Sherwood Meeting

April 14-16, 1986
New York

# PRACTICAL GENERALIZED ENERGY PRINCIPLES FOR DETERMINING LINEAR AND NONLINEAR STABILITY

P. J. Morrison and M. Kotschenreuther
*IFS, UNIVERSITY OF TEXAS, AUSTIN 78712*

We present generalizations of the ideal MHD energy principle, $\delta W$, for fluid and kinetic models, that can handle equilibrium flow and FLR effects. The usual energy principle arises because the Hamiltonian has standard kinetic energy and potential energy terms, in which case stability is determined by the potential alone. More generally in Hamiltonian systems, the total energy can serve as a Liapunov functional. Although the Hamiltonian structure of continuous media expressed in Eulerian variables is not of the canonical form, it still has this built in Liapunov functional feature. There is a generalization of the Hamiltonian, a generalized free energy (F), that has equilibria as stationary points and for which definiteness of the second variation, $\delta^2F$, is sufficient for stability. This definiteness of $\delta^2F$ is a more dependable criterion for practical stability than conventional linear spectral stability. Indeed, sometimes linear theory is highly misleading because nonlinear instability for arbitrarily small pertubations can arise. This can occur only when $\delta^2F$ is indefinite. We will show that $\delta^2F$ - not the second variation of the physical energy-is the appropriate perturbed energy. When $\delta^2F$ is indefinite there exists either instability or a negative energy mode (direction in function space). (Comparison with the Sommerfeld-Brillouin condition, $\omega\partial\varepsilon/\partial\omega < 0$, will be made.) The later can result in explosive instability even though spectral theory indicates stability. One can use $\delta^2F$ in much the same spirit as $\delta W$ ; i.e. insert trial functions and then vary parameters to search for indefiniteness. A further test is used to distinguish linear instability from negative energy directions. We emphasize that $\delta^2F$ is applicable even if the dielectric functional is intractable or not even defined. We will present examples. In particular we find that the $\delta^2F$ velocity threshold for the warm two-stream instability is lower than that of conventional linear theory. We also examine some MHD equilibria with flow and FLR effects.

# SUMMARY OF NEW DEVELOPMENTS

→ Explanation of $\delta^2 F$ - Spectral theory disagreement

→ Basic definition of negative energy wave

→ Bracket Perturbation Theory

→ Generalization of $\delta W$

REVIEW

## II. Noncanonical or Generalized Hamiltonian Mechanics (finite $N$)

Hamiltons Eqs. :

$$\dot{q}_k = \frac{\partial H}{\partial p_k} = [q_k, H]$$

$$\dot{p}_k = -\frac{\partial H}{\partial q_k} = [p_k, H] \qquad k = 1, 2, \dots N$$

Poisson Bracket :

$$[f, g] = \sum_{k=1}^{N} \left( \frac{\partial f}{\partial q_k} \frac{\partial g}{\partial p_k} - \frac{\partial g}{\partial q_k} \frac{\partial f}{\partial p_k} \right)$$

Cosymplectic Form :

let

$$z^i = \begin{cases} q_k & i = 1, 2, \dots N = k \\ p_k & i = k+N = N+1, \dots 2N \end{cases}$$

obtain

$$[f, g] = \frac{\partial f}{\partial z^i} J_c^{ij} \frac{\partial g}{\partial z^j}$$

$$(J_c^{ij}) = \begin{bmatrix} 0 & I_N \\ -I_N & 0 \end{bmatrix}$$

$$\dot{z}^i = J_c^{ij} \frac{\partial H}{\partial z^j}$$

kinematics or phase space

dynamics

Bracket Properties:

(i) bilinear $[g+h,f] = [g,f] + [h,f]$

(ii) $-[f,g] = [g,f]$

(iii) Jacobi $[f,[g,h]] + [g,[h,f]] + [h,[f,g]] = 0$

(iv) $[fg,h] = f[g,h] + [f,h]g$

Lie Algebra

Transformations:

$z^i \longrightarrow z'^i$ coordinate change

$J_c^{ij} \longrightarrow J^{ij}(z')$ contravariant tensor

$J_c^{ij} \longrightarrow J_c^{ij}$ canonical transformation

bracket properties are invariant

Converse outlook:

bracket properties $\Longrightarrow$ $z'^i \longrightarrow z^i$, $J^{ij} \longrightarrow J_c^{ij}$

Darboux ( local , $\det J^{ij} \neq 0$ )

# Noncanonical or Generalized Hamiltonian Mechanics :

**Definition.** A system of ordinary differential equations is Hamiltonian in the generalized sense if it can be cast into the form

$$\dot{z}^i = J^{ij}\frac{\partial H}{\partial z^j} = [z^i, H] \qquad i,j = 1,2,\ldots M$$

need not be even

where

$$[f,g] = \frac{\partial f}{\partial z^i} J^{ij} \frac{\partial g}{\partial z^j}$$

has bracket properties.

## Generalized Phase Space :

Since definition allows $\underline{\det(J^{ij}) = 0}$ the structure of phase space is changed.

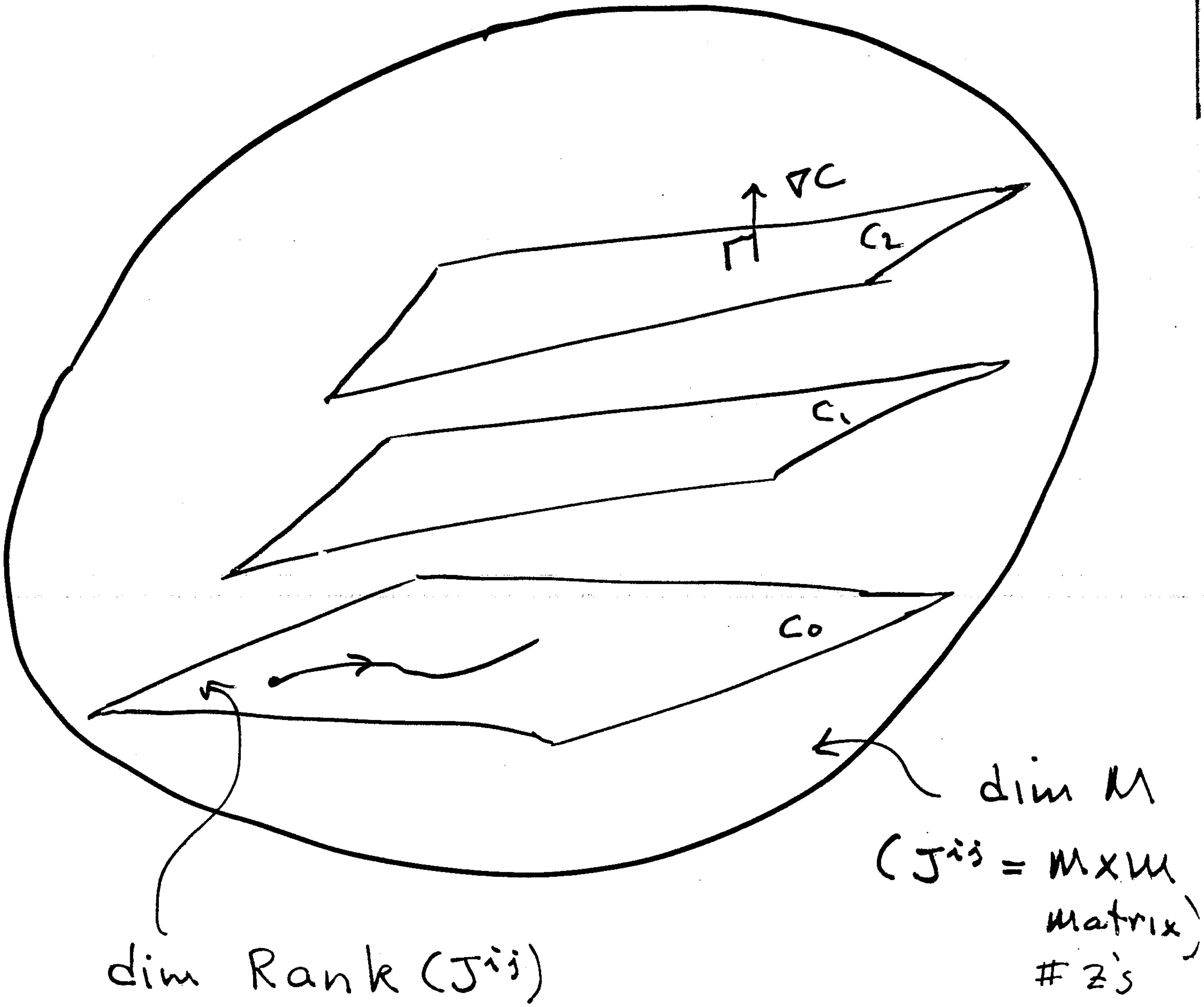
Corank of $(J^{ij})$ = dimension of null space

Null space spanned by gradients : $\underline{\frac{\partial C}{\partial z^i} J^{ij} = 0}$

The quantities $C$ are Casimirs — phase space constants; built into phase space

$$[C, g] = \frac{\partial C}{\partial z^i} J^{ij} \frac{\partial g}{\partial z^j} = 0 \qquad \text{for all } g$$

# Phase Space (Poisson Manifold):




For any hamiltonian the trajectory is confined to symplectic leaf.

# Generalization - Noncanonical Brackets

$$\{F, G\} = \int \frac{\delta F}{\delta \psi^i} O^{ij} \frac{\delta G}{\delta \psi^j} \, d\tau$$

$$= \left\langle \frac{\delta F}{\delta \psi^i}, \underbrace{O^{ij}}_{\text{cosymplectic operator}} \frac{\delta G}{\delta \psi^j} \right\rangle$$

(1) Antisymmetry $\Rightarrow$ $O^{ij}$ anti-self-adjoint

(2) Jacobi - stiff requirement!
(Bracket must be Lie product for algebra of functionals)

Equations of Motion:

$$\frac{\partial \psi^i}{\partial t} = \{\psi^i, H\} = O^{ij} \frac{\delta H}{\delta \psi^j}$$

---

Canonical Case $\quad (O^{ij}) = \begin{pmatrix} 0 & I_M \\ -I_M & 0 \end{pmatrix}$

# III. Field Theory

Canonical bracket:

$$\{F, G\} = \sum_{k=1}^{L} \int \left( \frac{\delta F}{\delta \eta_k} \frac{\delta G}{\delta \pi_k} - \frac{\delta G}{\delta \eta_k} \frac{\delta F}{\delta \pi_k} \right) d\underline{x}$$

bracket acts on functionals of the field variables, $\eta_k, \pi_k$; e.g.

$$H = \int \mathcal{H} \, d\underline{x}$$

Hamiltonian density $(\frac{1}{2} \rho v^2)$

phase space derivatives become functional derivatives

$$\frac{\partial}{\partial q_k} \longrightarrow \frac{\delta}{\delta \eta_k}$$

defined by

$$\delta F = \frac{d}{d\varepsilon} F[\eta + \varepsilon \delta \eta] \Big|_{\varepsilon = 0} = DF \cdot \delta \eta = \left\langle \frac{\delta F}{\delta \eta}, \delta \eta \right\rangle$$

$$= \int \frac{\delta F}{\delta \eta} \delta \eta \, d\underline{x}$$

Canonical Fields:

Klein-Gordon etc.

$$(O^{ij}) = \begin{bmatrix} 0 & I_N \\ -I_N & 0 \end{bmatrix}$$

Continuous Media Fields:

Ideal MHD, Vlasov, etc.

$$(O^{ij}) = \left( \psi^k c_k^{ij} \right)$$ linear in the field variables

$c_k^{ij}$ are structure operators for some Lie algebra on functions

Lie-Poisson Brackets:

$$\{F, G\} = \int \psi^k \left[ \frac{\delta F}{\delta \vec{\psi}}, \frac{\delta G}{\delta \vec{\psi}} \right]_k dz$$

outer algebra on functionals

inner algebra on functions

# STABILITY

**Spectral** $\qquad \psi = \psi_e + \delta\psi\, e^{+i\omega t}$

linearize $-$ $\mathrm{Im}\,\omega < 0$ ?

$\mathrm{Im}\,\omega = 0$ stable ?

**Linear Stability** $\qquad$ secular growth - Linear eqs. still

**Formal Stability** $\qquad$ Liapunov Function

$\delta W$ ; $\delta^2 F$ definite

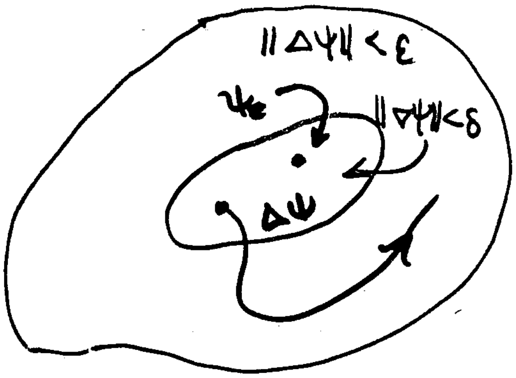
$\| \; \|$ Norm

**Nonlinear Stability**

Definition. An equilibrium $\psi_e$ is nonlinearily stable if for all $\varepsilon > 0$ there is a $\delta > 0$ such that for $\psi(t=0) = \psi_e + \Delta\psi(t=0)$ with $\|\Delta\psi\| < \delta$ (at $t=0$), then $\|\Delta\psi\| < \varepsilon$ for all time.

Dynamics determined by
Nonlinear equations ($\Delta\psi$ finite)

requires Norm
$\| \; \|$




Nonlinearily stable

# Thermodynamic Variational Principles

Fowler, Newcomb, Oberman & Kruskal, Rosenbluth, Gardner, Taylor, Arnold

approach: Energy is minimized subject to some constraint like constant entropy or helicity. It is then noted that the Euler-Lagrange eq. thus obtained corresponds to the equation for equilibria.

comments: This approach is ad hoc. No connection between the dynamics and energy minimization is made. Why does this approach yield the correct equilibria?

The noncanonical Hamiltonian formalism fills in this gap. To see this note that

$$\frac{\partial \psi^i}{\partial t} = \{\psi^i, H\} = \{\psi^i, F\} = O^{ij}\frac{\delta(H+C)}{\delta\psi^j}$$

$$F = H + C\,, \quad \{\psi^i, C\} = 0$$

Therefore

More general equilibria →

$$\frac{\delta(H+C)}{\delta\psi^j} = 0 \implies \frac{\partial\psi^i}{\partial t} = 0 \quad !$$

# CLASSIFICATION

| EQUATIONS | HAMILTONIANS | BRACKET | CASIMIRS |
|---|---|---|---|
| KdV<br>MKdV | $\int\left(\frac{u^{\alpha}}{6}-\frac{1}{2}u_x^2\right)dx$ | Gardner | $\int u\,dx$ |
| Liouville Eq.<br>Vlasov-Poisson<br>2-D Euler<br>Guiding Center | $\int hf\,dz$<br>$\int h_1 f + \int h_2 ff$<br>$\int \omega\phi$<br>$\int \rho\phi$ | Canonical Transformations of $\mathbb{R}^{2n}$ | $\int F(\psi)\,dz$ |
| RMHD<br>Tokamak Models | $\int \lvert\nabla\phi\rvert^2+\lvert\nabla\psi\rvert^2$ | Above extended by sem-direct pdt. | $\int F(\psi)$<br>$\int \omega F(\psi)$ |
| MHD<br>CGL Theory | $\int \frac{1}{2}\rho v^2+\rho U(\sigma,\rho)+\frac{B^2}{2}$<br>$U(\sigma,\rho,B)$ | Diffeomorphisms of $\mathbb{R}^3 \times$ fns. | $\int A\cdot B$, $\int v\cdot B$<br>& others |

Just as many fields are naturally canonical, there are many equations that have the same generalized Poisson bracket. They have different Hamiltonians.

Casimirs are bracket constants. They are independent of the Hamiltonian. If $C$ is a casimir then $\{C,F\}=0$ for all $F$.

Casimirs are useful for obtaining variational principles for equilibria. They are an ingredient in the algorithm for constructing Liapunov functionals.

# $\delta^2 F$ Stability $\quad F = H + C$

finite canonical systems : $\dfrac{\partial^2 H}{\partial z^i \partial z^j}$ definite

$\Rightarrow$ Nonlinear stability $\quad$ standard K.E. $\dfrac{\partial^2 V}{\partial q_i \partial q_j}$

Noncanonical finite systems :

$$\dot{z}^i = J^{ij} \frac{\partial F}{\partial z^j}$$

$F = H + C$

$\dfrac{\partial F}{\partial z^i} = 0$ equil.

$\dfrac{\partial^2 F}{\partial z^i \partial z^j}$ definite

Noncanonical Fields :

$F = H + C$

$\delta F = 0 \qquad$ equilibria

$\delta^2 F =$ quadratic form in $(\delta \psi^i)$
definite ?

---

$$\delta F = \int \frac{\delta F}{\delta \psi^i} \delta \psi^i \, dz = 0$$

$$\delta^2 F = \int \delta \psi^i \frac{\delta^2 F}{\delta \psi^i \delta \psi^j} \delta \psi^j \, dz$$

# EXAMPLES (with flow)

→ REDUCED MHD

→ COMPRESSIBLE REDUCED MHD

# REDUCED MHD (helical symmetry)

**EQS:** $\psi_t = [\psi, \phi]$ $\qquad U_t = [\psi, J] - [\phi, U]$

scalar vort. $U = \nabla^2\phi$ $\qquad J = \nabla^2\psi$ // current

$[f,g] = \hat{z} \cdot \nabla f \times \nabla g$

**Constants:**

$$H = \int \tfrac{1}{2}|\nabla\phi|^2 + \tfrac{1}{2}|\nabla\psi|^2 \leftarrow \text{energy}$$

Casimirs $\rightarrow$ $C_1 = \int \mathcal{F}(\psi)\, d\vec{x}$ $\qquad C_2 = \int U \mathcal{G}(\psi)\, d\vec{x}$

**Free Energy:**

$$F = \int \frac{|\nabla\phi|^2}{2} + \frac{|\nabla\psi|^2}{2} + U\mathcal{G}(\psi) + \mathcal{F}(\psi)$$

**Equilibria:**

$$\delta F = 0 \iff \nabla^2\psi = U\mathcal{G}'(\psi) + \mathcal{F}'(\psi) = J$$

flow term

$$\phi = \mathcal{G}(\psi)$$

**$\delta^2 F$:**

$$\boxed{\delta^2 F = \int \underbrace{|\nabla\delta\phi - \nabla\mathcal{G}'\delta\psi|^2}_{\text{equil. relative K.E.}} + \underbrace{|\nabla\delta\psi|^2 (1-\mathcal{G}'^2)}_{\text{flow modified line bending}} + \underbrace{(\delta\psi)^2\left[\mathcal{G}''\nabla^2\mathcal{G} + \mathcal{F}'' + \mathcal{G}'\nabla\cdot(\mathcal{G}''\nabla\psi)\right]}_{\text{flow modified kink}}}$$

# Compressible Reduced MHD (Single helicity)

CRMHD ⊂ 4-Field Model - HKM

Scalar vorticity $\quad \underbrace{U_t = [U, \phi]}_{} + \underbrace{[\psi, J] + 2[p, h]}_{}$

$\nabla h \sim$ curvature

Ohm's Law $\quad \underline{\psi_t = [\psi, \phi]}$

//- motion $\quad \underline{v_t = [v, \phi] + [\psi, p]}$

pressure $\quad \underline{p_t = [p, \phi] + \beta[\psi, v] + 2\beta[h, \phi]}$

//- compressibility $\qquad$ ⊥- compressibility

$\beta = \text{const.}$

$$[f, g] = \hat{z} \cdot \nabla f \times \nabla g = \frac{1}{r}\left(\frac{\partial f}{\partial r}\frac{\partial g}{\partial \theta} - \frac{\partial f}{\partial \theta}\frac{\partial g}{\partial r}\right) = \frac{\partial(f, g)}{\partial(r, \theta)}$$

$$\vec{v}_\perp = \hat{z} \times \nabla \phi \qquad \boxed{\vec{B}_\perp = -\hat{z} \times \nabla \psi}$$

$$U = \nabla^2 \phi \qquad \boxed{J = \nabla^2 \psi}$$

Energy: $\quad H = \int \frac{|\nabla_\perp \psi|^2}{2} + \frac{|\nabla_\perp \phi|^2}{2} + \frac{v^2}{2} + \frac{p^2}{2\beta}$

Casimirs: $\quad C_1 = \int F(\psi) \qquad C_2 = \int \mathcal{L}(\psi)\,(p/\beta + 2h)$

$$C_3 = \int v\, \mathcal{N}(\psi)$$

$$C_4 = \int \mathcal{G}(\psi)\, U - v\, \mathcal{G}'(\psi)\left(\frac{p}{\beta} + 2h\right)$$

## $\partial J/\delta\psi$ Results

**Case I**

$$F = H + C_1 \qquad \delta F = 0 \quad \Rightarrow \quad \nabla_\perp^2 \psi = F'(\psi)$$
$$U = 0$$

$$\delta^2 F = \int |\nabla \delta\phi|^2 + |\nabla \delta\psi|^2 + F''(\psi_e)(\delta\psi)^2 + (\delta v)^2 + \frac{(\delta p)^2}{\beta}$$

Easy result : $F''(\psi_e) > 0 \Rightarrow$ definite
$\Leftrightarrow$ $J(\psi_e)$ monotonic

Can do better: $|\nabla \delta\psi|^2 > 0$ [redacted]
trial function $\delta\psi \sim f(r)\, e^{im\theta}$
$\Leftrightarrow$ Necomb $f(r) \sim k_\parallel \xi(r)$

**Case II** $\qquad$ curvature

$$F = H + C_1 + C_2$$

$$\delta F = 0 \quad \Rightarrow \quad \boxed{\nabla_\perp^2 \psi = A(\psi) + B(\psi)\, h}$$
$$v = U = 0 \qquad B \sim p'(\psi)$$

Easy result:
$\delta^2 F$ def. if $\quad A'(\psi_e) + B'(\psi_e) h > 0 \quad ; \quad \dfrac{\partial J}{\partial \psi} > 0$

**Case III**

$$F = H + C_1 + C_2 + C_3$$
$\parallel$-flow

$$\delta F = 0 \quad \Rightarrow \quad \boxed{\nabla_\perp^2 \psi = A(\psi) + B(\psi)\, h}$$
$$U = 0 \qquad p(\psi), \quad v(\psi)$$

$\delta^2 F$ def. $\quad \boxed{A'(\psi_e) + B'(\psi_e)\, h > 0}$ $\quad$ insensitive to $v$ !

Case IV $\quad$ Poloidal ($\perp$) flow

# WHEN $\delta^2 F$ AND SPECTRAL THEORY DON'T AGREE

# PERTURBED ENERGY – What is neg. energy wave?

Linear Theory :

$$\dot{z}^i = J^{ij}(z)\,\frac{\partial F}{\partial z^j}$$

$$F = H + C$$

$$z = z_e + \delta z$$

Equil: $\dfrac{\partial F(z_e)}{\partial z^i} = 0$

$$\delta\dot{z}^i = J^{ij}(z_e)\,\frac{\partial^2 F(z_e)}{\partial z^j \partial z^k}\,\delta z^k$$

$$= \left\{ \delta z^i,\ \frac{\delta^2 F}{2} \right\}_L$$

perturbed Hamiltonian

Not! $\delta^2 H$.

$\dfrac{\delta^2 F}{2}$ = linearized Hamiltonian

Should it be the energy?

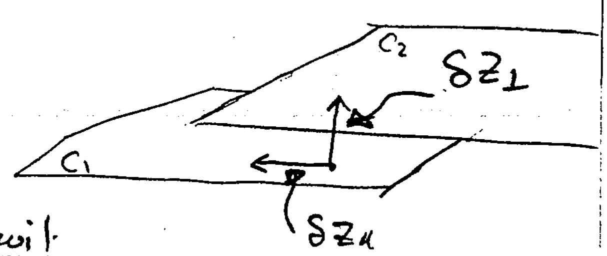
Why should the linearized energy depend on $C$?

Add Source term } $\delta z_{\parallel}$ relevant

Analogy } $dW = dU + T\,dS$

work done on — input E. — heat

change in Equil.



Casimir Constrained Source:

$$H \to H + H_{ext}$$

guarantees motion on the leaf

$\Longleftrightarrow$ Sommerfeld / Brillouin

Having identified the perturbed energy we transform to action angle variables and obtain:

$$\frac{\delta^2 F}{2} = \sum_i \omega_i J_i$$

action

freq.

A <u>negative energy wave</u> (mode) occurs when $\omega_i < 0$. The sign of the $\omega_i$'s cannot be changed by transformation (Sylvester's theorem).

This <u>basic definition</u> agrees with usual case when comparison can be made, i.e. when $\exists$ $\mathcal{E}(k,\omega)$.

<u>Assume:</u> $H_{ext} = z^j S_j(t)$

linear source term

Examples: (1) $-qF_{ext}$ 1-degree of freedom

(2) $-f\,\phi_{ext}$ Vlasov-Poisson

<u>Power Input:</u>

$-\dot{z}^j S_j$ motion // to leaf

<u>Energy Input:</u>

$$\Delta H = -\int_{-\infty}^{\infty} \dot{z}^j S_j \, dt$$

// to leaf

<u>Can Prove:</u>

$$\boxed{\Delta H = \delta^2 F/2}$$

# THE DANGER OF SPECTRAL THEORY
## (Cherry's Example)

o.d.e.'s

$$\dot{z}_1 = z_2 - \alpha(z_2 z_3 + z_1 z_4)$$

$$\dot{z}_2 = -z_1 + \alpha(z_2 z_4 - z_1 z_3)$$

$$\dot{z}_3 = -2z_4 - \alpha z_1 z_2$$

$$\dot{z}_4 = 2z_2 + \frac{\alpha}{2}(z_2^2 - z_1^2)$$

linear analysis $\Rightarrow$ real frequencies, yet unstable. Solution diverges in finite time.

$$z_1 = \frac{\sqrt{2}}{\alpha(t-\varepsilon)} \sin(t+\gamma) \qquad z_2 = \frac{\sqrt{2}}{\alpha(t-\varepsilon)} \cos(t+\gamma)$$

$$z_3 = \frac{1}{\alpha(t-\varepsilon)} \sin(2t+\gamma) \qquad z_4 = \frac{-1}{\alpha(t-\varepsilon)} \cos(2t+\gamma)$$

---

Cherry's Hamiltonian: $H = \underbrace{\tfrac{1}{2}(p_1^2 + q_1^2)}_{\omega_1 = 1} - \underbrace{(q_2^2 + p_2^2)}_{\omega_2 = -2}$

$$+ \frac{\alpha}{2}\left\{ q_2(q_1^2 - p_1^2) - 2q_1 p_1 p_2 \right\}$$

two features:

(i) $O(3)$ resonance: $2\omega_1 + \omega_2 = 0$

$\Downarrow$

generic behavior

(ii) Negative energy mode

$\delta^2 F$ indefinite

# Two-Stream Instability (warm ions & electrons)

$$\frac{\partial v_\alpha}{\partial t} + v_\alpha \frac{\partial v_\alpha}{\partial x} = \frac{e_\alpha}{m_\alpha} E - \frac{1}{\rho_\alpha}\frac{\partial p_\alpha}{\partial x}$$

$$\frac{\partial n_\alpha}{\partial t} + \frac{\partial}{\partial x}(n_\alpha v_\alpha) = 0 \qquad \frac{\partial E}{\partial x} = 4\pi e\,(n_i - n_e)$$

**equil.** $n_{0i}$, $n_{0e}$, $v_D$ ← drifting electrons

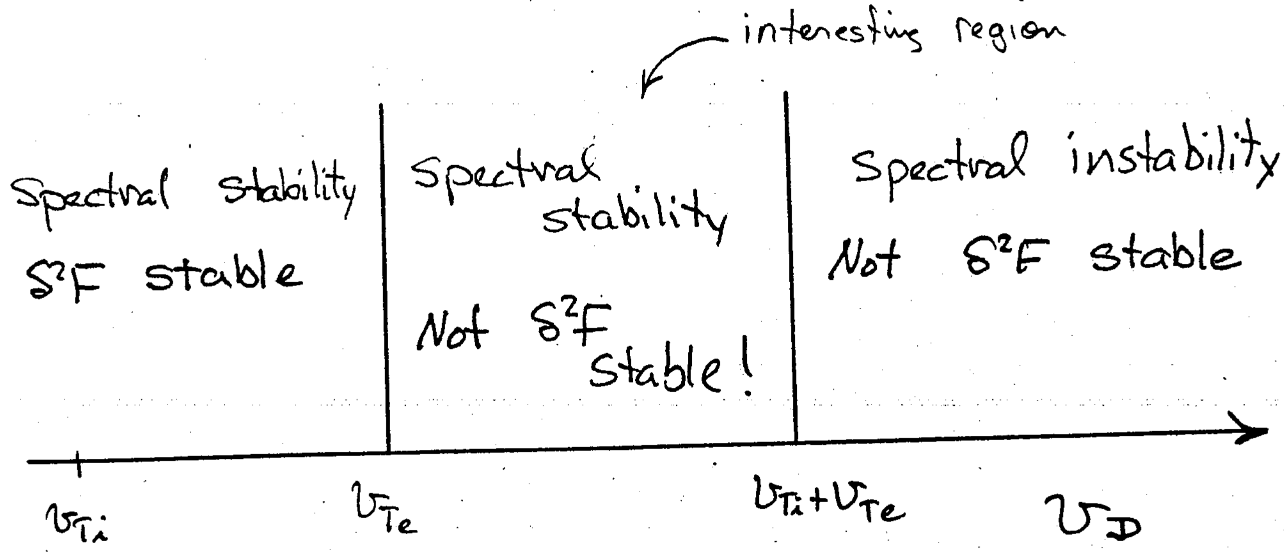
spectral stability condition given via

$$0 = 1 - \frac{\omega_{pi}^2}{\omega^2 - k^2 v_{Ti}^2} - \frac{\omega_{pe}^2}{(\omega - k v_0)^2 - k^2 v_{Te}^2}$$

Threshold: $v_D > v_{Ti} + v_{Te} \Rightarrow$ instab.

$\delta^2 F$:

threshold: $v_D < v_{Te} \Rightarrow$ $\delta^2 F$ positive definite



# Fourier Transform & "Get on the leaf"

e.g.
$$\delta m_\alpha = \sum_k N_k^{s(\alpha)} \sin kx + N_k^{c(\alpha)} \cos kx$$
etc.

## One set of canonical variables →

$$q_{k1}^{(\alpha)} = \frac{\sqrt{\pi}\, V_k^{c(\alpha)}}{v_{T\alpha}} m_\alpha \qquad P_{k_1}^{(\alpha)} = \frac{\sqrt{\pi}}{k} N_k^{s(\alpha)} v_{T\alpha}$$

$$q_{k2}^{(\alpha)} = \frac{\sqrt{\pi}}{k} N_k^{c(\alpha)} \qquad P_{k_2}^{(\alpha)} = \sqrt{\pi}\, V_k^{s(\alpha)} m_\alpha$$

## Another set when $v_{Te} < v_D < v_{Ti} + v_{Te}$

$$h = \frac{1}{2}\delta^2 F = \sum_{\substack{k \\ i=1,4}} \omega_i \frac{(P_j^2 + Q_j^2)}{2}$$

$\omega(k)$

$\underline{\omega_1 > 0}$, $\underline{\omega_2 > 0}$, $\underline{\omega_3 > 0}$

$\underline{\omega_4 < 0}$ ← Negative energy wave

Signature agrees w/ $\boxed{\operatorname{sgn}\left(\omega \frac{\partial \varepsilon}{\partial \omega}\right)}$

can show generally

# D. Bracket Perturbation Theory

$$\dot{z}^i = J^{ij} \frac{\partial I}{\partial z^j} \quad ; \quad z = z_e + \delta z$$

Expand to second order

$$\delta \dot{z} = \left( J + \frac{\partial J}{\partial z} \delta z \right) \left[ \cancel{\frac{\partial I}{\partial z}}^{\,0} + \frac{\partial^2 I}{\partial z^2} \delta z^2 + \frac{1}{2} \frac{\partial^3 I}{\partial z^3} \times \delta z^3 \right. + \cdots$$

truncates if Lie-Poisson

$$\delta \dot{z} = \{ \delta z, h_3 \}_2$$

$$\{ f ; g \}_2 = \frac{\partial f}{\partial \delta z} \left[ J + \frac{\partial J}{\partial z} \delta z \right] \frac{\partial g}{\partial \delta z}$$

$$h_3 = \frac{1}{2} \frac{\partial^2 I}{\partial z^2} \delta z^2 + \frac{1}{6} \frac{\partial^3 I}{\partial z^3} \delta z^3$$

## Nonlinearity in Bracket & Hamiltonian

Clean up bracket to $\mathcal{O}(\delta z^3)$ by

$$\delta \bar{Z} = A\, \delta z + \frac{1}{2} B (\delta z)^2$$

Puts all cubic (quad.) nonlinearity in Hamiltonian

# E. Hamiltonian Version of Resonant Three Wave Coupling Via Averaging for Two-Stream

Bracket Perturbation theory yields

$$H = H_0 + H_1$$

$$H_0 = \sum_{k=1}^{\infty} \left( \omega_1^k J_1^k + \omega_2^k J_2^k + \omega_3^k J_3^k + \omega_4^k J_4^k \right)$$

$q^2 + p^2$

Many degrees of freedom in plasma make it possible for resonance, i.e. $\exists$ k's s.t.

$$\boxed{\omega_4^{k_4} + \omega_1^{k_1} + \omega_2^{k_2} = 0}$$

recall $\omega_4 < 0$

$$H_1 \sim J_1^{1/2} J_2^{1/2} J_4^{1/2} \cos(\theta_1 + \theta_2 + \theta_4)$$

$$k_4 = k_1 + k_2 \qquad + \quad \langle \rangle \to 0$$

---

Explosive instability, e.g

$$J_1^{1/2} \sim \frac{A}{t_0 - t}$$ like Cherry's example

# $\delta W$ versus $\delta^2 F$

$\delta W$: Requires standard K.E. & P.E. terms (Lagrange's Theorem)

$$\ddot{X}^i = -\frac{\partial V}{\partial X^i} \qquad \text{linearize} \qquad X = X_e + \delta X e^{i\omega t}$$

$$\Rightarrow \quad \omega^2 \delta X^i = \frac{\partial^2 V}{\partial X^i \partial X^j} \delta X^j \equiv V_{ij} \delta X^j$$

$$\omega^2 \doteq \frac{\delta X^i V_{ij} \delta X^j}{\|\delta X\|^2}$$

2nd order accurate

insert trial function & search for $\omega^2 < 0$ $\Rightarrow$ unstable

$\delta^2 F$: Does not require standard K.E. & P.E.

$$i\omega \delta z^i = J^{ij} F_{jk} \delta z^k \qquad \delta z^i = J^{im} Y_m$$

projects onto leaf

$$\omega = \frac{\bar{Y}_i J^{ij} F_{jk} J^{k\ell} Y_\ell}{\bar{Y}_i (i J^{im}) Y_m}$$

2nd order accurate — symplectic inner product

Scenario: Suppose trial functions & equilibrium have parameters. Fix equil. parameters & vary trial function searching for $\omega = 0$ (vanishing numerator). First suppose $\delta^2 F > 0$, step equil. until $\omega = 0 \Rightarrow$ instability or neg. energy wave. Use denominator to distinguish ($\omega = i\gamma \Rightarrow \bar{Y} J Y = 0$)