIGURE 1 that $\omega_1 > \omega_2, \omega_3$, so the relative signatures of this triplet are immune to a Galilean shift. Note from FIGURE 1 that a resonant triplet involving two PEMs and one NEM would not have robust relative signatures under a frame shift since the PEMs have larger frequencies.

The lowest-order nonlinear terms come from the third variation of H expanded about the dynamical equilibrium,

$$\delta^3 H = \frac{1}{M} \sum_{j=1}^{2} \int_0^L dx \left(\frac{M^2}{2} \delta\rho_j \delta\phi_{jx}^2 + \frac{\gamma}{6} \delta\rho_j^3 \right). \tag{27}$$

In terms of the Fourier amplitudes this expression is,

$$\begin{aligned} \delta^3 H = \frac{L}{2} \sum_{j=1}^{2} \sum_{m,n=1}^{\infty} \Big[& -M k_m k_n (\rho_{m+n}^{(j)} \phi_{-m}^{(j)} \phi_{-n}^{(j)} - \rho_{m-n}^{(j)} \phi_{-m}^{(j)} \phi_{-n}^{(j)} + \text{c.c.}) \\ & + \frac{\gamma}{3M} (\rho_m^{(j)} \rho_n^{(j)} \rho_{-m-n}^{(j)} - \rho_m^{(j)} \rho_{-n}^{(j)} \rho_{n-m}^{(j)} + \text{c.c.}) \Big] \,. \end{aligned} \tag{28}$$

We then effect the same transformations on (28) that led to the diagonalized free energy (25). This spawns a myriad of nonlinear terms, only some of which survive an averaging process that leads to the Birkhoff normal form [20].

For a three-wave resonance, one finds after near-identity transformations that the only higher-order terms contributing to the normal form are of the type [14],

$$\mathcal{O}(3)\ \text{Terms} \sim J_1^{|l|/2} J_2^{|m|/2} J_3^{|n|/2} \sin(l\theta_1 + m\theta_2 + n\theta_3)\,, \tag{29}$$

with $|l| + |m| + |n| = 3$. The Hamiltonian up to third-order terms for the resonant NEM/PEM triplets can be written,

$$H = \omega_1 J_1 - \omega_2 J_2 - \omega_3 J_3 + \alpha\sqrt{J_1 J_2 J_3}\sin(\theta_1 + \theta_2 + \theta_3)\,, \tag{30}$$

where $\alpha = \alpha(k_m, M, L)$ is a nonlinear coupling constant that is neither especially large or small in the parameter regime of the three-wave resonance considered here.

Since the angles $(\theta_1,\theta_2,\theta_3)$ appear in only one combination in H, further simplification of (30) is possible via the generating function,

$$F_2(\mathbf{I}, \boldsymbol{\theta}) = \frac{1}{2} I_1(\theta_1 + \theta_2) + I_2\theta_2 + I_3\theta_3, \tag{31}$$

with $\psi_j = \partial F_2 / \partial I_j$ and $J_j = \partial F_2 / \partial \theta_j$. This canonical transformation yields,

$$H = \frac{1}{2}\tilde{\omega}_1 I_1 - \omega_3 I_3 + \frac{\alpha}{2}\sqrt{I_1(I_1 + 2I_2)I_3}\sin(2\psi_1 + \psi_3) \tag{32}$$

with $2\tilde{\omega}_1 \equiv \omega_1 - \omega_2$. If one chooses initial conditions satisfying $I_2 \equiv J_2 - J_1 = 0$, then H is identical to the normal form of a two-wave interaction originally presented by Cherry [21], [22]. In terms of the $(\mathbf{q}, \mathbf{p})$ variables, Cherry's Hamiltonian is

$$H = \frac{1}{2}\tilde{\omega}_1(p_1^2 + q_1^2) - \frac{1}{2}\omega_3(p_3^2 + q_3^2) + \frac{\epsilon}{2}(2q_1 p_1 p_3 - q_3[q_1^2 - p_1^2])\,, \tag{33}$$

where $\epsilon = \sqrt{2}\alpha/4$.

The dynamical system generated by the Cherry Hamiltonian is integrable. In the special case of a third-order resonance with $\omega_3 = 2\tilde{\omega}_1$, there exists a family of two-parameter solutions,

$$\begin{aligned} q_1 &= \frac{\sqrt{2}}{\xi - \epsilon t}\sin(\tilde{\omega}_1 t + \eta)\,, \quad p_1 = -\frac{\sqrt{2}}{\xi - \epsilon t}\cos(\tilde{\omega}_1 t + \eta)\,, \\ q_3 &= -\frac{1}{\xi - \epsilon t}\sin(2\tilde{\omega}_1 t + 2\eta)\,, \quad p_3 = -\frac{1}{\xi - \epsilon t}\cos(2\tilde{\omega}_1 t + 2\eta)\,, \end{aligned} \tag{34}$$

where ξ and η are constants depending on the initial conditions.

The solutions (34) show the possibility of *finite-time density singularities* when two negative energy modes interact resonantly with a positive energy mode. A system exhibiting this behavior is said to undergo *explosive growth*, and it could be an important mechanism for structure formation in galactic and cosmological settings when relative motion between different fluid species is involved. If the resonance is detuned, separatrices bounding stable orbits emerge in phase space, but the dynamics are still prone to finite-amplitude instability.

5. DISSIPATIVE INSTABILITY

We close our investigation of the consequences of negative energy modes by examining the effects of dissipation on the linear stability of the system. With negative energy modes propagating through a dissipative medium, we may expect new instabilities since the damping can pump more negative energy into the wave. This somewhat counterintuitive effect of frictional forces in other contexts was first pointed out by Kelvin and Tait [23] (see also Zajac [24]).

Suppose that collisions are important at some stage in the development of a gravitationally bound structure. A simple model of this effect incorporates a dynamical friction term $(-1)^j\nu(\mathbf{u}_1 - \mathbf{u}_2)$ on the right side of the momentum equations (2), where ν is a positive damping coefficient.

In the dimensionless symmetric case the dispersion relation becomes,

$$\omega^4 - 2i\nu\omega^3 + [1 - 2k^2(M^2+1)]\omega^2 + 2i\nu[k^2(M^2+1) - 1]\omega + k^2(M^2-1)[k^2(M^2-1)+1] = 0\,. \tag{35}$$

If we assume the damping is weak, $\nu \ll 1$, we may develop (35) in a regular perturbation series in ν to find the lowest-order corrections to the frequencies (10),

$$\omega_1^{\pm} = \frac{i}{2}\left(1 \pm \frac{1}{\sqrt{1 - 8k^2M^2 + 16M^2k^4}}\right), \tag{36}$$

where we assume $k^2 \gg (M \pm \sqrt{M^2-1})/(4M)$ to avoid the singularity accompanying the vanishing denominator in (36) (see Casti [16] for the details). If we assume $M >$ 1 so that negative energy modes are present, then a close examination of the corrections reveals that the dissipation promotes instability in the wavenumber band $(M \pm \sqrt{M^2-1})/(4M) \ll k < \sqrt{2}/2$ for any $M \geq 1$ *no matter how weak the damping*. Since the instability as $k^2 \to ½$ is realized only in the $M \to \infty$ limit of the undamped problem, we see that the dissipation indeed has the effect of destabilizing modes that were stable in the conservative case. A numerical investigation revealed that this result holds for any $\nu > 0$.

The modal bands destabilized by the damping become more significant in the asymmetric case. As remarked in Section 3.2, bubbles of unstable two-stream modes can pinch off from the Jeans-unstable bubble and result in well-separated instability bands. The inclusion of dissipation can destabilize the entire band of modes separating the bubbles, as well as some higher-k modes beyond the undamped two-stream bubble. This is illustrated in FIGURE 4.

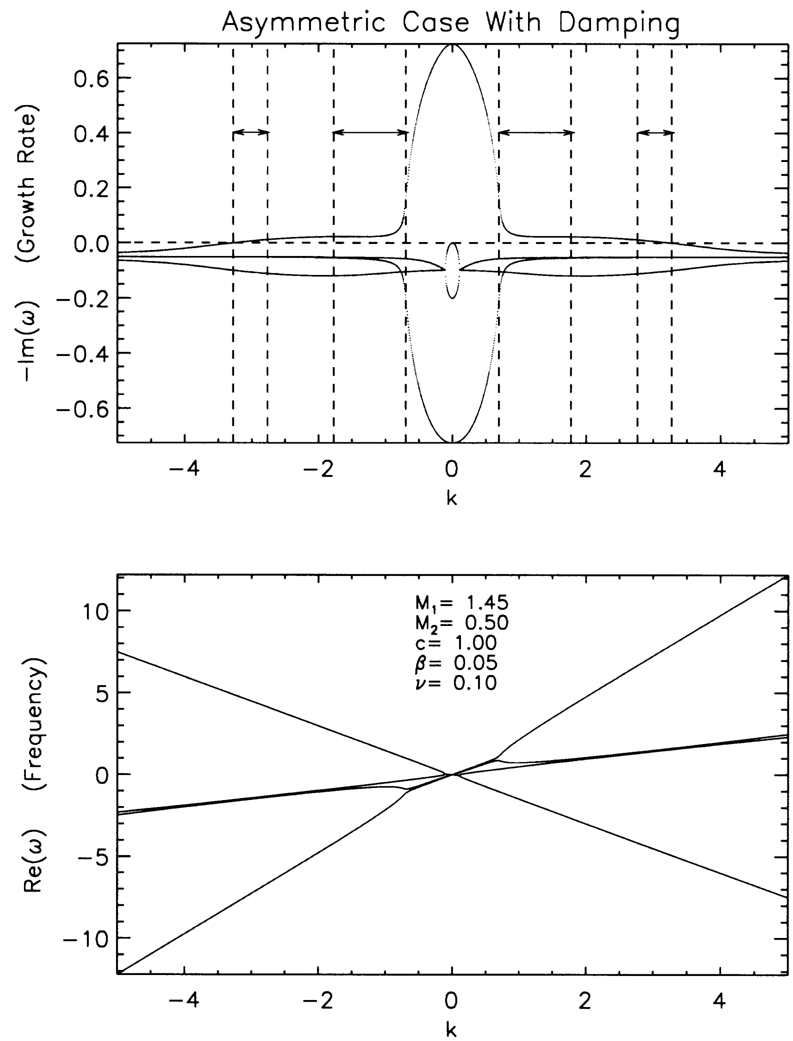

FIGURE 4. Dispersion curves for damped supersonic motion, $\nu = 0.1$, $M_1 = 1.45$, $M_2 = 0.5$, in the asymmetric case with $\beta = .05$ and $c = 1$. The entire k-band separating the Jeans bubble and two-stream bubbles of FIGURE 2 are now unstable. The modal bands destabilized by the dissipation are demarcated by the arrows between the dashed vertical lines.

One should not assume that any form of dissipation will destabilize negative energy modes. For instance, if each fluid feels only a drag proportional to its own velocity, there are no new instabilities even with relative motion. In other words, the dissipation must in some sense project onto the eigenspace spanned by the negative energy modes in a way that decreases their energies. This depends not only on the nature of the dissipation, but also upon the initial equilibrium about which one perturbs.

The effect of damping can be understood by examining the time evolution of the free energy. If the dissipation acts to increase the energy of a positive energy mode or decrease the energy of a negative energy mode, then one can show that pure imaginary eigenvalues take on a positive real part [25]. To see that this is possible here, consider the temporal change in the free energy, which in the symmetric case can be written,

$$\delta^2\dot{H} = -\nu M\int_0^L dx(\delta\phi_{1x} - \delta\phi_{2x})^2 - \nu M\int_0^L dx(\delta\phi_1 - \delta\phi_2)(\delta\rho_{1x} + \delta\rho_{2x})\,. \tag{37}$$

Since the first term of $\delta^2\dot{H}$ is negative definite, the conditions for which the free energy decays or grows in time are determined by the relative phasings of the velocity and density perturbations comprising the second term. One may deduce the effect of the dissipation on any particular mode of the conservative problem by inserting the undamped modes into (37), which yields a formula for $\delta^2\dot{H}$ valid up to $\mathcal{O}(\nu^2)$. For subsonic relative motion, $M < 1$, the expression (37) is negative definite and the damping lives up to its name and causes the PEMs to decay in time. When $M > 1$, $\delta^2\dot{H}$ can be either positive or negative for an NEM depending on the value of k, which explains why some NEMs are destabilized and others are damped in the usual sense.

6. DISCUSSION

The formation of structures through the action of gravity is much analyzed in cosmology, galactic structure and cosmogony. Most of this analysis is centered on the operation of gravitational instability, though streaming fluids can resonantly interact via the gravitational field to cause linear instability in spectral ranges inaccessible to the traditional Jeans instability. As we have brought out here, the distinction between the two types of unstable modes, the Jeans and the two-stream, becomes sharper when one constituent is far denser than the other. Much of the previous work on the subject failed to take advantage of this crucial feature by focusing attention on situations where each component exists in equal abundance. Even when the two-stream instability does not occur, if the total energy of the gravitational two-stream interaction is indefinite, the positive and negative energy modes that are *stable* in the linear theory can interact to produce explosive development of disturbances of arbitrarily small amplitude. This can be a significant aspect of the theory of structure formation.

There are many clear instances where the dynamics of interpenetrating fluids may play a role in developing structures, but we close here by suggesting that even when the streaming is not apparent, two-stream dynamics may be relevant. An interesting example is provided by the coexistence of dark matter and luminous (baryonic) matter that is generally believed to occur throughout the cosmos. The locations of the two kinds of matter seem to be well correlated, which would not be the case if they were now streaming through each other. On the other hand, it might be reasonable to ask why there is this apparent correlation (or anticorrelation in the case of negative gravitational density) of the two kinds of material. Even if they had once been in relative motion, this situation would not long persist, as we have seen. But the outcome, as far as large-scale structure is concerned, could be quite different if the kinematic history of the interaction of the two matters had been richer than has been supposed hitherto. Given the indefiniteness of the free energy if the initial streaming is large enough, waves of short-length scale could have interacted in an explosive manner to quickly produce highly nonlinear density fluctuations. This is a feature of gravitational structure development that could be profitably studied, particularly in situations where the background Hubble expansion cannot be ignored. The dynamics of a two-fluid system with initially *time-dependent* relative motion is currently under investigation.

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