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Computing Casimir invariants from Pffafian systems

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Abstract

We describe a method for computing Casimir invariants that is applicable to both finite and infinite-dimensional Poisson brackets. We apply the method to various finite and infinite-dimensional examples, including a Poisson bracket embodying both finite and infinite-dimensional structure. © 1999 Published by Elsevier Science B.V. All rights reserved.

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1. Introduction

Non-canonical Hamiltonian, or Poisson, systems have been studied extensively by mathematical physicists, and with good reason. They arise in fields as diverse as dynamical systems theory [1,2], fluid dynamics and plasma physics [3,4], and condensed matter physics [5]. (For more references see [6,7].)

One of the key features of Hamiltonian systems is the existence of a Poisson bracket. This object finds many applications, including computing perturbative solutions [8], determining stability of steady states [9], and studying integrable systems [2,10,11].

The ubiquity of the Poisson bracket has led to study of it in its own right. In particular, a good deal of work has gone into exploring its geometric theory [12,13] (also see references on the related subject of symplectic geometry [14]). In this letter, we consider only one facet of this rich geometry, that related to Casimir invariants. These are functions whose Poisson bracket with any other function vanishes. Their existence for any finite-dimensional degenerate Poisson bracket follows from the Frobenius theorem of differential geometry [15].

The Casimir invariants of a given Poisson bracket are important because they are conserved quantities in any Hamiltonian system that uses that bracket. As such, they play a vital part in reducing the order [16], or even

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integrating some systems [17]. In addition, they are central to both the Energy-Casimir method of determining stability [9], and the Semenov–Tian–Shansky scheme of constructing integrable systems [11]. In all these applications, the actual computation of Casimir invariants is essential.

We present a method for computing Casimir invariants. This method amounts to integrating the null covectors of the second rank tensor (the cosymplectic form) defined by the Poisson bracket. This is a natural strategy: many of the Casimir invariants in [7] were calculated in this fashion, a fact not noted there. This method *has* been presented explicitly, for infinite-dimensional brackets in [18] and finite-dimensional brackets in [19]. Here, we show that these papers both treat special cases of a single method by presenting the method in a geometrical framework that includes both cases (Section 2). This presentation has a partly pedagogical function: we present a transparent example of differential geometry applied to Hamiltonian dynamics. But also, it offers a clear framework for arriving at a novel result: the form of Casimir invariants for Poisson brackets with a nested Lie–Poisson structure, involving both finite and infinite-dimensional brackets. The method is used to compute these Casimir invariants (and those of several other Poisson brackets) in Sections 3 and 4.

2. Casimir invariants and Pfaffian systems

The Poisson bracket is a mathematical structure common to all Hamiltonian systems. If z^{α} denotes a coordinate of a point in a phase space M, the equations of motion generated by a Hamiltonian H:M $\to \mathbb{R}$ are $\dot{z}^{\alpha} = \{z^{\alpha}, H\}$ where $\{\cdot,\cdot\}: C^{\infty}(\mathrm{M}) \times C^{\infty}(\mathrm{M}) \to C^{\infty}(\mathrm{M})$.

The bracket that appears in the equation of motion is a Poisson bracket if it is antisymmetric, bilinear, and it satisfies two more conditions. These are the Leibnitz rule, $\{f,gh\} = g\{f,h\} + \{f,g\}h$ and the Jacobi identity,

$$\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0. \tag{1}$$

The most familiar example of a bracket satisfying these properties is given by the so-called canonical bracket:

$$\{f,g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q^i}\frac{\partial f}{\partial p_i}. \tag{2}$$

But there are the other, non-canonical brackets that obey the above conditions; some are given in the examples. However, such non-canonical brackets, even when they are degenerate, can be locally represented in the form (2) (where n is equal to half the rank of the bracket), as shown by the Darboux theorem (see also [12]). In this sense, they are true generalizations of the more familiar canonical brackets.

By virtue of the above properties, a Poisson bracket defines a second rank tensor called the cosymplectic form, $\mathrm{J}:\mathrm{T}^*\mathrm{M} \to \mathrm{TM}$, in the following way:

$$\langle \mathrm{d}f(z), \mathrm{J}(z)\mathrm{d}g(z)\rangle := \{f,g\}(z) \tag{3}$$

where the angular brackets denote the pairing between the cotangent and tangent spaces at the point $z \in \mathrm{M}$. In a given set of coordinates z^{α}, for a finite-dimensional system, Eq. (3) has the form

$$\frac{\partial f}{\partial z^{\alpha}}\mathrm{J}^{\alpha\beta}\frac{\partial g}{\partial z^{\beta}} := \{f,g\}. \tag{4}$$

(We use the summation convention.) The vector field given by $\mathrm{J}\mathrm{d}g$ is called the Hamiltonian vector field generated by g.

Casimir invariants are functions that have a zero Poisson bracket with any other function. In other words, given a Poisson bracket, if, for every g, a function C satisfies $\{g,C\} = 0$, then C is a Casimir invariant.

This definition is the starting point of any method to compute Casimir invariants. In terms of the tensor J, it becomes $\langle \mathrm{d}g, \mathrm{J}\mathrm{d}C\rangle = 0$. Since g is arbitrary, solving this equation amounts to finding the functions C from the condition that the vector field $\mathrm{J}\mathrm{d}C$ vanishes. In the case of a finite-dimensional system, this computation requires solving a set of coupled partial differential equations for the function C: $\mathrm{J}^{ij}\partial C/\partial z^j = 0$. However, a more efficient method for computing Casimir invariants follows from an alternate geometric interpretation of $\{g,C\} = 0$. Making use of the antisymmetry of the Poisson bracket, we rewrite $\langle \mathrm{d}g,\mathrm{J}\mathrm{d}C\rangle = 0$ as $\langle \mathrm{d}C,\mathrm{J}\mathrm{d}g\rangle = 0$. Since this equation must hold for arbitrary g, we can interpret it as saying that the differential one-form given by $\mathrm{d}C$ is annulled by every vector field in the image of J.

Geometrically, this means that the vector field $\mathrm{J}\mathrm{d}g$ must lie everywhere tangent to the level set of the function C. In this case, any integral curve of this (or any other) Hamiltonian vector field must lie within a single level set of C, which is a way of saying that C is a constant of motion.

These considerations help us to compute Casimir invariants in the following way. The information needed to construct the level sets of Casimir invariants is contained in the differential one-forms that are annuled by the vectors tangent to these sets. Obviously, if we explicitly knew a Casimir invariant C, $\langle \mathrm{d}C,\mathrm{J}\mathrm{d}g\rangle = 0$ tells us that $\mathrm{d}C$ is such a one-form. However, we do not need to know the Casimir invariants to find differential one-forms that are annulled by these vectors. In fact, from $\langle \mathrm{d}g,\mathrm{J}\mathrm{d}C\rangle = \langle \mathrm{d}C,\mathrm{J}\mathrm{d}g\rangle = 0$, it is clear that any element in the kernel of J will do. And so, we can get the information we need to determine all of the Casimir invariants by finding all the linearly independent null covectors of J.

The null covectors of J, call them $\gamma^{(i)}$ (suppose there are n), are differential one-forms. They are not necessarily exact: that is, it is not necessarily true that there exists an F such that $\gamma^{(i)} = \mathrm{d}F$.

But once we have the $\gamma^{(i)}$, we can find the Casimir invariants from the condition that the $\gamma^{(i)}$ vanish when paired with any vector tangent to a level set of a Casimir invariant: $\gamma^{(i)} = 0$ where i varies from 1 to n. A system of equations, like this one, given by setting a collection of one-forms to zero, is called a Pfaffian system. The level sets of the functions on which the restrictions of these one-forms vanish are called the integral manifolds of the system.

Finding the integral manifolds is conceptually (but not always computationally) simple. If we were able to linearly combine the $\gamma^{(i)}$ into n independent exact differentials, we would obtain a system of equations equivalent to $\gamma^{(i)} = 0$. This system would have the form $\mathrm{d}F^{(j)} = 0$ where j varies from 1 to n. The solution to this system of equations is obviously given by $F^{(j)} = \text{constant}$. In other words, the integral manifolds are just the level sets of the $F^{(j)}$. By the chain rule, we can then write the general solution of $\gamma^{(i)} = 0$ in terms of an arbitrary function $k = k(F^{(1)}, \cdots, F^{(n)})$.

It is not always possible to transform a Pfaffian system in this way. The conditions for when it is possible are given by the Frobenius theorem. In [15] it was shown that the possibility of doing this with the set of null covectors of a finite-dimensional J follows from Eq. (1). Later, in [18], an argument was given that extends this proof to continuum systems. However, there are some infinite-dimensional spaces in which the Frobenius theorem does not hold (see [20]), so this should be seen as a formal argument.

Examples of applying the above process are given in the next two sections. In the finite-dimensional case, the greatest complication is finding integrating factors to make the resulting equations exact differential equations (see Section 3). The infinite-dimensional case is typically more complicated. Finding components of the null covectors requires solving partial differential equations, and it is not always obvious how to integrate the Pfaffian system.

3. Casimir invariants of finite-dimensional brackets

For our first example, we will find the Casimir invariant of a familiar Hamiltonian system: the free rigid body. The components of J are given by: $\mathbf{J}^{ij} = \epsilon_k^{ij} z^k$ where $i,j = 1\ldots3$, and ϵ denotes the Levi-Civita symbol, as usual. In fact, this bracket is the Lie–Poisson bracket associated to the Lie algebra so(3) (see Section 4). Note that $\mathrm{rank(J)} = 2$ everywhere except at the origin. The null covector of J is immediately found to be $\gamma = z^1\mathrm{d}z^1 + z^2\mathrm{d}z^2 + z^3\mathrm{d}z^3$. And the Pfaffian system $\gamma = 0$ is immediately solved for the well-known Casimir invariant $C = (z^1)^2 + (z^2)^2 + (z^3)^2 = \text{constant}$.

As a second example, we consider the brackets of arbitrary dimension d introduced by Plank [2] for Lotka–Volterra equations (see also [21] for an application of such brackets to more general systems). In this case, the components of J are given by (not summed):

$$\mathrm{J}^{ij} = c_{(ij)} z^i z^j, \quad i,j = 1 \dots d \tag{5}$$

where $c_{(ij)} \in \mathbb{R}$ and $c_{(ij)} = -c_{(ji)}$ for all i,j. It can be shown [2] that the skew-symmetry of the c_{ij} is enough to ensure that J in (5) indeed defines a cosymplectic form. It is simple to prove that every null covector of the matrix J in (5) is of the form:

$$\gamma = a_i \frac{\mathrm{d}z^i}{z^i}, \quad a_i \in \mathbb{R}. \tag{6}$$

Every Pfaffian equation $\gamma = 0$, with γ given by expression (6), can be integrated to give one Casimir invariant:

$$C = a_i \log z^i = \text{constant}. \tag{7}$$

Therefore, all Casimir invariants of (5) can be easily found in this way.

An example that involves solving a less trivial Pfaffian system is the Lie–Poisson bracket associated with the Lie algebra su(3) (see Section 4). This arises in the study of finite-mode analogs of two-dimensional hydrodynamics [22]. With respect to (i times the) Gell-Mann basis of su(3) [23], the components of J are given by

$$\begin{pmatrix}
0 & 2z^3 & -2z^2 & z^7 & -z^6 & z^5 & -z^4 & 0 \\
-2z^3 & 0 & 2z^1 & z^6 & z^7 & -z^4 & -z^5 & 0 \\
2z^2 & -2z^1 & 0 & z^5 & -z^4 & -z^7 & z^6 & 0 \\
-z^7 & -z^6 & -z^5 & 0 & z^3+\sqrt{3}\,z^8 & z^2 & z^1 & -\sqrt{3}\,z^5 \\
z^6 & -z^7 & z^4 & -z^3-\sqrt{3}\,z^8 & 0 & -z^1 & z^2 & \sqrt{3}\,z^4 \\
-z^5 & z^4 & z^7 & -z^2 & z^1 & 0 & -z^3+\sqrt{3}\,z^8 & -\sqrt{3}\,z^7 \\
z^4 & z^5 & -z^6 & -z^1 & -z^2 & z^3-\sqrt{3}\,z^8 & 0 & \sqrt{3}\,z^6 \\
0 & 0 & 0 & \sqrt{3}\,z^5 & -\sqrt{3}\,z^4 & \sqrt{3}\,z^7 & -\sqrt{3}\,z^6 & 0
\end{pmatrix}. \tag{8}$$

The null vectors (and resulting Pfaffian system) of this J were found using *Mathematica* to be

$$\begin{aligned}
\gamma^{(1)} = \Delta^{-1}\Big[& 2\sqrt{3}\left(z^1z^2z^4 - (z^1)^2 z^5 + z^1z^3z^7 + z^4z^6z^7 + z^5(z^7)^2 + \sqrt{3}\,z^1z^7z^8\right)\mathrm{d}z^1 \\
& + 2\sqrt{3}\left((z^2)^2 z^4 - z^1z^2z^5 + z^2z^3z^7 + z^5z^6z^7 - z^4(z^7)^2 + \sqrt{3}\,z^2z^7z^8\right)\mathrm{d}z^2 \\
& + \sqrt{3}\Big(2z^2z^3z^4 - 2z^1z^3z^5 + 2(z^3)^2 z^7 + (z^4)^2 z^7 + (z^5)^2 z^7 - (z^6)^2 z^7 - (z^7)^3 \\
& + 2\sqrt{3}\,z^3z^7z^8\Big)\mathrm{d}z^3 + 2\sqrt{3}\left(z^2(z^4)^2 - z^1z^4z^5 + 2z^3z^4z^7 + z^1z^6z^7 - z^2(z^7)^2\right)\mathrm{d}z^4 \\
& + 2\sqrt{3}\left(z^2z^4z^5 - z^1(z^5)^2 + 2z^3z^5z^7 + z^2z^6z^7 + z^1(z^7)^2\right)\mathrm{d}z^5 \\
& + 2\sqrt{3}\,(z^2z^4z^6 - z^1z^5z^6 + z^1z^4z^7 + z^2z^5z^7)\mathrm{d}z^6 + \Big(2(z^1)^2 z^7 + 2(z^2)^2 z^7 + 2(z^3)^2 z^7 \\
& - (z^4)^2 z^7 - (z^5)^2 z^7 - (z^6)^2 z^7 - (z^7)^3 + 2\sqrt{3}\,(z^2z^4 - z^1z^5 + z^3z^7)\,z^8\Big)\mathrm{d}z^8\Big] = 0,
\end{aligned} \tag{9}$$

and

$$\gamma^{(2)} = \Delta^{-1}\Big[\Big(2(z^1)^3 + 2z^1(z^2)^2 + 2z^1(z^3)^2 - z^1(z^4)^2 - z^1(z^5)^2 - z^1(z^6)^2 - z^1(z^7)^2 - 2\sqrt{3}\,z^4z^6z^8$$
$$- 2\sqrt{3}\,z^5z^7z^8 - 6z^1(z^8)^2\Big)\mathrm{d}z^1 + \Big(2(z^1)^2z^2 + 2(z^2)^3 + 2z^2(z^3)^2 - z^2(z^4)^2 - z^2(z^5)^2$$
$$- z^2(z^6)^2 - z^2(z^7)^2 - 2\sqrt{3}\,z^5z^6z^8 + 2\sqrt{3}\,z^4z^7z^8 - 6z^2(z^8)^2\Big)\mathrm{d}z^2 + \Big(2(z^1)^2z^3 + 2(z^2)^2z^3$$
$$+ 2(z^3)^3 - z^3(z^4)^2 - z^3(z^5)^2 - z^3(z^6)^2 - z^3(z^7)^2 - \sqrt{3}\,(z^4)^2z^8 - \sqrt{3}\,(z^5)^2z^8$$
$$+ \sqrt{3}\,(z^6)^2z^8 + \sqrt{3}\,(z^7)^2z^8 - 6z^3(z^8)^2\Big)\mathrm{d}z^3 + \Big(2(z^1)^2z^4 + 2(z^2)^2z^4 + 2(z^3)^2z^4 - (z^4)^3$$
$$- z^4(z^5)^2 - z^4(z^6)^2 - z^4(z^7)^2 - 2\sqrt{3}\,z^3z^4z^8 - 2\sqrt{3}\,z^1z^6z^8 + 2\sqrt{3}\,z^2z^7z^8\Big)\mathrm{d}z^4$$
$$+ \Big(2(z^1)^2z^5 + 2(z^2)^2z^5 + 2(z^3)^2z^5 - (z^4)^2z^5 - (z^5)^3 - z^5(z^6)^2 - z^5(z^7)^2 - 2\sqrt{3}\,z^3z^5z^8$$
$$- 2\sqrt{3}\,z^2z^6z^8 - 2\sqrt{3}\,z^1z^7z^8\Big)\mathrm{d}z^5 + \Big(2(z^1)^2z^6 + 2(z^2)^2z^6 + 2(z^3)^2z^6 - (z^4)^2z^6$$
$$- (z^5)^2z^6 - (z^6)^3 - z^6(z^7)^2 - 2\sqrt{3}\,z^1z^4z^8 - 2\sqrt{3}\,z^2z^5z^8 + 2\sqrt{3}\,z^3z^6z^8\Big)\mathrm{d}z^6$$
$$+ \big(2(z^1)^2z^7 + 2(z^2)^2z^7 + 2(z^3)^2z^7 - (z^4)^2z^7 - (z^5)^2z^7 - (z^6)^2z^7 - (z^7)^3$$
$$+ 2\sqrt{3}\,z^8(z^2z^4 - z^1z^5 + z^3z^7)\mathrm{d}z^7\Big] = 0, \tag{10}$$

where

$$\Delta = 2(z^1)^2z^7 + 2(z^2)^2z^7 + 2(z^3)^2z^7 - (z^4)^2z^7 - (z^5)^2z^7 - (z^6)^2z^7 - (z^7)^3$$
$$+ 2\sqrt{3}\,(z^2z^4z^8 - z^1z^5z^8 + z^3z^7z^8). \tag{11}$$

Neither Eq. (9) nor (10) can be integrated by itself. So to obtain two independent Casimir invariants C_1 and C_2, we must find exact linear combinations of the equations. For example,

$$z^8\gamma^{(1)} + z^7\gamma^{(2)} = z^1\mathrm{d}z^1 + z^2\mathrm{d}z^2 + z^3\mathrm{d}z^3 + z^4\mathrm{d}z^4 + z^5\mathrm{d}z^5 + z^6\mathrm{d}z^6 + z^7\mathrm{d}z^7 + z^8\mathrm{d}z^8 = 0. \tag{12}$$

This equation is exact, and has the solution

$$C_1 = (z^1)^2 + (z^2)^2 + (z^3)^2 + (z^4)^2 + (z^5)^2 + (z^6)^2 + (z^7)^2 + (z^8)^2. \tag{13}$$

Another independent solution can be found by taking the linear combination

$$a\gamma^{(1)} + b\gamma^{(2)} = \big(-18z^4z^6 - 18z^5z^7 - 12\sqrt{3}\,z^1z^8\big)\mathrm{d}z^1 + \big(-18z^5z^6 + 18z^4z^7 - 12\sqrt{3}\,z^2z^8\big)\mathrm{d}z^2$$
$$+ \Big(-9(z^4)^2 - 9(z^5)^2 + 9(z^6)^2 + 9(z^7)^2 - 12\sqrt{3}\,z^3z^8\Big)\mathrm{d}z^3$$
$$+ \big(-18z^3z^4 - 18z^1z^6 + 18z^2z^7 + 6\sqrt{3}\,z^4z^8\big)\mathrm{d}z^4$$
$$+ \big(-18z^3z^5 - 18z^2z^6 - 18z^1z^7 + 6\sqrt{3}\,z^5z^8\big)\mathrm{d}z^5$$
$$+ \big(-18z^1z^4 - 18z^2z^5 + 18z^3z^6 + 6\sqrt{3}\,z^6z^8\big)\mathrm{d}z^6$$
$$+ \big(18z^2z^4 - 18z^1z^5 + 18z^3z^7 + 6\sqrt{3}\,z^7z^8\big)\mathrm{d}z^7$$
$$+ \Big(-6\sqrt{3}\,(z^1)^2 - 6\sqrt{3}\,(z^2)^2 - 6\sqrt{3}\,(z^3)^2 + 3\sqrt{3}\,(z^4)^2 + 3\sqrt{3}\,(z^5)^2$$
$$+ 3\sqrt{3}\,(z^6)^2 + 3\sqrt{3}\,(z^7)^2 + 6\sqrt{3}\,(z^8)^2\Big)\mathrm{d}z^8 = 0, \tag{14}$$

where

$$a = -3\sqrt{3}\left[-2(z^1)^2 - 2(z^2)^2 - 2(z^3)^2 + (z^4)^2 + (z^5)^2 + (z^6)^2 + (z^7)^2 + 2(z^8)^2\right], \tag{15}$$

$$b = 6\left[3z^2z^4 - 3z^1z^5 + 3z^3z^7 + \sqrt{3}\,z^7z^8\right]. \tag{16}$$

Integrating Eq. (14) yields the solution

$$\begin{aligned} C_2 = & -18z^1z^4z^6 - 18z^1z^5z^7 - 6\sqrt{3}(z^1)^2z^8 + 18z^2z^4z^7 - 18z^2z^5z^6 - 6\sqrt{3}(z^2)^2z^8 - 9z^3(z^4)^2 \\ & -9z^3(z^5)^2 + 9z^3(z^6)^2 + 9z^3(z^7)^2 - 6\sqrt{3}(z^3)^2z^8 + 3\sqrt{3}(z^4)^2z^8 + 3\sqrt{3}(z^5)^2z^8 \\ & +3\sqrt{3}(z^6)^2z^8 + 3\sqrt{3}(z^7)^2z^8 + 2\sqrt{3}(z^8)^3. \end{aligned} \tag{17}$$

We should note here that for Lie–Poisson brackets based on semi-simple Lie algebras (as in this example), a formula exists for the Casimir invariants. It follows immediately from results in [24] that if the basis elements of the algebra are given as matrices $X^{(i)}$ in any representation of the Lie algebra, then $C = \mathrm{Tr}(X^{(i)} \cdots X^{(j)})z^i \cdots z^j$ is a Casimir invariant (here, z^i is the coordinate dual to the basis vector $X^{(i)}$). Using the Gell-Mann matrices in this formula yields Casimir invariants proportional to (13) and (17).

As complicated as this last example might seem, the method used is certainly more economical than solving the system of coupled PDEs mentioned in Section 3, which is the only systematic alternative known for the computation of Casimir invariants. And while for a small number of PDEs, solutions can often be guessed, this becomes difficult for large systems. The reader is referred to [19] for additional examples.

4. Casimir invariants of infinite-dimensional brackets

For infinite-dimensional examples, we first consider Poisson brackets that occur in the Hamiltonian formulation of the Vlasov–Poisson system and the 2D Euler equation [4,25]. Then, we consider a modification of our first example that might arise in other types of kinetic theories, such as that which describes a spin gas.

All of the examples for this section arise in systems that can be described by a single field variable f. This field is taken to be a function on an n-dimensional space D, with coordinates z^i, and of time. In these examples, the space D is endowed with a Poisson bracket.

The brackets in these examples have the following general form:

$$\{F,G\}[f] = \int_{\mathrm{D}} f\left[\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f}\right]\mathrm{d}^n z. \tag{18}$$

In this expression, The variables F,G represent functionals of the field f, and $\delta F/\delta f$ denotes the functional derivative of F with respect to f. The square brackets in the integrand denote the (finite-dimensional) Poisson bracket defined on the space D. This will be referred to as the inner bracket. This bracket is an infinite-dimensional generalization of the Lie–Poisson bracket, first considered by Lie [26]. (The bracket is sometimes also known as the Kostant–Kirillov bracket [27].)

Furthermore, the brackets we consider can be rewritten in the form

$$\{F,G\} = -\int_{\mathrm{D}} \frac{\delta F}{\delta f}\left[f, \frac{\delta G}{\delta f}\right]\mathrm{d}^n z. \tag{19}$$

This is not true for all brackets of the form (18). Whether it holds or not depends on the nature of the inner bracket, as will be seen below.

Now, from Eq. (19), we can read off the cosymplectic operator as $\mathrm{J} = -[f,\cdot\,]$. So the condition for finding the components γ of the null covector $\Gamma := \gamma\delta f$ of J is simply

$$[f,\gamma] = 0. \tag{20}$$

To solve this completely for γ we need to know the details of the inner bracket. These will be provided for two cases below.

If the inner bracket in (18) is canonical, that is, takes the form (2), an integration by parts (assuming that the field vanishes on the boundary of D) shows that the bracket obeys Eq. (19). Hence, in this case, we can use Eq. (20) to determine its Casimir invariants.

By antisymmetry of the bracket, $\gamma = f$ solves Eq. (20). Hence $\gamma = k(f)$ is also a solution, for an arbitrary function k. Since the variation $k(f)\delta f$ is an exact variation (the formal analogue in infinite dimensions of an exact differential), the Pfaffian system, $k(f)\delta f = 0$, is easily integrated. This yields the expression for the Casimir invariant $\mathscr{C}$:

$$\mathscr{C}[f] = \int_D K(f)\,\mathrm{d}^n z\,, \tag{21}$$

where $K(f)$ is the primitive with respect to f of $k(f)$. (Note: we use the caligraphic font to distinguish Casimir invariants of infinite-dimensional brackets from those of finte dimensional brackets.)

As another, slightly more complicated example, we consider a bracket of the form (18) with an inner bracket of Lie–Poisson type. We suggest that brackets of this form are applicable to the Hamiltonian formulations of nontraditional kinetic theories in analogy to their use in the Vlasov–Poisson system mentioned above. Notice that the inner bracket for the Vlasov–Poisson system is that appropriate to the phase space of a point particle. If, however, we wished to describe a 'gas' composed of freely spinning (fixed) rigid bodies instead of point particles, we expect that the inner bracket for this kinetic theory would be the so(3) bracket mentioned in Section 3. Another, more exotic, possibility would be a kinetic theory that describes a gas of Kida vortices. The dynamics of a Kida vortex takes place on the Lie algebra so(2,1) (see [17]), and is governed by the corresponding Lie–Poisson bracket.

$$[f,g](z) = c_i^{jk} z^i \frac{\partial f}{\partial z^j}\frac{\partial g}{\partial z^k}\,, \tag{22}$$

where c_i^{jk} are the structure constants of the Lie co-algebra.

Integration by parts (assuming f is constant on the boundary) shows that the bracket (18) with inner bracket of form (22) satisfies Eq. (19) if and only if the structure constants obey the condition $c_i^{ik} = 0$ for all k. This happens to be satisfied by the structure constants of any semi-simple Lie algebra. (Incidentally, any Hamiltonian flow generated by a Lie–Poisson bracket satisfying this condition obeys the naive Liouville's theorem in non-canonical coordinates [7].)

Now, to calculate the Casimir invariants for these examples, we again must solve Eq. (20) for γ. As in the above example, one solution is given by $\gamma = f$. However, the inner bracket in this example *can* be degenerate, and thus have Casimir invariants of its own. Suppose there are m such independent functions and denote them by $C^{(i)}$. Then a solution of (20) is $\gamma = k(C^{(1)},\dots,C^{(m)},f)$, where k is an arbitrary function. From this we see that

$$\Gamma = k\big(C^{(1)},\dots,C^{(m)},f\big)\delta f \tag{23}$$

is a null covector of J. Not only that, it is an exact variation, making $\Gamma = 0$ easy to integrate. Indeed, we find as a Casimir invariant

$$\mathscr{C}[f] = \int_D K\big(C^{(1)},\dots,C^{(m)},f\big)\mathrm{d}^n z \tag{24}$$

where $k(f) = \partial K(f)/\partial f$.

Applying this result to the proposed spin gas bracket, we use the result from Section 3, and obtain the Casimir invariant

$$\mathscr{C}[f] = \int_D K\Big((z^1)^2 + (z^2)^2 + (z^3)^2, f\Big) \mathrm{d}^3 z. \tag{25}$$

We could also envision a kinetic theory describing particles evolving on the Lie algebra su(3). Its Casimir invariant would be:

$$\mathscr{C}[f] = \int_D K(C_1, C_2, f) \mathrm{d}^8 z, \tag{26}$$

where C_1, C_2 are given respectively by Eqs. (13, 17). In both (25) and (26), K is an arbitrary function.

Note the distinct origins of the two types of functional dependence K in these invariants. Its dependence on f arises from the structure of the infinite-dimensional bracket alone, and so is common to any kinetic theory to which bracket (18) is applicable: it might be called the 'macroscopic' part of the Casimir invariant. On the other hand, the dependence of K on the $C^{(i)}$ arises from the structure of the finite-dimensional inner bracket, and so reflects the nature of the 'microscopic' theory.

5. Conclusion

We have presented a method for computing Casimir invariants applicable to both finite and infinite-dimensional Poisson brackets. We believe that presenting this method geometrically has two benefits: it enhances pictorial intuition of the behavior of finite-dimensional Hamiltonian systems, and it gives insight into how infinite-dimensional Hamiltonian systems can be treated analogously to finite-dimensional ones.

We have also computed some new examples of Casimir invariants. Most interesting, perhaps, are those that are the results of Poisson structures at finite and infinite-dimensional levels simultaneously. In these, we have pointed out what parts of the invariants correspond to which Poisson structure.

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References

[1] H. Gümral, Y. Nutku, J. Math. Phys. 34 (1993) 5691; S.A. Hojman, J. Phys. A: Math. Gen. 29 (1996) 667.
[2] M. Plank, J. Math. Phys. 36 (1995) 3520.
[3] E.A. Kuznetsov, A. Mikhailov, Phys. Lett. 77A (1980) 37; P.J. Morrison, J.M. Greene, Phys. Rev. Lett. 45 (1980) 790; 48, 569 (E); R.G. Spencer, A.N. Kaufman, Phys. Rev. A 25 (1982) 2437; D.D. Holm, B.A. Kupershmidt, Physica D 6 (1983) 347; P.J. Morrison, R.D. Hazeltine, Phys. Fluids 27 (1984) 886; R.D. Hazeltine, C.T. Hsu, P.J. Morrison, Phys. Fluids 30 (1987) 3204.
[4] P. J Olver, J. Math. Anal. Appl. 89 (1982) 233.
[5] I.E. Dzyaloshinskii, G.E. Volovick, Ann. Physics 125 (1980) 67; D.D. Holm, B.A. Kupershmidt, Phys. Lett. A 93 (1983) 177.
[6] J.E. Marsden, T. Ratiu, Introduction to Mechanics and Symmetry: a Basic Exposition of Classical Mechanical Systems, Springer, New York, 1994.

[7] P.J. Morrison, Reviews of Modern Physics 70 (1998) 467.
[8] R.G. Littlejohn, J. Math. Phys. 20 (1979) 2445; R.G. Littlejohn, J. Math. Phys. 23 (1982) 742; J.R. Cary, R.G. Littlejohn, Ann. Physics 151 (1983) 1.
[9] D.D. Holm, J.E. Marsden, T. Ratiu, A. Weinstein, Phys. Rep. 123 (1985) 1; P.J. Morrison, S. Eliezer, 1986, Phys. Rev. A 33 (1986) 4205; T.G. Shepherd in: H.K. Moffatt (Ed.), Topological Aspects of the Dynamics of Fluids and Plasmas, Kluwer Academic, Dordrecht (1992) 275.
[10] F. Magri, J. Math. Phys. 19 (1978) 1156; P.J. Olver, Phys. Lett. A 148 (1990) 177.
[11] Y. Kosmann-Schwarzbach in: Y. Kosmann-Schwarzbach, B. Grammaticos, K.M. Tamizhmani (Eds.), Integrability of Nonlinear Systems, Springer, Berlin, 1997, 104.
[12] A. Weinstein, J. Diff. Geom. 18 (1983) 523, (1985) 22 255 (E)
[13] I. Vaisman, Lectures on the Geometry of Poisson Manifolds, Birkhauser, Basel, 1994.
[14] P. Libermann, C-M. Marle, trans B.E. Schwarzbach, Symplectic Geometry and Analytical Mechanics (D. Reidel, Boston, 1987); H. Hofer, E. Zehnder, Symplectic Invariants and Hamiltonian Dynamics, Birkhauser, Basel, 1994.
[15] E. Sudarshan, N. Mukunda, Classical Dynamics: a Modern Perspective, John Wiley, New York, 1974; A. Lichnerowicz, J. Diff. Geom. 12 (1977) 253; R.G. Littlejohn, AIP Conf. Proc. 88 (1982) 47.
[16] Dynamical Systems III, ed. V.I. Arnold (Springer, Berlin, 1988); D. David, D.D. Holm, J. Nonlinear Sci. 2 (1992) 241; P.J. Olver, Applications of Lie Groups to Differential Equations, 2nd Ed., Springer-Verlag, New York, 1993.
[17] S.P. Meacham, P.J. Morrison, G.R. Flierl, Phys. Fluids 9 (1997) 2310.
[18] Y. Kuroda, J. Phys. Soc. Jap. 60 (1991) 727.
[19] B. Hernández–Bermejo, V. Fairén, Phys. Lett. A 241 (1998) 148.
[20] A. Kriegl, P.W. Michor, The Convenient Setting of Global Analysis, American Mathematical Society, Providence, 1997.
[21] B. Hernández–Bermejo, V. Fairén, J. Math. Phys. 39 (1998) 6162.
[22] V. Zeitlin, Physica D 49 (1991) 353.
[23] M. Gell-Mann, Phys. Rev. 125 (1962) 1067.
[24] B. Gruber, L. O'Raifeartaigh, J. Math. Phys. 5 (1964) 1796.
[25] P.J. Morrison, AIP Conf. Proc. 88 (1982) 13.
[26] S. Lie, Theorie der Transformationsgruppen, 2nd ed, B.G. Teubner, Leipzig, 1890, reprinted by Chelsea, New York, 1970.
[27] A.A. Kirillov, Russian Math. Surveys 17 (1962) 53; Kostant, Proc. US-Japan Seminar on Diff. Geom, Kyoto (1966) 77.