Strong echo effect and nonlinear transient growth in shear flows

J. Vanneste[a)]
Department of Physics, University of Toronto, Toronto, Canada

P. J. Morrison
Department of Physics and Institute for Fusion Studies, University of Texas at Austin, Austin, Texas 78712

T. Warn
Department of Oceanic and Atmospheric Sciences, McGill University, Montreal, Canada

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The nonlinear interaction of two disturbances excited successively in a two-dimensional Couette flow is shown to lead to a transient energy growth. This phenomenon, which is called the echo effect and exists in several other physical systems, is interesting because the energy growth appears long after the energy associated with the original disturbances has decayed. Here, the echo effect is studied analytically and numerically in a situation where the nonlinear response has the same order of magnitude as the two excitations. A system of amplitude equations describing the nonlinear interactions between three sheared modes is derived and employed to examine the physical mechanism of the echo. The qualitative validity of this system is confirmed by numerical simulations. The influence of viscous dissipation on the echo effect is also considered. © *1998 American Institute of Physics.* [S1070-6631(98)01406-8]

I. INTRODUCTION

It is now widely recognized that the kinetic energy of an infinitesimal disturbance in a shear flow can be significantly amplified even if the flow is spectrally stable. Such an amplification, which is then followed by a decay in the long-time limit and is generally referred to as transient growth, was originally identified by Orr[1] in a two-dimensional Couette flow. In recent years, it has been studied in connection with various problems in hydrodynamics stability[2–7] and in meteorology.[8]

The physical mechanism behind transient growth is particularly transparent for two-dimensional incompressible flows, which are governed by a vorticity equation. In such flows, the disturbance vorticity is advected and sheared by the basic-flow velocity, and the disturbance energy crucially depends on the phase mixing that appears when the streamfunction is derived from the vorticity. In the long-time limit, the scale of the vorticity field systematically decreases with time, leading to an enhanced phase mixing and thus to a decrease of the disturbance streamfunction and energy[9]—this decrease is sometimes referred to as Landau damping, by analogy with the similar phenomenon of decrease of electrostatic energy in plasmas.[10] However, if the initial disturbance is dominated by vorticity lines tilted against the shear, the phase mixing does not evolve monotonically: it temporarily cancels when, after some time, the vorticity lines are perpendicular to the shear, leading to a peak in the disturbance streamfunction and the disturbance energy (the analogue of the plasma electrostatic energy), which is the mark of transient growth.

A crucial feature of (linear) transient growth is the fact that it requires very specific initial conditions: the initial disturbances must be coherent, have a very small scale and a specific orientation for the amplification to be significant.[11] In this paper, we discuss a different mechanism that also leads to a transient amplification of the disturbance, but does not require such a small-scale excitation of the flow. This mechanism, called the echo effect, is essentially nonlinear; it is characterized by a transient growth of the disturbance energy following two successive excitations, as illustrated schematically in Fig. 1. The two successive (spatially-periodic, impulsive) excitations, denoted by a and b, are applied at times $t=\epsilon^{-1}\tau_a<0$ and $t=0$, respectively. Through nonlinear interaction, they produce a delayed response, denoted by c, the echo. The energy peak associated with the echo is isolated, because it appears when the direct responses to the excitations a and b have already decayed away through Landau damping.

The echo effect is in fact common to a variety of physical systems;[12–14] in particular, as noted above, it has been studied in plasmas modelled by the Vlasov–Poisson equations.[13,15] In view of the analogy between these equations and the equations describing the evolution of disturbances in two-dimensional shear flows, the existence of an echo effect in shear flows is not entirely surprising. It has in fact been previously studied by Lifschitz[16] who considered a two-dimensional Couette flow in a channel. Following the plasma derivation, he assumed weak amplitudes for the two excitations and used a regular perturbation expansion to calculate the nonlinear response leading to the echo. When this treatment is valid, the amplitude of the energy peak corresponding to the echo is much smaller than the maximum

[a)]Present address: DAMTP, University of Cambridge, Silver Street, Cambridge CB3 9EW, United Kingdom. Telephone: 44(0)1223 337886; Fax: 44(0)1223 337918; Electronic mail: J.Vanneste@damtp.cam.ac.uk

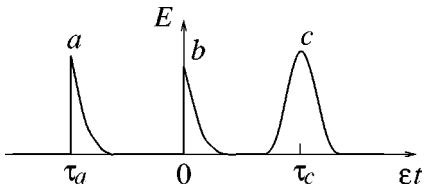

FIG. 1. Schematic representation of the echo effect showing the time evolution of the disturbance energy. At $t=\epsilon^{-1}\tau_a$ and $t=0$, the shear flow is disturbed by spatially periodic excitations. The energy of the response to those excitations decays through Landau damping, but because of nonlinear effects, a third peak—the echo—appears in the energy after a time $t=\epsilon^{-1}\tau_c$.

amplitude of the forced responses a and b (although when the echo appears the amplitude of these responses may have decayed sufficiently and be negligible).

In this paper, we shall be concerned with a somewhat different—and more spectacular—phenomenon, which we refer to as the *strong* echo and which arises when the three energy peaks in the energy have comparable amplitudes. This is possible provided that the amplitude of the two excitations is $O(\epsilon)$, with $\epsilon \ll 1$, while the time-lag between them is $O(\epsilon^{-1})$ (τ_a and τ_c introduced in Fig. 1 are then $O(1)$ quantities). However, in that case, the perturbative treatment of Ref. 16 is not strictly valid; we therefore employ numerical simulations to demonstrate the existence of a strong echo. We also considerably simplify the problem by dealing with a Couette flow in an unbounded domain as opposed to a channel. We note, however, that the physical mechanism at work is one of generality. With the simplified Couette geometry, the disturbance vorticity and streamfunction can be expanded in terms of sheared modes (the exact solutions originally discovered by Kelvin[17]) whose form is very simple; the echo effect may then be interpreted as the nonlinear interaction between three sheared modes. Exploiting this, we derive a system of three amplitude equations (analogous to the three-wave equations for wave triads) which capture the essence of the echo effect. Although these amplitude equations cannot be obtained completely rigorously, they provide results which compare fairly well with numerical simulations. As will be seen, the echo effect involves disturbances with small spatial scales, and hence may be expected to be significantly affected by viscous dissipation. We investigate this influence by deriving an estimate for the viscous damping of the echo and confirm our findings numerically.

II. SHEARED MODES IN COUETTE FLOW

The nonlinear evolution of disturbances in a two-dimensional Couette flow $U=\Lambda y$ is governed by the vorticity equation

$$\left(\frac{\partial}{\partial t}+y\frac{\partial}{\partial x}\right)\omega+\frac{\partial(\psi,\omega)}{\partial(x,y)}=\mu\nabla^2\omega, \quad \text{with} \quad \nabla^2\psi=\omega, \tag{1}$$

where ψ is the streamfunction and μ the inverse of a Reynolds number. This equation has been rendered dimensionless using Λ^{-1} as a timescale and a reference length L as a lengthscale. The dimensional viscosity ν is thus related to μ through $\mu=\nu/(\Lambda L^2)$. A standard procedure to study Couette flows is to introduce the convected coordinates (the importance of which was noted, e.g., in Ref. 18 for this problem),

$$X:=x-yt, \quad Y:=y, \quad T:=t,$$

which transform (1) into

$$\frac{\partial\omega}{\partial T}+\frac{\partial(\psi,\omega)}{\partial(X,Y)}=\mu\nabla^2\omega, \tag{2}$$

and give the form

$$\nabla^2=(1+T^2)\frac{\partial^2}{\partial X^2}-2T\frac{\partial^2}{\partial X\partial Y}+\frac{\partial^2}{\partial Y^2},$$

for the Laplacian. Considering a periodic domain in X and Y, we can expand the vorticity and the streamfunction in Fourier series according to

$$\begin{aligned}\omega&=\epsilon\sum_{k_a=-\infty}^{+\infty}\sum_{l_a=-\infty}^{+\infty}A_a(T)\exp[\mathrm{i}(k_aX+l_aY)],\\ \psi&=\epsilon\sum_{k_a=-\infty}^{+\infty}\sum_{l_a=-\infty}^{+\infty}\frac{-A_a(T)}{k_a^2+(l_a-k_aT)^2}\exp[\mathrm{i}(k_aX+l_aY)],\end{aligned} \tag{3}$$

where the subscript a of A_a designates the pair (k_a,l_a) and $A_{-a}=(A_a)^*$. Here, we have introduced a formal parameter $\epsilon\ll 1$ as we are concerned with weak amplitude disturbances. Note that the extension of (3) to an unbounded domain is immediately obtained by replacing the summations by integrals. Introducing (3) into (2) leads to the amplitude equations

$$\begin{aligned}\frac{dA_a}{dT}=\frac{\epsilon}{2}&\sum_{k_b=-\infty}^{+\infty}\sum_{l_b=-\infty}^{+\infty}\\ &\times\left[\frac{1}{k_b^2+(l_b-k_bT)^2}-\frac{1}{k_c^2+(l_c-k_cT)^2}\right]\\ &\times(k_cl_b-k_bl_c)\,A_b^*A_c^*-\mu[k_a^2+(l_a-k_aT)^2]A_a,\end{aligned} \tag{4}$$

where c is defined by the interaction conditions

$$k_a+k_b+k_c=0, \quad l_a+l_b+l_c=0. \tag{5}$$

(We consider sum interactions only, allowing for both positive and negative values of the wavenumbers.)

We now concentrate on a strictly inviscid fluid with $\mu=0$. As is well known, a single mode with constant amplitude $A_a(T)=A_a(0)$ is then an exact solution of (4). In the original variables (x,y,t), this solution consists of a sheared mode whose vorticity

$$\omega=\epsilon A_a(0)\exp\{\mathrm{i}[k_ax+(l_a-k_at)y]\} \tag{6}$$

behaves like a passive tracer: it is constant along straight lines that are simply tilted by the shear—the slope of these lines evolves as $1/(t-l_a/k_a)$. The time evolution of the disturbance kinetic energy $\frac{1}{2}\int\int|\nabla\psi|^2dxdy$ of a sheared mode is given by

$$E_a(t)=\frac{\epsilon^2|A_a(0)|^2}{k_a^2+(l_a-k_at)^2}=\frac{k_a^2+l_a^2}{k_a^2+(l_a-k_at)^2}E_a(0).$$

Although in the long-time limit this energy decays like t^{-2}, it can be temporarily amplified if $k_a/l_a>0$, i.e., if the vorticity lines are initially tilted against the shear. It is this transient growth, identified originally by Orr,[1] that has motivated much of the literature on sheared modes.[2–7]

III. ECHO EFFECT

A superposition of sheared modes is generally not an exact solution of the nonlinear equation (1). Analyzing spectral equations analogous to (4), Tung[19] nevertheless concluded that it represents an $O(\epsilon)$ approximation to an exact solution and is uniformly valid in time, because of the explicit decrease of the nonlinear terms. However, as pointed out by Haynes[20] in his study of the stability of sheared modes, this conclusion assumes that all the wavenumbers l_a are $O(1)$. If at least one wavenumber l_a is $O(\epsilon^{-1})$, i.e., if initially the disturbance partly consists of strongly tilted vorticity lines, the nonlinearity can have nontrivial consequences at leading order that cannot be captured by a regular perturbation expansion. The echo effect which we now describe relies on this fact and requires the simultaneous presence of two modes: a mode a with $l_a=O(\epsilon^{-1})$ and a mode b with $l_b=O(1)$. A natural way to achieve this configuration is to force the two modes at two successive instants separated by a time of $O(\epsilon^{-1})$. Successive excitations have been traditionally considered to study other echo effects both theoretically and experimentally;[12–16] however, in these studies, the time lag is assumed to be much smaller than ϵ^{-1}, leading to an echo amplitude much smaller than the maximum amplitudes of the forced responses.

Consider mode a forced at $t=\epsilon^{-1}\tau_a<0$, $\tau_a=O(1)$, with vorticity lines initially perpendicular to the shear and initial amplitude $A_a(\epsilon^{-1}\tau_a)$. The subsequent evolution of the vorticity is given by (6), with

$$l_a=\epsilon^{-1}k_a\tau_a, \quad A_a(0)=A_a(\epsilon^{-1}\tau_a), \tag{7}$$

while the energy decays according to

$$E_a(t)=\frac{\epsilon^2|A_a(\epsilon^{-1}\tau_a)|^2}{k_a^2[1+(t-\epsilon^{-1}\tau_a)^2]}=\frac{1}{1+(t-\epsilon^{-1}\tau_a)^2}E_a(\epsilon^{-1}\tau_a).$$

At $t=0$, when mode a has vorticity lines strongly tilted in the direction of the shear (their slope is $-\epsilon\tau_a^{-1}>0$) and a very small energy $[E_a(0)\approx\epsilon^2E_a(\epsilon^{-1}\tau_a)/\tau_a^2]$, we excite the second mode b, with $k_b\neq k_a$ and $l_b=O(1)$. It is only essential that $l_b\ll\epsilon^{-1}$, and for simplicity we take $l_b=0$ in what follows. The linear evolution of this mode is again a simple tilt of the vorticity lines, with a decay of the energy. However, the nonlinear interaction of a and b generate a third mode c, whose wavenumbers satisfy (5). Defining,

$$\epsilon^{-1}\tau_c:=\frac{l_c}{k_c}=\epsilon^{-1}\frac{k_a\tau_a}{k_a+k_b}, \tag{8}$$

where τ_c is $O(1)$, it can be seen that the vorticity of c takes the form

$$\begin{aligned}\omega_c&=\epsilon A_c(t)\exp\{\mathrm{i}[k_cx+(l_c-k_ct)y]\}\\&=\epsilon A_c(t)\exp\{\mathrm{i}k_c[x+(\epsilon^{-1}\tau_c-t)y]\},\end{aligned}$$

where the amplitude $A_c(t)$ is determined by the nonlinear interactions. Correspondingly, the energy of c is given by

$$E_c(t)=\frac{\epsilon^2|A_c(t)|^2}{k_c^2[1+(t-\epsilon^{-1}\tau_c)^2]}.$$

If $\tau_c>0$, which is achieved if

$$\frac{k_a\tau_a}{k_a+k_b}>0, \quad \text{i.e., if } k_ak_b<0 \text{ and } |k_a|<|k_b|, \tag{9}$$

one can expect the appearance of a peak in E_c for $t\approx\epsilon^{-1}\tau_c$. At that time, both E_a and E_b have decreased to $O(\epsilon^4)$ while E_c is $O(\epsilon^2)$ since, as shown below, $A_c(\epsilon^{-1}\tau_c)=O(1)$. The peak in E_c can thus be viewed as the echo of two modes which have already damped away (see Fig. 1). Note that, because sheared modes are exact nonlinear solutions of (1) (with $\mu=0$), we can treat the generation of the echo as an initial-value problem, assuming that both disturbances a and b are initialized at $t=0$ with l_a given by (7) and $l_b=0$.

IV. TRUNCATED MODEL

To investigate the echo effect in more detail, we first consider the truncation of (4) to the triad of modes a,b,c. Neglecting $O(\epsilon)$ terms and using (7) and (8), the evolution equations for A_a,A_b,A_c can be written as

$$\begin{aligned}\frac{dA_a}{dt}&=-J\left\{\frac{1}{k_b^2(1+t^2)}-\frac{1}{k_c^2[1+(t-\epsilon^{-1}\tau_c)^2]}\right\}A_b^*A_c^*,\\\frac{dA_b}{dt}&=-\frac{J}{k_c^2[1+(t-\epsilon^{-1}\tau_c)^2]}A_c^*A_a^*,\\\frac{dA_c}{dt}&=\frac{J}{k_b^2(1+t^2)}A_a^*A_b^*,\end{aligned} \tag{10}$$

where $J:=k_bk_c\tau_c=O(1)$. Clearly, the initial forcing of A_c is $O(1)$ and, in general, one can expect the mode c to quickly reach an $O(1)$ amplitude. This can be confirmed by direct numerical solution of (10). As an example, we consider the modes with

$$k_a=-2, \quad l_a=10, \quad k_b=3, \quad l_b=0, \quad k_c=-1, \quad l_c=-10,$$

which satisfy (5) and (9), such that $\epsilon^{-1}\tau_a=-5$ and $\epsilon^{-1}\tau_c=10$. We take $\epsilon=0.1$ and choose the initial amplitudes $A_a=2$ and $A_b=2\mathrm{i}$. Figure 2 displays the time evolution of the three amplitudes. As expected, mode c attains an amplitude comparable to that of a and b due to their nonlinear forcing. For $t=O(\epsilon^{-1})$, A_c is approximately constant (because of the explicit time decrease of the nonlinear interaction term), but A_a and A_b are strongly modulated for $t\approx\epsilon^{-1}\tau_c=10$. The echo effect appears in the cumulative energy, given by

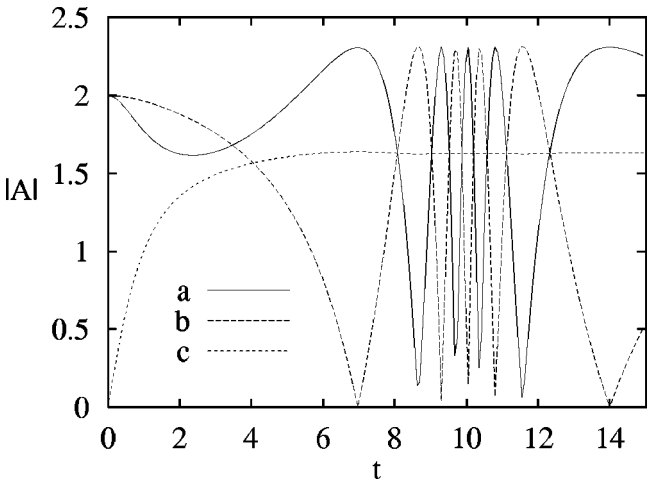

FIG. 2. Time evolution of the amplitudes of the three modes a, b and c as predicted by the truncated system (10).

$$E=\frac{\epsilon^2|A_a|^2}{k_a^2[1+(t-\epsilon^{-1}\tau_a)^2]}+\frac{\epsilon^2|A_b|^2}{k_b^2(1+t^2)}+\frac{\epsilon^2|A_c|^2}{k_c^2[1+(t-\epsilon^{-1}\tau_c)^2]},$$

whose evolution is shown in Fig. 3 (dashed curve). Initially, the energy is dominated by the contribution of mode b and decays as predicted by the linear theory (dotted curve). For longer time, however, the contribution of mode c is dominant and the energy exhibits a clear peak for $t\approx\epsilon^{-1}\tau_c$ corresponding to the echo. (The energy had also peaked after the excitation of mode a, at $t=\epsilon^{-1}\tau_a<0$.)

It is possible to derive an approximate analytical solution of (10) and thus to obtain an estimate for the echo amplitude by noting that with $\epsilon\ll 1$ the nonlinear terms are significant only for $t\leqslant O(1)$ and $t-\epsilon^{-1}\tau_c\leqslant O(1)$. For $t\leqslant O(1)$, A_b is almost constant and simply plays a catalytic role in the interaction between modes a and c. Similarly, for $t-\epsilon^{-1}\tau_c\leqslant O(1)$, A_c is almost constant; the evolution of A_a and A_b during this second time period is irrelevant for the echo effect. Focusing on $t\leqslant O(1)$, we approximate (10) by the linear system

$$\frac{dA_a}{dt}=\frac{-J}{k_b^2(1+t^2)}A_b^*A_c^*,\qquad \frac{dA_c}{dt}=\frac{J}{k_b^2(1+t^2)}A_a^*A_b^*,\tag{11}$$

where $A_b=A_b(0)$ is kept constant. The solution corresponding to the initial conditions $A_a=A_a(0)$ and $A_c=0$ is

$$A_a(t)=A_a(0)\cos(\alpha\ \arctan\ t),$$

$$A_c(t)=\frac{A_a^*(0)A_b^*(0)}{|A_b(0)|}\sin(\alpha\ \arctan\ t),$$

where $\alpha:=|A_b|J/k_b^2$. Since A_c does not change significantly for $t\gg 1$, we obtain the following estimates for the echo amplitude:

$$|A_c(\epsilon^{-1}\tau_c)|\approx|A_a(0)\sin(\alpha\pi/2)|\rightarrow E_{\text{echo}}\approx\frac{\epsilon^2|A_a(0)|^2}{k_c^2}\sin^2(\alpha\pi/2).\tag{12}$$

In the above example, $\alpha=2/3$, so that the estimates are $|A_c(\epsilon^{-1}\tau_c)|\approx\sqrt{3}=1.73$ and $E_{\text{echo}}\approx 0.03$. Both values compare well with what has been found by solving (10) numerically, as seen from Figs. 2 and 3, although ϵ is only marginally small.

A few remarks can be made about the approximate result (12). First, it indicates that the echo amplitude depends on the initial amplitudes of modes a and b in two distinct manners: the echo amplitude is directly proportional to $A_a(0)$, whereas $A_b(0)$ merely determines a time scale for the evolution of c. Note also the particular dependence of $|A_c|$ on α, i.e., on $|A_b|$ for fixed wavenumbers (and thus fixed J); the echo is maximized for $\alpha=(2n+1)$, where n is an integer, and it disappears for $\alpha=2n$. Although these results only hold for the truncated system (11), one can expect an analogous complex dependence of the echo amplitude on $|A_b|$ in a more realistic situation. Finally, we mention that standard results about (weak) echo, which neglect the feedback of c on a and b, correspond to the limit $\alpha\ll 1$ in (12). This leads to the estimates

$$|A_c(\epsilon^{-1}\tau_c)|\approx\frac{\pi k_c\tau_c|A_a(0)A_b(0)|}{2k_b}=2.09\rightarrow E_{\text{echo}}\approx E_c(\epsilon^{-1}\tau_c)\approx\frac{\epsilon^2\pi^2\tau_c^2|A_a(0)A_b(0)|^2}{4k_b^2}=0.044,$$

which provide the correct order of magnitude, although they significantly overestimate the amplitude of the echo as one might expect.

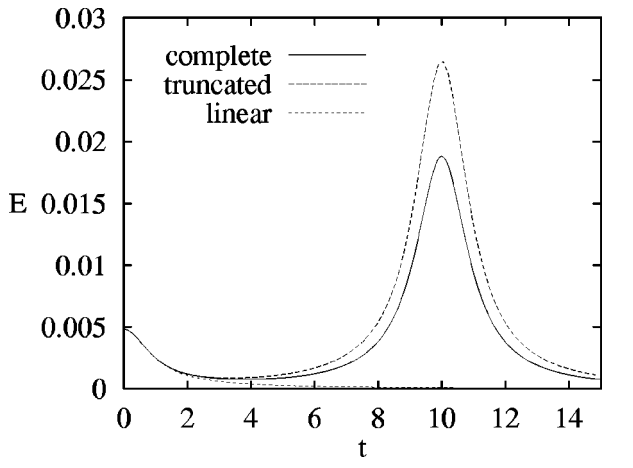

FIG. 3. Time evolution of the total disturbance energy in the system governed by the complete equations (4) with $\mu=0$, in the truncated system (10) and according to the linear theory.

V. NUMERICAL RESULTS

The system (10), which has been derived as an ad hoc truncation of (4), misses important parts of the dynamics of

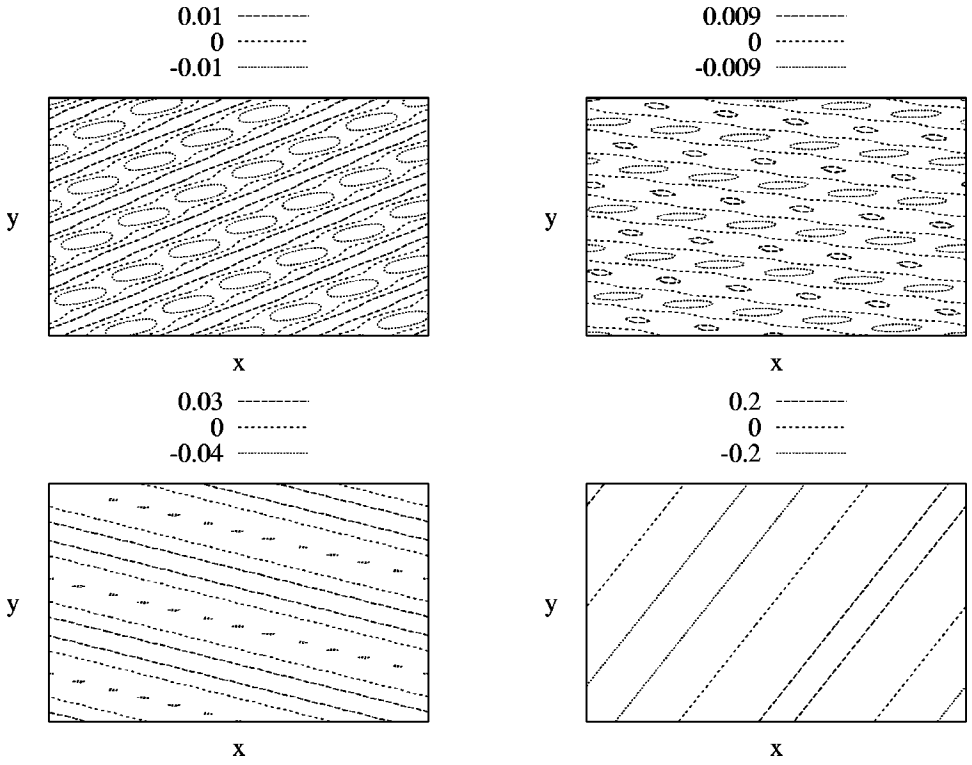

FIG. 4. Disturbance streamfunction for $t=1.5,4.5,7.5$ and 10.5. Note the change in the contour levels.

(4), in particular the generation of a mode d, with $k_d=k_b-k_a$ and $l_d=l_b-l_a$, corresponding to the interaction of a with $-b$. This and other neglected interactions are likely to weaken the echo. It is therefore important to study the echo effect directly with the complete system (4), or in practical terms, with a truncation of (4) keeping many modes. It turns out that convergent results are obtained with a limited number of modes; those presented below have been obtained for the same parameters as before, with a 25×25 truncation, and reflect the behavior of the complete system (the disturbance energy is particularly stable when resolution is varied). The evolution of the energy so obtained is displayed in Fig. 3 (solid curve). As anticipated, the truncated model overestimates the energy peak at $t\approx\epsilon^{-1}\tau_c$. Nevertheless, these results confirm the qualitative validity of the truncated model and the existence of the strong echo effect. Note that Fig. 3 also shows the energy evolution predicted by the linear theory; this theory only captures the Landau damping of mode b and thus completely misses the appearance of the echo.

It is interesting to examine the evolution of the structure of the streamfunction and vorticity during the simulation. Figure 4 shows the streamfunction at $t=1.5,4.5,7.5$ and 10.5. At $t=1.5$, the streamfuction is dominated by mode b, since Landau damping has already acted strongly on mode a, while mode c only begins to emerge. At $t=4.5$, the streamfunction amplitudes of modes b and c are similar, but weak because both have relatively small spatial scales. At $t=7.5$ and $t=10.5$, little remains of mode b in terms of the streamfunction which is dominated by mode c. The amplitude of the streamfunction increases as the spatial scale of this mode increases; it reaches its maximum for $t=10$ before decreasing. Note the factor of 20 between the amplitude of the streamfunction for $t=1.5$ and for $t=10.5$, which indicates the importance of the echo effect.

It is difficult to plot the evolution of the vorticity field in a similar manner, since it contains very small scales associated with mode a, and, after some time, those associated with mode b. Figure 5, which shows the vorticity field with x-wavenumber k_c only, focuses on mode c. The two panels correspond to $t=7.5$ (before the echo) and $t=12.5$ (after the echo). Although a dominant orientation of the vorticity lines can be distinguished and corresponds to the behavior expected from mode c, this figure emphasizes that several sheared modes, with wavenumber $k=k_c$ but different wavenumbers l, are in fact superposed.

As Figs. 4 and 5 clearly illustrate, the echo effect relies on the presence in the flow of disturbances with very small scales. Since such disturbances are strongly affected by dissipation, it is important to consider the echo effect with a nonzero viscosity. A rough estimate of the influence of viscosity can be derived from (4) by noting that, in the linear approximation, any mode amplitude is damped by a factor $\exp[-\mu(k_a^2+l_a^2-k_al_at+k_a^2t^2/3)t]$. The modes a and c involved in the echo have wavenumbers l_a and l_c which are $O(\epsilon^{-1})$, and mode c must remain excited until $t\approx\epsilon^{-1}\tau_c$. Therefore, at the moment of the echo, dissipation is responsible for an overall damping factor of $\exp(-\mu\epsilon^{-3}k_a^2\tau_c^3/3)$. Since $k_a^2\tau_c^3/3=O(1)$, an approximate condition for the echo to occur in a viscous fluid is $\mu\epsilon^{-3}\leqslant1$. The validity of this estimate can be confirmed by direct numerical solution of (4) (again with a 25×25 truncation) for different values of the viscosity parameter μ. The corresponding evolution of the total energy is

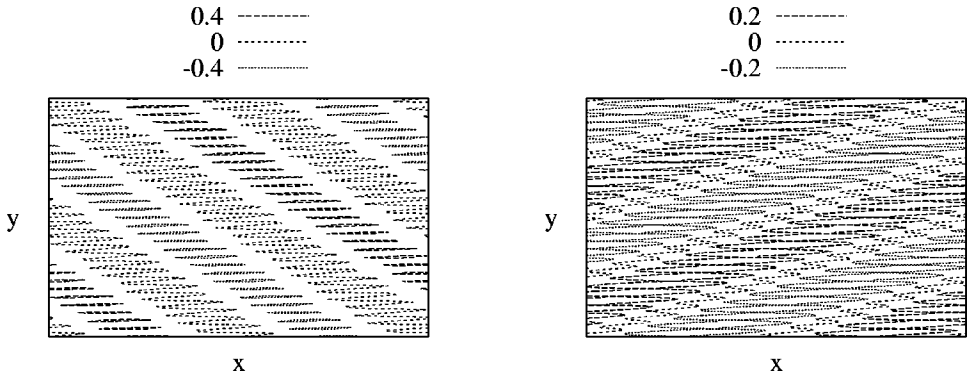

FIG. 5. Vorticity field with wavenumber k_c for $t=7.5$ and 12.

shown in Fig. 6. With the nonzero values chosen for μ, which are $O(\epsilon^3)$, the energy peak significantly decreases, although it remains well defined. It is clear that for $\mu \ll \epsilon^3 = 0.001$, the echo would be virtually unaffected by dissipation, whereas for $\mu \gg \epsilon^3$ it would entirely disappear.

VI. CONCLUSION

In this paper, we have studied the echo effect in a two-dimensional Couette flow both analytically and numerically. This phenomenon can be regarded as a nonlinear mechanism of transient growth: a disturbance strongly tilted against the shear, and thus susceptible to experience a significant energy amplification, is generated by the nonlinear interaction of two other disturbances which are excited successively. The time lag between these two excitations, when large as is assumed here, provides a natural way of introducing in the system disturbances with very different scales. This leads to a significant nonlinear effect although the initial excitation amplitude is weak. The echo effect, as the instability of sheared modes,[20] thus illustrates the difficulties that may arise when linearizing evolution equations for disturbances in stable shear flows: stability guarantees that some norm of the disturbance—the enstrophy for Couette flows—is bounded, but this does not preclude the nonlinear terms to be large since they involve vorticity gradients which increase with time.

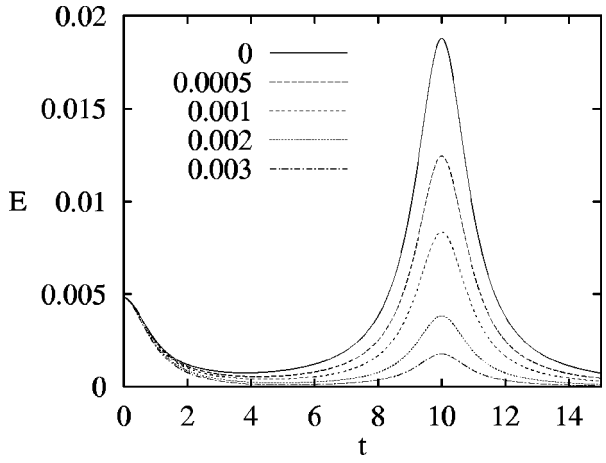

FIG. 6. Time evolution in the system governed by the complete equations (4) for different values of the viscosity parameter μ.

Our investigation of the echo effect is particularly simple, mainly thanks to the very simple form taken by sheared modes in two-dimensional Couette flows when the domain is unbounded. The physical mechanisms involved, however, are generic to all monotonic shear flows, so that echoes can be expected to occur in a variety of situations. Geophysical flows seem especially interesting in this respect, since they are only slightly affected by dissipation; this motivates the extension of our work to include the effects of rotation, curvature (β-effect) and stratification. Spatial echoes[15] also deserves investigation, notably because this phenomenon would be well suited for an experimental demonstration of the echo effect in shear flows.

ACKNOWLEDGMENTS

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