SHERWOOD THEORY MEETING 1984

April 11 to 13

Multiple Hamiltonian Structure of Fluid and Kinetic Theory*

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The most natural formulation of the equations of plasma physics as Hamiltonian systems involves the use of non-canonical variables.[1] The Hamiltonian form for the equations for a field $\psi(\underset{\sim}{x},t)$ is $\dot{\psi} = \{\psi,H\}$, where { , } is the Poisson bracket operator and H is the energy functional. Typically the Poisson bracket is degenerate: there exist functionals, C_k, such that $\{C_k,F\} = 0$ <u>for all</u> functionals F. The C_k are therefore invariant, and are called Casimir invariants from the theory of Lie Algebras.

We have reformulated the equations of motion of a system with N-1 Casimir invariants in terms of a non-degenerate, multilinear bracket with N+1 slots; that is, the equations of motion can be written $\dot{\psi} = \{\psi,C_1,C_2,\ldots C_{N-1},H\}$ where the bracket is completely antisymmetric. Such a system has effectively N Hamiltonians--the distinction between Hamiltonian and Casimir invariant is artificial. We show that if the multi-bracket satisfies a generalized Jacobi Identity (multi-Jacobi) then <u>any</u> N-1 Functionals can be used as Casimir invariants to reduce the multi-bracket to an ordinary Poisson bracket.

The simplest example of this structure is the rigid rotor which has a non-degenerate 3-bracket. We obtain 3-brackets for the Vlasov-Poisson and fluid equations as well. The former appeared as part of a formulation for dissipative systems[2,3]. Reduced MHD, on the other hand, has two simple Casimir invariants[3] leading to a 4-bracket.

[1] P.J. Morrison "Poisson Brackets for Fluids and Plasmas" in <u>Mathematical Methods in Hydrodynamics</u>, M. Tabor and Y. Treve (editors) AIP, New York, 1982.
[2] P.J. Morrison, Phys. Rev. Lett. A, "Bracket Formulation for Irreversible Classical Fields," to be published.
[3] P.J. Morrison and R.D. Hazeltine, to be published in Phys. Fluids.

*Work supported by USDOE.

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POSTER

Multiple Hamiltonian Structures

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Given:

1) Generalized Poisson Bracket

$$\dot{\psi} = \{\psi, H\}_1$$

2) Degeneracies:

$$\{\ , C\}_1 \equiv 0$$

C is a Casimir Invariant

Does there exist another bracket $\{\ ,\ \}_2$ for which C is the Hamiltonian?

$$\dot{\psi} = \{\psi, C\}_2$$

Prototypical Example: Free Rigid Body

See e.g. Sudarshan + Mukunda

$\underset{\sim}{S} = (S_1, S_2, S_3)$ is spin vector

I_1, I_2, I_3 are principle moments of inertia

$$H(S) = \frac{1}{2} \sum_{i=1}^{3} \frac{1}{I_i} S_i^2$$

Poisson Structure: $f(\underset{\sim}{S}), g(\underset{\sim}{S})$

$$\{f, g\} = \underset{\sim}{S} \cdot \underset{\sim}{\nabla} f \times \underset{\sim}{\nabla} g$$

e.g. $\{S_i, S_j\} = \varepsilon_{ijk} S_k$

Hamilton's Equations

$$\dot{S}_1 = \{S_1, H\} = \left(\frac{1}{I_2} - \frac{1}{I_3}\right) S_2 S_3$$

Casimir : $\frac{1}{2} S^2 =$ total spin $\equiv G$ *

$$\{f, S^2\} = 2\, \underset{\sim}{S} \cdot \nabla f \times \underset{\sim}{S} \equiv 0$$

* Any function of S^2 is a Casimir

2^{nd} Bracket

$$\{ f, g \}_\bullet = \varepsilon_{ijk} \frac{S_i}{I_i} \frac{\partial f}{\partial S_j} \frac{\partial g}{\partial S_k}$$

Note:

1) $\dot{S}_i = \{ S_i, G \}_\bullet$

2) $\{ \quad \}_\bullet$ satisfies Jacobi

$$\{\{a,b\},c\} + \{\{b,c\},a\} + \{\{c,a\},b\} = 0$$

This works because of the (obscure?) identity

$$0 = \varepsilon_{\alpha\beta i}\, \varepsilon_{\gamma jk} + \varepsilon_{\alpha\beta j}\, \varepsilon_{\gamma ki} + \varepsilon_{\alpha\beta k}\, \varepsilon_{\gamma ij} + (\alpha \Leftrightarrow \gamma)$$

* Actually the above holds for any completely antisymmetric matrix in 3-D.

This suggests that Casimirs and Hamiltonians are <u>not</u> really different!

To put them on an equal footing:

3-Bracket

$$\{f, g, h\} = \nabla f \cdot \nabla g \times \nabla h$$

This bracket is

1) tri-linear
2) completely anti-symmetric
3) a derivation
4) satisfies a generalized Jacobi

see below

Rigid Body Equations are obtained from

$$\dot{S}_i = \{ S_i, H, C \}$$
$$= \{ S_i, H_1, H_2 \}$$

$$H_1 = \frac{1}{2} \sum_i \frac{1}{I_i} S_i^2$$
$$H_2 = \frac{1}{2} \sum_i S_i^2$$

"Bi-Hamiltonian Theory"

Note:

Reduction of 3-Bracket gives a good 2-Bracket

$$\{ \ , \ , C \} = \{ \ , \ \}_H$$
$$\{ \ , H , \ \} = \{ \ , \ \}_C$$

Both are "Jacobi"

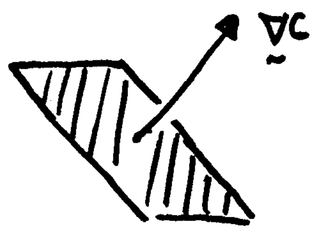
Note: The 3-bracket is <u>non-degenerate</u>

e.g. there is no C s.t.

$$\{f, g, C\} = 0 \quad \forall f, g$$

This would imply $\nabla C \cdot (\nabla f \times \nabla g) = 0$

or $\nabla C \perp$ to <u>all</u> vectors



which implies $C =$ constant

Non-degeneracy of the three-bracket implies there was precisely 1 Casimir

General Theory

Define a multibracket $\{f_1, f_2, \dots f_n\}$

1) multi-linear

$$\{f_1, \dots af_j, \dots f_N\} = a\{f_1, \dots f_j, \dots f_N\}$$

2) Antisymmetric

$$\{f_1, \dots f_k, \dots f_j \dots, f_N\} = (-1)^P \{f_1 \dots f_j \dots f_k \dots f_N\}$$

3) Derivation

$$\{\ , \dots fg, \dots\} = f\{\dots, g, \dots\} + g\{\dots, f, \dots\}$$

4) Generalized - Jacobi

Require that Hamiltonian flow preserve the group structure :
"Time development is a canonical transformation"

If the group is $T(t)$ such that

$$\frac{d}{dt} T = \{T, H_1, \dots H_{n-1}\}, \quad T(0) = Id$$

Then we require

$$T\{f_1, f_2, \dots f_n\} = \{Tf_1, Tf_2, \dots Tf_n\}$$

Differentiate w.r.t. time

$$\{\{f_1, f_2 \dots f_n\}, H_1, \dots H_{n-1}\} = \{\{f_1, H_1, \dots H_{n-1}\}, f_2, \dots f_n\}$$
$$+ \{f_1, \{f_2, H_1, \dots H_{n-1}\}, \dots f_n\}$$
$$+ \dots + \{f_1, f_2 \dots \{f_n, H_1, \dots H_{n-1}\}\}$$

<u>Generalized Jacobi</u>: require ⇑ for all functions $H_1, \dots H_{n-1}$

3-Jacobi

$$\{\{a,b,c\},d,e\} = \{\{a,d,e\},b,c\} + \{a,\{b,d,e\},c\}$$
$$+ \{a,b,\{c,d,e\}\}$$

Note: Rigid Body Bracket satisfies 3-Jacobi

Given a 3-Bracket:

$$\{f, g, h\} = \sum_{i,j,k} J^{ijk} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j} \frac{\partial h}{\partial x^k}$$

where J^{ijk} is completely antisymmetric

Then 3-Jacobi is satisfied if

1) $0 = J^{ijk} J^{lmn} + J^{ijm} J^{lnk} + J^{ijn} J^{lkm} + (i \leftrightarrow l)$

permute on (k, mn)

exchange (il)

2) $J^{ijk} J^{lmn}{}_{,k} = J^{lij}{}_{,k} J^{kmn} + J^{mij}{}_{,k} J^{knl} + J^{nij}{}_{,k} J^{klm}$

Note these are <u>covariant</u>: if 1) + 2) are satisfied in one coordinate system, then will be in any.

e.g. ordinary derivatives may be replaced by covariant ones.

1) **Thm:** Given an N-Bracket, satisfying N-Jacobi, then any N-1 bracket obtained by reduction, satisfies N-1 Jacobi:

$$\{f_1, \cdot, \cdot, \cdot\} = \{\cdot, \cdot, \cdot, \cdot\}_{f_1}$$ is a good N-1 bracket

2) Thm: If $H_1, \dots H_{n-1}$ generate a particular flow then any set of functions $F_i(H_1, \dots, H_{n-1})$ $i = 1, N-1$ generate the <u>same</u> <u>flow</u> if $\left\| \frac{\partial F_i}{\partial H_j} \right\| = 1$

(unit Jacobian)

3) Suppose $\underset{\sim}{\omega} = \omega_{ijk}\, dx^i \wedge dx^j \wedge dx^k$ is the "symplectic" 3-form

a) It is closed $d\underset{\sim}{\omega} = 0 \Rightarrow \omega_{ijk,\ell} + \circlearrowleft = 0$

b) It is preserved by the flow

$$\frac{d}{dt}\underset{\wedge}{\omega} = 0 \Rightarrow \omega_{ijk} J^{\ell mn} = \delta_j^m \delta_k^n - \delta_k^m \delta_j^n$$

Reduced MHD
(2-D, Low β)

$$\frac{\partial}{\partial t} U + [\phi, U] = [\psi, J] \qquad \text{(Vorticity, } \nabla\cdot J = 0 \text{ equation)}$$

$$\frac{\partial \psi}{\partial t} + [\phi, \psi] = 0 \qquad \text{(Ohm's Law)}$$

$$J = \nabla_\perp^2 \psi \,, \quad U \equiv \nabla_\perp^2 \phi \qquad [f,g] = \hat{z}\cdot\nabla f \times \nabla g$$

2-Bracket (Morrison + Hazeltine PF, 1984)

$$\{F,G\} = \int d^2\underline{x} \left(\psi \left[\frac{\delta F}{\delta \psi}, \frac{\delta G}{\delta u}\right] + \psi \left[\frac{\delta F}{\delta u}, \frac{\delta G}{\delta \psi}\right] + u \left[\frac{\delta F}{\delta u}, \frac{\delta G}{\delta u}\right]\right.$$

$$H = \int (\nabla_\perp \psi)^2 + (\nabla_\perp \phi)^2 \, d^2x$$

$$\left.\begin{array}{l} C_1 = \int \psi \, d^2x \\ C_2 = \int u\psi \, d^2x \end{array}\right\} \quad \underline{\text{KNOWN Casimir's}}$$

Thus we expect a <u>4-Bracket</u>:

$$\{F^1, F^2, F^3, F^4\} = \frac{1}{2}\int d^2x\; \varepsilon_{ijkl}\, \frac{\delta F^i}{\delta \psi}\left[\frac{\delta F^j}{\delta \psi}, \frac{\delta F^k}{\delta u}\right]\frac{\delta F^l}{\delta u}$$

Equations of motion are

$$\frac{df}{dt} = \{f, H, C_1, C_2\}$$

<u>Questions</u>:

Does it obey 4-Jacobi?

Is it non-degenerate?

One-Dimensional Fluid

$$v_t = -v v_x - \frac{1}{\rho} P_x$$
$$\rho_t = -(\rho v)_x \qquad\qquad P \equiv (\rho^2 U_\rho)$$
$$S_t = -v S_x$$

$$H = \int dx \left(\tfrac{1}{2}\rho v^2 + \rho U(\rho, S) \right)$$

$$\mathcal{S} = \int dx\, \rho S \quad \longleftarrow \text{Casimir}$$

The 2-bracket was given by Morrison & Greene (1980)

3-Bracket:

$$\{F^1, F^2, F^3\} = \frac{1}{2}\int dx \,\frac{1}{\rho}\, \varepsilon_{ijk} \left(\frac{\partial}{\partial x}\frac{\delta F^i}{\delta \rho}\right)\left(\frac{\delta F^j}{\delta S}\frac{\delta F^k}{\delta v} - \frac{\delta F^j}{\delta v}\frac{\delta F^k}{\delta S}\right)$$

$$\{\ ,\mathcal{S},\ \} = \text{2-Bracket of M.-G.}$$

$$\{\ ,\ ,H\} = \text{"Entropy" 2-Bracket}$$

Vlasov - Poisson

$$\{F,G\} = \int dx\,dv\, f \left[\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f} \right]$$

$$H = \int \tfrac{1}{2} v^2 f \, dz + \int \phi f \, dz$$

Casimirs $C = \int \mathcal{C}(f)$ for any function $\mathcal{C}$

Three Bracket

$$\{E,F,G\} = \int \frac{\delta E}{\delta f} \left[\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f} \right] dz$$

This reduces to V-P Bracket using $\mathcal{C} = f$

$$\{E, F, \textstyle\int f\, dz\} = \{E,F\}$$

This 3-Bracket appears <u>not</u> to satisfy 3-Jacobi...

SYMMETRIC BRACKETS FOR DISSIPATIVE FIELDS

Motivation: Just as any operator can be split into self-adjoint and anti-self-adjoint parts we split brackets into an antisymmetric generalized Poisson bracket and a symmetric bracket.

Recall: GPB's have the form

$$\{F, G\} = \int \frac{\delta F}{\delta \psi^i} O^{ij} \frac{\delta G}{\delta \psi^j} \, dz$$

where $(O^{ij})^+ = -O^{ij}$.

Symmetric Brackets: For these we suppose

$$(F, G) = \int \frac{\delta F}{\delta \psi^i} A^{ij} \frac{\delta G}{\delta \psi^j} \, dz$$

where $(A^{ij})^+ = A^{ij}$.

goal: Desire to cast any equation into the form

$$\Psi_t^i = \{\Psi^i, \hat{T}\}_{\hat{T}} = \{\Psi^i, \hat{T}\} + (\Psi^i; \hat{T})$$

where $\hat{T}$ is the <u>generator of time translation</u>

<u>questions:</u>

① What should one use for $\hat{T}$?

② What algebraic properties should (F, G) possess? (e.g. $\{ , \}$ has Jacobi)

Generators of Time Translation

Natural choices for $\hat{T}$ are the thermodynamic potentials: Energy, Entropy, Free Energy, .. etc.
E S F

Extrema of these quantities correspond to thermodynamic equilibria.

Brackets using E, F, & S exist. (see below).

Generalized Free Energy

A particularly appealing choice for $\hat{T}$ is something we call generalized free energy

Recall thermo. equilibria arise by varying the energy at constant entropy

$$F = H - \bar{T}S$$

where $\bar{T}$ is a Lagrange multiplier.

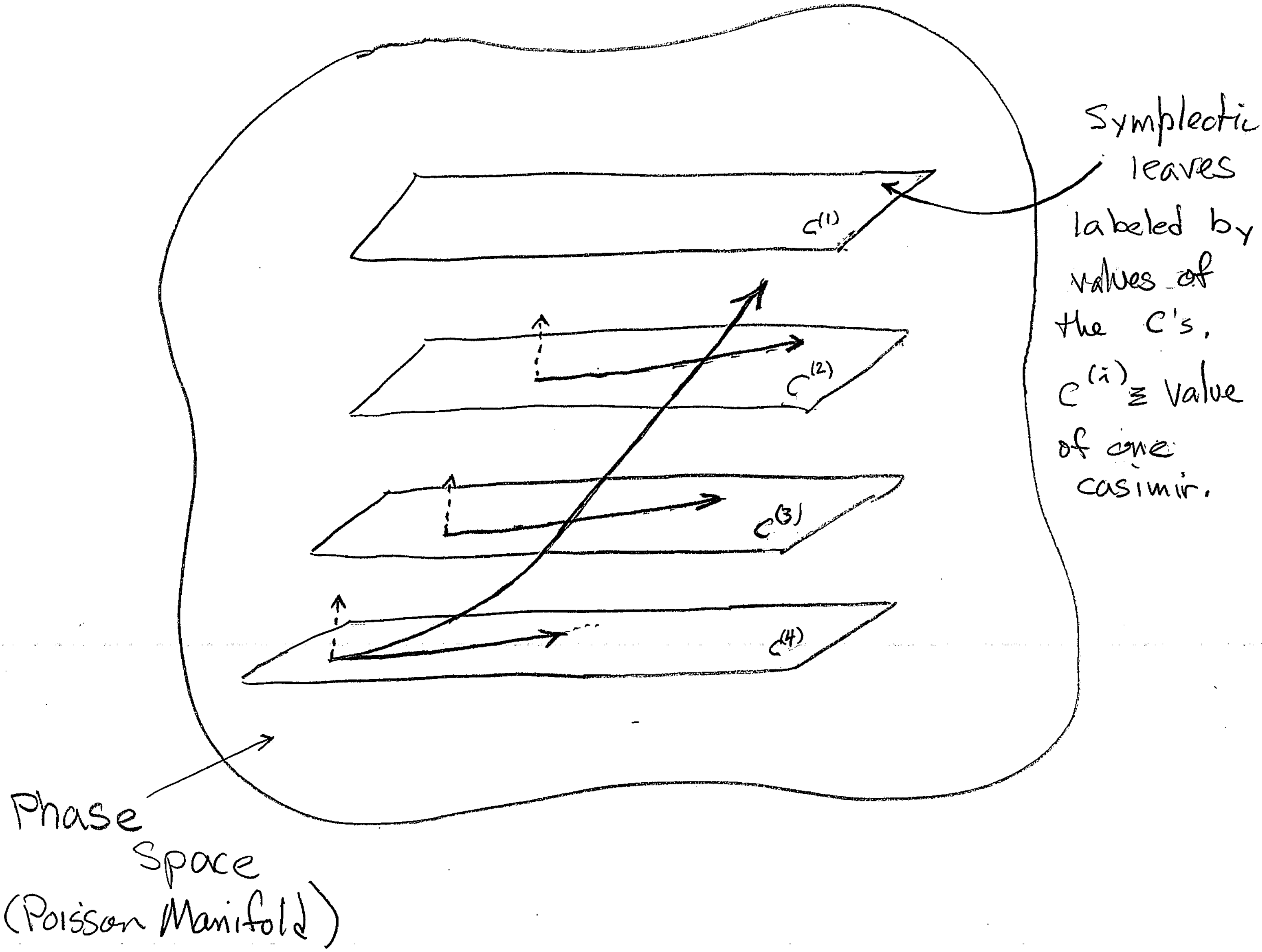
GPB's possess generalized entropy functionals called Casimirs. (e.g. $S = \int s(f)\,dz$ for V-Poisson)

The generalized free energy is

$$F = H + \sum_i \underbrace{C_i}_{\text{Casimirs}}$$

This is an ideal quantity since fixed points correspond to both Thermodynamic and Dynamic Equilibria

PICTORIAL REPRESENTATION



The <u>blue</u> trajectories are generated by $\{,\}$

The dotted <u>black</u> trajectories are generated by $(,)$

The total traj. is <u>solid black</u>

EXAMPLES Vlasov Eq. w/ Collisions

$$\frac{\partial f}{\partial t} = -\vec{v}\cdot\frac{\partial f}{\partial x} + \frac{\partial \phi}{\partial \vec{x}}(\vec{x}; f)\cdot\frac{\partial f}{\partial \vec{v}}$$

$$+ \frac{\partial}{\partial v_i}\int \omega_{ij}(z,z')\left[\frac{\partial f(z)}{\partial v_j} f(z') - \frac{\partial f}{\partial v_j'}(z') f(z)\right] dz'$$

$$\text{e.g. } \omega_{ij} = \frac{L}{g}\left(\delta_{ij} - \frac{g_i g_j}{g^2}\right)\delta(\vec{x}-\vec{x}')$$

Energy as $\hat{T}$: $E[f] = \int T(z) f(z) dz + \frac{1}{2}\iint V(z,z') f(z) f(z') dz dz'$

$$\{A, B\}_E = \{A, B\} + (A, B)$$

$$\{A, B\} = \int f \left[\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f}\right] dz$$

$$(A, B) = \int \left(\frac{\partial}{\partial v_j}\frac{\delta A}{\delta f(z)} - \frac{\partial}{\partial v_j'}\frac{\delta A}{\delta f(z')}\right)\left(\frac{\partial}{\partial v_i}\frac{\delta B}{\delta f(z)} - \frac{\partial}{\partial v_i'}\frac{\delta B}{\delta f(z')}\right)$$

$$\times\; T_{ij}^{(E)}(z,z')\, dz\, dz'$$

$$T_{ij}^{(E)} = \frac{1}{2}\left[f(z')\frac{\partial f(z)}{\partial v_k}\frac{\partial \omega_{ij}}{\partial v_k} + f(z)\frac{\partial f(z')}{\partial v_k'}\frac{\partial \omega_{ij}}{\partial v_k'}\right]$$

Similar expressions exist for entropy and free energy w/ $S[f] = \int f \ln f\, dz$. (See below)

Note one can also use generalized $S = \int s(f)\, dz$!

GENERALIZED COLLISION OPERATOR (w/ W.B. Thompson)

Symmetric and Antisymmetric tri-brackets exist for the preceeding. The following generalized Collison operator evolved out of these considerations

$$\left.\frac{\partial f}{\partial t}\right)_c = \frac{\partial}{\partial v_i}\int \omega_{ij}\left[F(f(v))\frac{\partial f(v')}{\partial v_j'} - F(f(v'))\frac{\partial f(v)}{\partial v_j}\right] d^3v'$$

- F is an arbitrary function of f.
- $\omega_{ij}(v,v')$ symmetric in indices and arguements
- $(v_i - v_i')\,\omega_{ij} = 0$

Properties

- Conserves Mass
- Conserves Momentum
- Conserves Energy

— Has compatible <u>Generalized entropy</u>

$$S[f] = \int s[f] \, d^3v$$

where $\underline{\dfrac{\partial^2 s}{\partial f^2} F(f) = 1}$

Compatible means generalized entropy obeys <u>H - theorem</u> !

<u>Examples</u>

① pick $s = f \ln f \Rightarrow F = f$

obtain Landau form for $\left.\dfrac{\partial f}{\partial t}\right)_c$

② pick s s.t. $F = f(\eta - f)$ then

$\left.\dfrac{\partial f}{\partial t}\right)_c$ is the operator that relaxes

to Lynden-Bell statistics (F. Dirac)

(agress w/ Kadomtsev)

SYMMETRIC BRACKET FOR GENERALIZED COLLISION OPERATOR

Generator:

$$\hat{T} = E[f] + S[f]$$
$$= \int T(z) f(z)\, dz + \iint \tfrac{1}{2} V(z,z') f(z) f(z')\, dz\, dz' + \int \mathcal{S}(f)\, dz$$

Symmetric Bracket:

$$(A,B) = \iint \left[\frac{\partial}{\partial v_j}\frac{\delta A}{\delta f(z)} - \frac{\partial}{\partial v_j'}\frac{\delta A}{\delta f(z')}\right]\left[\frac{\partial}{\partial v_i}\frac{\delta B}{\delta f(z)} - \frac{\partial}{\partial v_i'}\frac{\delta B}{\delta f(z')}\right] T_{ij}(z,z')\, dz\, dz'$$

$$T_{ij}(z,z') = \frac{1}{2}\,\omega_{ij}(z,z')\, F(f(z))\, F(f(z'))$$

compatibility $\quad F(f(z))\, \mathcal{S}''(f(z)) = 1$

$$\left.\frac{\partial f}{\partial t}\right)_c = (f, \hat{T}) = \text{generalized Collision operator}$$