WAVE Action,
Action-Angle Variables,
& Adiabatic Invariance

for the
Continuous Spectrum of Vlasov Poisson

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WHY WAVE Action?

It is useful for describing wave propagation in inhomogeneous time varying media, for example. For slowly varying media wave action is nearly constant, but the energy & frequency can change a lot.

WHENCE WAVE Action?

* Wave quanta, e.g. plasmons; analogy w/ q. mechanics. (where there?)
* Average Lagrangian techniques — Whitham e.g.
* Fooling around

* Hamiltonian idea: Wave Action = Action Variable
  Fundamental? = Adiabatic Invariant
LIMITATIONS

* quantum analogy is silly - where there?
* fooling around is good method for smart people
* Lagrangian techniques: 1) £ strange action principles
e.g. £\psi = 0 has

Where is \( \text{F=ma} \)?

\[ S_1[\psi] = \int (\dot{\psi})^2 \]
\[ S_2[\psi] = \int \psi + \dot{\psi} \text{ etc.} \]

2) Must develop methods of proof, beyond all orders etc. - lack Hamiltonian intuition

3) Methods for continuous spectrum?

These tools exist for Hamiltonian approach! The fundamental degrees of freedom are not "waves"

4) Problem is easy for fluid theories w/o wave-particle resonance
Vlasov–Poisson = Hamiltonian Field Theory

\[ \frac{\delta f (x, v, t)}{\delta t} + \{ f, \phi \} = 0 \]

\[ \{ f, \phi \} = \frac{\partial f}{\partial x} \frac{\partial \phi}{\partial t} - \frac{\partial f}{\partial t} \frac{\partial \phi}{\partial x} \]

\[ H[f] = \int \frac{m v^2}{2} f \, dx \, dv + \frac{1}{8\pi} \int E^2 \, dx \]

\[ \frac{\delta H}{\delta f} = \varepsilon = \frac{m v^2}{2} \]

\[ \frac{\partial f}{\partial t} = \{ f, H \} \]

\[ \{ f, H \} = \int f \left[ \frac{\partial F}{\partial f} \delta G \right] \, dx \, dv \]

\[ L \text{ noncanonical Poisson Bracket} \]

\[ C[f] = \int c(t) \, dx \, dv \]

\[ \{ C, F \} = 0 \quad \forall F \]
Linear Theory (about homogeneous equilibria)

\[ f = f_0(u) + \sum_{k} \frac{i}{2} f_k(u, t) e^{ikx} \]

\[ 2 F, G F_L = \frac{4i}{mv} \sum_{k=1}^{\infty} \int_{-\infty}^{\infty} du \frac{d f_0}{d u} \left( \frac{\partial F \partial G - \partial F \partial G}{\partial f_k \partial f_k - \partial f_k \partial f_k} \right) \]

\[ \frac{df_k}{dt} = [f_k, H_L] \]

\[ H_L = S^2 F = S^2 H + S^2 C \]

Kruskal-Oberman energy

\[ S^2 F = -\frac{m}{4} \int_{-\infty}^{\infty} \sum_{k} v \frac{d f_0}{d u} \frac{d f_k}{d u} \frac{|f_k|^2}{2} du \]

\[ + \frac{1}{16} \sum_{k} |E_k|^2 \]

\[ = \int \sum_{k} f_k(u) \Theta(v/u') f_k(u') du' du \]

Not Diagonal
Diagonalization

Mixed Variable Generating Function:

\[ F[P_k, Q_k] = \sum_k \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv \, Q_k(u) P_k(u) Y_k(u,v) \]

New \quad Old

\[ Y_k(u,v) = E_{I(k,v)} \frac{1}{\Pi} \frac{1}{u-v} + E_{R(k,v)} S(u-v) \]

\[ \frac{SF}{S_{Q_k(u)}} = P_k(u) = \int_{-\infty}^{\infty} P_k(u) Y_k(u,v) \, du \]

\[ \frac{SF}{S_{P_k(u)}} = Q_k(u) = \int_{-\infty}^{\infty} Q_k(u) Y_k(u,v) \, dv \]

Basic Identity: \[ \int_{-\infty}^{\infty} Y_k(u,u') Y_k(u,v) \, du = S(u-v') \]
\[ G_k(u,v) = \frac{1}{|E_k(u)|^2} \{ E_I(k,u) \frac{1}{\Pi} \frac{P}{u-v} + E_R(k,u) S(u-v) \} \]

Inserting \( Q_k(u) \) & \( P_k(u) \) into into \( H \) into diagonal form

\[ H[Q_k, P_k] = \int du \sum_k (-i k u) Q_k P_k \]

Action-Angle Variables

\[ Q_k = \sqrt{J_k} e^{-i \theta_k(u)} \quad P_k = i \sqrt{J_k(u)} e^{i \theta_k(u)} \]

\[ H[J_k, \theta_k] = \int du \sum_k (k u) J_k(u) \]

(like SHO \( \sum_k \omega_k J_k \))
Adiabatic Invariance for Continuous Spectra?

Suppose \( f_0 (\nu, et) \) has a slow time dependence \( \Rightarrow \text{ slow} \)

\[(Q_x, P_x) \leftrightarrow (Q_x, P_x) \text{ is an explicitly time dependent transformation (Since } E_{I, R} (k, \nu, \epsilon t)) \Rightarrow \]

\[H(Q_x, P_x) = \int du \sum_k (-i k \nu) Q_x P_x + \frac{\partial F}{\partial \epsilon} \]

\[\frac{\partial F}{\partial \epsilon} = \int du \int du' \int du'' Q_x (u') P_x (u'') \bar{\Phi} (u, u') \frac{\partial \Phi (u, u')}{\partial \epsilon} \]

\[\frac{\partial F}{\partial \epsilon} = \int du \int du' \int du'' \sqrt{J_x (u')} \ e^{-i \Phi (u')} \ e^{i \Phi (u)} \sqrt{J_x (u'')} \ \bar{\Phi} (u', u'') \frac{\partial \Phi (u', u'')}{\partial \epsilon} \]

\[\dot{J}_x = - \frac{\delta H}{\delta \theta_x} = - \frac{\partial}{\partial \theta_x} \left( \frac{\partial F}{\partial \epsilon} \right) = O(\epsilon) \]

\[\text{Small} \]
Consider an arbitrary time $T$ & integrate $\Rightarrow$

\[
\Delta J_{kT} := \int_0^T \dot{J}_k \, dt = -\int_0^T \sum_{\theta} \frac{\partial F}{\partial \theta_k} \, dt \quad \text{W. T. S.}
\]

\[
1 \Delta J_{kT} \leq \frac{CE}{|\kappa u|} \sim \frac{d \phi_0}{T}
\]

$C$ must be indep. of $T$!

With this, the action remains arbitrarily small with large changes in $\Delta \phi_0$ if $T$ is large, i.e. the change in $\phi_0$ is made very slowly over a long time.

**Question**

Can other wave actions (Kaufman, Bizard, Tracy, Crawford, ...) that are functions of $x \text{ & } t$ be represented as a sum over these continuum action variables?